Online Supplementary Appendix to: Contracting with Word-of-Mouth Management

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May 8, 2019

D Discussion

In this section we discuss various extensions and their implications, as well as the social planner's problem. For generality, we base the discussion on the generalized model introduced in the Appendix.

D.1 Heterogeneous WoM Cost

In Section B, we have entirely focused on homogeneous costs of talking, in order to emphasize the core trade-off faced by a firm when encouraging senders to engage in WoM. In the Online Appendix, we consider an extension in which different senders have different costs of talking. With heterogeneous costs of talking, the optimal reward scheme is more complicated as it can be used to *fine-tune the amount of WoM*, while with homogeneous costs either everyone or no one talks. We analyze the optimal scheme for a fairly general class of cost distribution G, and discuss how our results from Section B change. Here, we summarize the main findings of that section.

We show that the results from Section B are robust in the following sense. Free contracts are not optimal for large α because in that case the benefit of free contracts given by $(1-\alpha)r$ is small compared to the cost CF^*. Referrals and free contracts remain strategic substitutes. We also show how the homogeneous cost case can be thought of as the limit of models with heterogeneous costs.

New insights can be derived in the heterogeneous cost model with respect to the reward scheme. The optimal reward scheme is not constant in α when a free contact is offered (as it is when the cost of talking is homogeneous), but is increasing in α. The reason is that expected profits are higher with higher α and hence, the seller has a stronger incentive to increase WoM. If no free

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contracts are offered, in addition to the aforementioned effect, there is an opposing effect (that is present also with homogeneous costs), as the seller only needs to pay less to senders if the expected externalities are large in order to induce the same number of senders to talk. Thus, if no free contracts are offered the effect of α on rewards is ambiguous, where rewards are decreasing in α if costs are sufficiently homogeneous.

D.2 Continuous Type Space

In the model with two receiver types, the optimal scheme results in the low-type customers experiencing zero value from the product, a feature that may not be realistic. Our intention in the main section was to provide the simplest model that highlights the role of free contracts as a way to incentivize WoM, and the unrealistic feature is an artifact of the simplification, not an implication of the effect we want to highlight. The aim of this section is to make this claim formal.

To this end, we provide an alternative model with a continuous type space and characterize the optimal scheme. In particular, the characterization shows that under an open set of parameter values, conditional on a customer purchasing a free contract (which happens with positive probability), with probability one the customer receives a strictly positive value from the product.

Formally, let us consider the same model as in the main section with a continuum of receiver types. The receivers' types θ are uniformly distributed on $[0,1]$ and type θ's valuation for quantity q is given by

$$v_\theta(q) = \begin{cases} 0 & \text{if } q = 0 \\ \theta \ln(q+1) - K & \text{if } q > 0 \end{cases}$$

for some constant $K > 0$ that is independent of θ and q. Since $\lim_{q \searrow 0} v_\theta(q) = -K$, one can think of K as the fixed cost of starting to use the product. To simplify the exposition, let us assume $c < K < -\ln(c) - 1 + c$ and $c < 1$. Moreover, let us define, for $\theta \in (0,1]$, $\underline{q}(\theta) = e^{\frac{K}{\theta}} - 1$ which is the smallest quantity that must be offered to a type-θ receiver to make her indifferent between using the product and not. Note that $\underline{q}(\theta) > 0$ for all $\theta \in (0,1]$ and the receivers with $\theta = 0$ would not like to use the product for any $q \geq 0$. For simplicity, let us also assume $N = 1$.

The seller solves

$$\Pi^*(\xi) = \max_{p_\xi(\cdot),\, q_\xi(\cdot),\, \underline{\underline{\theta}}_\xi,\, R_\xi} \quad \mathbf{1}_{\{a_1=\text{Refer}\}} \cdot \left(\int_{\underline{\underline{\theta}}_\xi}^1 (p_\xi(\theta) - q_\xi(\theta)c)\, d\theta - R_\xi \right) \tag{12}$$

where $p_\xi \in \mathbb{R}^{[0,1]}$ and $q_\xi \in \mathbb{R}_+^{[0,1]}$ are integrable functions, $\underline{\underline{\theta}}_\xi \in [0,1]$, and $R_\xi \in \mathbb{R}$ subject to the receiver's incentive compatibility and participation constraints which are given by[1]

$$\begin{aligned} &\max\{v_\theta(q_\xi(\theta)), 0\} - p_\xi(\theta) \geq \max\{v_\theta(q_\xi(\theta')), 0\} - p_\xi(\theta') && \forall \theta, \theta' && (\theta\text{-type's IC}) \\ &\max\{v_\theta(q_\xi(\theta)), 0\} - p_\xi(\theta) \geq 0 && \forall \theta \geq \underline{\underline{\theta}}_\xi && (\theta\text{-type's PC}) \end{aligned} \tag{13}$$

and the sender's incentive compatibility (IC) constraint which is given by

$$a_1 = \text{Refer} \quad \text{if and only if} \quad \xi \leq r(1 - \underline{\underline{\theta}}_\xi) + R_\xi. \tag{14}$$

Define a strengthening of the constraint (14) by imposing a condition that the sender must talk, i.e.,

$$a_1 = \text{Refer holds and } \ \xi \leq r(1 - \underline{\underline{\theta}}_\xi) + R_\xi. \tag{14'}$$

We denote by $\tilde{\Pi}(\xi)$ the optimal profit of the problem (12) subject to (13) and (14').

In order to characterize the optimal scheme, we first define several notations. First, for $\xi = 0$, there exists a unique (up to measure-zero set of θ) solution to (12) subject to (13) and (14'), which satisfies

$$q_0^*(\theta) := \begin{cases} q^{**}(\theta) & \text{if } \theta \geq \underline{\underline{\theta}}_0^* \\ 0 & \text{if } \theta < \underline{\underline{\theta}}_0^* \end{cases} \tag{15}$$

where

$$q^{**}(\theta) := \frac{2\theta - 1}{c} - 1$$

and a $\underline{\underline{\theta}}_0^*$ which is the unique solution to $(2\theta - 1)\left[\ln\left(\frac{2\theta-1}{c}\right) - 1\right] - K + c = 0$ (we will prove this below).

[1] Note that an analogous result to Lemma 1 holds in this setup.

Second, let us denote by θ' the unique solution to $q^{**}(\theta') = \underline{q}(\theta')$. Finally, if

$$(2\theta' - 1)\left[\ln\left(\frac{2\theta' - 1}{c}\right) - 1\right] - K + c + r \le 0, \tag{16}$$

let θ'' denote the unique value of θ that solves $(2\theta - 1)\left[\ln(\frac{2\theta-1}{c}) - 1\right] - K + c + r = 0$, which always exists.

Proposition 7. *Let $\xi > 0$.*

(i) Whenever $\tilde{\Pi}(\xi) > 0$, there exists a unique solution (up to measure-zero set of types)[2] *to the problem (12) subject to (13) and (14), and it is a solution to (12) subject to (13) and (14').*

(ii) There is a unique solution (up to measure-zero set of types) to the problem (12) subject to (13) and (14') given by $(p^*_\xi(\cdot), q^*_\xi(\cdot), \underline{\underline{\theta}}^*_\xi, R^*_\xi)$. It has the following properties:

1. *If $\xi < r(1 - \underline{\underline{\theta}}^*_0)$, then neither a free contract nor reward is offered, i.e., $p^*_\xi(\theta) > 0$ if and only if $q^*_\xi(\theta) > 0$, and $R^*_\xi = 0$. Moreover, $q^*_\xi(\cdot) = q^*_0(\cdot)$ for $\theta \in [0, 1]$ and $\underline{\underline{\theta}}^*_\xi = \underline{\underline{\theta}}^*_0$.*

2. *Suppose $r(1 - \underline{\underline{\theta}}^*_0) \le \xi$.*

   (a) *If (16) is satisfied, then the following hold.*

      i. *No free contract is offered, i.e., $p^*_\xi(\theta) > 0$, if and only if $q^*_\xi(\theta) > 0$.*

      ii. *$q^*_\xi(\theta) = q^{**}(\theta)$ for $\theta \ge \underline{\underline{\theta}}^*_\xi$ and $q^*_\xi(\theta) = 0$ otherwise.*

      iii. *$\underline{\underline{\theta}}^*_\xi = \theta''$*

      iv. *A positive reward is offered, i.e., $R^*_\xi = \xi - r(1 - \underline{\underline{\theta}}^*_\xi) > 0$, if and only if $\xi > r(1 - \theta'')$.*

   (b) *If (16) is not satisfied, then there exists a $\underline{\theta}_\xi > \theta'$ such that the following hold.*[3]

      i. *For $\theta > \underline{\theta}_\xi$, no free contract is offered, i.e., $p^*_\xi(\theta) > 0$. For $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$, a free contract is offered, i.e., $p^*_\xi(\theta) = 0$. Otherwise, $p^*_\xi(\theta) = 0$.*

      ii. *$q^*_\xi(\theta) = q^{**}(\theta)$ for $\theta > \underline{\theta}_\xi$, $q^*_\xi(\theta) = \underline{q}(\underline{\underline{\theta}}^*_\xi)$ for $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$, and $q^*_\xi(\theta) = 0$ otherwise.*

      iii. *$\underline{\underline{\theta}}^*_\xi < \theta'$.*

      iv. *A positive reward is offered, i.e., $R^*_\xi = \xi - r(1 - \underline{\underline{\theta}}^*_\xi) > 0$, if and only if $\xi > r(1 - \underline{\underline{\theta}}^*_\xi)$.*

[2] It is not payoff-relevant for the firm if for a zero-mass of types a different contract satisfying the constraints is offered.

[3] The type $\underline{\theta}_\xi$ is determined such that $\frac{2\underline{\theta}_\xi - 1}{c} = \underline{q}(\underline{\underline{\theta}}^*_\xi)$.

The proof is presented at the end of this section. The proposition highlights that, as in the two-type case that we consider in the main analysis, the optimal scheme exhibits a rich pattern of the use of free contracts and referral rewards. In particular, it allows for the parameter regions such that both are used, only free contracts are used, only referral rewards are used, and none are used. To see our main point about the size of the surplus the receiver purchasing a free contract experiences, first note that a free contract is offered under an open set of parameter values because it is offered whenever $r(1-\underline{\underline{\theta}}^*) \leq \xi$ holds and (16) is not satisfied, and those conditions hold (case 2b of Proposition 7) for an open set of parameter values. Second, whenever a free contract $(\underline{q}(\underline{\underline{\theta}}_\xi^*), 0)$ is offered, it is purchased with a positive probability as all types $[\underline{\underline{\theta}}_\xi, \underline{\theta}_\xi]$ purchase that contract and $\underline{\underline{\theta}}_\xi < \underline{\theta}_\xi$, but everyone but $\underline{\underline{\theta}}_\xi$ receives strictly positive surplus $v_\theta(\underline{q}(\underline{\underline{\theta}}_\xi))$ from it.

Proof. **(Proposition 7)** Part (i) is straightforward, so we prove part (ii). Fix a solution to the problem (12) subject to (13) and (14') and denote it by $(p_\xi^*(\cdot), q_\xi^*(\cdot), \underline{\underline{\theta}}_\xi^*, R_\xi^*)$. We first rewrite the firm's problem. To this end, let us denote the utility received by type θ under the contract $(p_\xi(\theta), q_\xi(\theta))$ by $U(\theta) = v_\theta(q_\xi(\theta)) - p_\xi(\theta)$. Then, by a standard argument in mechanism design, the receivers' IC constraints can be rewritten as

$$U(\theta) = \int_{\underline{\underline{\theta}}_\xi}^{\theta} \ln(q_\xi(\tilde{\theta}) + 1) d\tilde{\theta} + U(\underline{\underline{\theta}}_\xi)$$

for $\theta \geq \underline{\underline{\theta}}_\xi$, $q_\xi(\cdot)$ being non-decreasing and $q_\xi(\theta) \geq \underline{q}(\theta)$ for $\theta \geq \underline{\underline{\theta}}_\xi$. The PC constraint and optimality then imply $U(\underline{\underline{\theta}}_\xi^*) = 0$. Then, the seller's objective function can be rewritten by substituting $U(\theta) = v_\theta(q_\xi(\theta)) - p_\xi(\theta)$ into $\int_{\underline{\underline{\theta}}_\xi}^1 (p_\xi(\theta) - q_\xi(\theta)c)\, d\theta$:

$$\begin{aligned} \int_{\underline{\underline{\theta}}_\xi}^1 (\theta \ln(q_\xi(\theta)+1) - K - q_\xi(\theta)c)\, d\theta &= \\ \int_{\underline{\underline{\theta}}_\xi}^1 (\theta \ln(q_\xi(\theta)+1) - K - U(\underline{\theta}) - q_\xi(\theta)c)\, d\theta - \int_{\underline{\underline{\theta}}_\xi}^1 \int_{\underline{\underline{\theta}}_\xi}^1 \mathbf{1}_{\{\tilde{\theta} \leq \theta\}} \cdot \ln(q_\xi(\tilde{\theta})+1) d\tilde{\theta}\, d\theta &= \\ \int_{\underline{\underline{\theta}}_\xi}^1 ((2\theta - 1)\ln(q_\xi(\theta)+1) - K - q_\xi(\theta)c)\, d\theta & \end{aligned}$$

yielding

$$\tilde{\Pi}(\xi) = \max_{\underline{\underline{\theta}}_\xi} \max_{q_\xi(\cdot),\, R_\xi} \quad \int_{\underline{\underline{\theta}}_\xi}^1 ((2\theta - 1)\ln(q_\xi(\theta) + 1) - K - q_\xi(\theta)c)\, d\theta - R_\xi \tag{17}$$

subject to

$$q_\xi(\cdot) \text{ being non-decreasing and } q_\xi(\theta) \geq \underline{q}(\theta) \text{ for } \theta \geq \underline{\underline{\theta}}_\xi, \tag{13'}$$

(14') and $p^*(\theta) = v(q_\xi(\theta)) - \int_{\underline{\underline{\theta}}_\xi}^{\theta} \ln(q_\xi(\tilde{\theta}) + 1)d\tilde{\theta}$.

Next, we solve this maximization problem for $\xi = 0$. Point-wise maximization of the integral with respect to $q_0(\theta)$ for a fixed θ results in the first-order condition given by $\frac{2\theta-1}{q_0(\theta)+1} - c = 0$, i.e., $q_0(\theta) = \frac{2\theta-1}{c} - 1$ and a second-order condition given by $-\frac{2\theta-1}{(q_0(\theta)+1)^2} < 0$. Thus, the solution of the first-order condition gives a maximum if $\theta > \frac{1}{2}$ and otherwise the unique solution of the maximization problem is $q_0(\theta) = 0$.

If we plug this into $(2\theta - 1)\ln(q_0(\theta) + 1) - K - q_0(\theta)c$, we get for $\theta > \frac{1}{2}$,

$$(2\theta - 1)(\ln((2\theta - 1)/c) - 1) - K + 1$$

which is strictly greater than zero for $\theta = 1$ if $-\ln(c) - 1 - K + c > 0$ which we assumed. It is exactly zero at some $\underline{\underline{\theta}}_0^*$ as long as $K > c$. Thus, (15) is a solution to the maximization problem as it is increasing. Also, note that θ' given by $q_0(\theta') = \underline{q}(\theta')$ is well defined as the equation has a unique solution no more than 1 as long as $K < -\ln(c)$ which is implied by the parameter restriction $K < -\ln(c) - 1 + c$ and $c < 1$. Then, $q_0^*(\theta) > \underline{q}(\theta)$ if and only if $\theta > \theta'$.

Part 1: If $\xi < r(1 - \underline{\underline{\theta}}_0^*)$, then the unconstrained solution (the solution to (12) subject to (13)) is also achievable with the constraint (the solution to (12) subject to (13) and (14')), so it is the unique optimum and no free contracts or rewards are provided under the optimal scheme.

Part 2: If $\xi \geq r(1 - \underline{\underline{\theta}}_0^*)$, then profits are zero unless some reward is paid or the good is sold to more buyers. It is immediate that the sender's IC (14') must be binding. To find the optimal scheme, we can, hence, substitute $\xi - r(1 - \underline{\underline{\theta}}_\xi)$ for R_ξ in the optimization problem, yielding

$$\tilde{\Pi}(\xi) = \max_{\underline{\underline{\theta}}_\xi} \max_{q_\xi(\cdot)} \quad \int_{\underline{\underline{\theta}}_\xi}^1 ((2\theta - 1)\ln(q_\xi(\theta) + 1) - K - q_\xi(\theta)c + r)\, d\theta - \xi$$

subject to (13'), $R^*_\xi = \xi - r(1 - \underline{\underline{\theta}}_\xi)$ and $p^*(\theta) = v(q_\xi(\theta)) - \int_{\underline{\underline{\theta}}_\xi}^{\theta} \ln(q_\xi(\tilde{\theta}) + 1) d\tilde{\theta}$. Point-wise maximization of $((2\theta - 1)\ln(q_\xi(\theta) + 1) - K - q_\xi(\theta)c + r)$ with respect to $q_\xi(\theta)$ yields $q^*_\xi(\theta) = q^{**}(\theta)$ for $\theta \geq \underline{\underline{\theta}}^*_\xi$ where $\underline{\underline{\theta}}^*_\xi$ solves

$$(2\theta - 1)\ln\left(\frac{2\theta - 1}{c}\right) - K - 2\theta + 1 - c + r = 0$$

as long as the solution satisfies $q^{**}(\underline{\underline{\theta}}^*_\xi) \geq \underline{q}(\underline{\underline{\theta}}^*_\xi)$ (i.e., $\underline{\underline{\theta}}^*_\xi \geq \theta'$), which is equivalent to (16).

Otherwise, since $\underline{q}(\cdot)$ is strictly decreasing, we need to apply bunching and offer a free contract at the bottom because the pointwise solution $\max\{q^{**}(\theta), \underline{q}(\theta)\}$ is decreasing for $\theta \in (0, \theta')$. More precisely, there exist $\underline{\theta}_\xi$ and $\underline{\underline{\theta}}^*_\xi$ such that for $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$, a free contract is offered, i.e., $p^*_\xi(\theta) = 0$ and $q^*_\xi(\theta) = \underline{q}(\underline{\underline{\theta}}^*_\xi)$ for $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$ under the optimal scheme.

A strictly positive reward is paid if and only if ξ is strictly higher than the induced externalities $r(1 - \underline{\underline{\theta}}^*_\xi)$. This concludes the proof of (ii). $\square$

D.3 Two-Sided Externalities

In the main analysis we assumed that only the senders receive externalities, and claimed that even if we assumed the receivers would receive externalities as well, the essence of the analysis would not change. The goal of this subsection is to make this formal. Consider a model as in Section 2, with an additional feature that if receiver i uses the product, she receives externalities r. In this model, for each $\theta \in \{H, L\}$, if a type-θ receiver uses quantity q, she experiences utility $v_\theta(q) + r$.

Note that this is a change that shifts the valuation functions by a constant, i.e., they change from $v_\theta(q)$ to $v_\theta(q) + r$ for each $\theta = H, L$. Hence, it does not alter the nature of the optimal contract scheme *under each fixed* r, assuming that our restrictions are met for the new valuation functions. This implies that all comparative statics with respect to parameters that are not r (e.g., Proposition 6) are robust. Below we show that our main comparative statics with respect to r (provided in Theorem 4) goes through as well.[4]

Note that Theorem 4 states that the use of free contracts is optimal if and only if the condition $r \in \left[\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right]$ is met. Then, the use of rewards is determined by conditions given by the bounds independent of the size of r (the conditions are $r < \xi$ in the presence of free contracts and

[4]We keep assuming that our restrictions are satisfied after the shifts of the valuation functions.

$r < \frac{\xi}{\alpha}$ otherwise, and ξ and $\frac{\xi}{\alpha}$ do not depend on r). It is immediate that the same characterization goes through in our modified model, but now the size of CF^* depends on r. If we show that CF^* is nonincreasing and $CF^* + \alpha r$ is nondecreasing in r, then the region of r such that free contracts are used is still given by a convex interval, guaranteeing that the essence of the comparative statics does not change. We first show that CF^* is strictly decreasing in r. To show this, let us write down the modified CF^* as follows:

$$CF^*(r) = \alpha(v_H(\underline{q}(r)) + r) + (1-\alpha)c\underline{q}(r),$$

where $CF^*(r)$ and $\underline{q}(r)$ denote the cost of free contracts under r and the break-even quantity for low-types under r (i.e., $v_L(\underline{q}(r)) + r = 0$), respectively. It is immediate that the second term is strictly decreasing in r because $v_L'(q)$ is strictly increasing in q and thus $\underline{q}(r)$ is strictly decreasing in r. The first term is strictly decreasing in r for the following reason: Take r and r' with $r < r'$. Then, by the assumption that $v_H'(q) > v_L'(q)$ and the definition of the $\underline{q}(\cdot)$ function, it must be the case that:

$$(v_H(\underline{q}(r')) + r') - (v_H(\underline{q}(r)) + r) < \big(v_L(\underline{q}(r')) - v_L(\underline{q}(r))\big) + (r' - r) = \big((-r') - (-r)\big) + (r' - r) = 0$$

Overall, $CF^*(r)$ is strictly decreasing in r. We next show that $CF^*(r) + \alpha r$ is strictly increasing in r under an additional assumption about the valuation functions. Specifically, suppose that $2v_L'(q) > v_H'(q) + \frac{1-\alpha}{\alpha}c$ for all $q > 0$. That is, the marginal values of the two types are not too different from each other, which ensures that the information rent $v_H(\underline{q}(r))$ does not vary too much with r. Then, taking the first-order condition of CF^* with respect to r and by noting $\underline{q}'(r) = -\frac{1}{v_L'(\underline{q}(r))}$ (by the Implicit Function Theorem), one can show that $CF^*(r) + \alpha r$ is strictly increasing in r. All in all, free contracts are used if and only if r is in a convex interval.

Note that this analysis provides an interesting observation that the cost of free contracts decreases in the size of externalities because both the production cost and the information rent decrease. The reason is that if low types receive externalities it becomes easier for the firm to make them willing to use the product (implying low production cost) and high types have less incentives to switch to the low-type contract at such a level of quantity provided to low types (implying lower information rent).

To sum up, the model of two-sided externalities provides qualitatively equivalent comparative statics as our main model with one-sided externalities.

D.4 Quantity-Dependent Externalities

The main analysis is based on a model in which the magnitude of externalities is captured by a single parameter r. As Theorem 4 shows, this is the key parameter that determines the optimal scheme. However, one can imagine that a Dropbox user who wants to refer his co-author receives higher positive externalities from joint usage if the co-author uses Dropbox more. The objective of this section is to formalize the idea of quantity-dependent externalities and discuss how such dependencies affect our predictions.

To this end, consider a function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}_+$ that assigns to each quantity level consumed the value of externalities generated. We employ the normalization that $\bar{r}(0) = 0$. Note that our main model corresponds to the case in which $\bar{r}(q) = r$ for all $q > 0$. In this section we assume that $\bar{r}$ is differentiable, strictly concave, $\bar{r}'(q) > 0$ for all $q \geq 0$ and $\lim_{q\to\infty} \bar{r}'(q) = 0$.

Fix an optimal scheme $((\bar{p}^*_L, \bar{q}^*_L), (\bar{p}^*_H, \bar{q}^*_H), \overline{R}^*)$. Then, the L-type's PC constraint and the H-type's IC constraint must be binding. First, consider the case when the sender's IC constraint is binding. In that case, (generically) positive rewards are being paid. Then, if a contract is offered to the low types ($\bar{q}^*_L > 0$), then the optimal scheme must satisfy the following first-order conditions:

$$\alpha(v'_H(\bar{q}^*_H) - c + \bar{r}'(\bar{q}^*_H)) \quad = \quad 0$$

and $\overline{q}^*_L \in \{0, \underline{q}\}$ (as in the main model) if

$$(1-\alpha)(v'_L(q_L) - c + \bar{r}'(q_L)) + \alpha(v'_L(q_L) - v'_H(q_L)) \quad < \quad 0 \tag{18}$$

holds for $q_L = \underline{q}$, and $\bar{q}^*_L$ satisfies the above inequality with equality otherwise.[5] For simplicity, we focus the discussion on the case when the inequality in (18) is satisfied for $q_L = \underline{q}$.

Otherwise, the contract has a positive price. If low types are not served under the optimal

[5]The solution exists and is unique as we assume $\bar{r}$ is strictly concave and the limit of its slope is zero.

contract scheme, then only the first-order condition for $\bar{q}_H^*$ need be satisfied. Thus, as in the main model, there are only three possible levels of realized externalities corresponding to the three contracts that the firm optimally chooses conditional on rewards being paid, $r_H := \bar{r}(\bar{q}_H^*)$, $r_L := \bar{r}(\underline{q})$ and $\bar{r}(0) = 0$. Note that in this case, $q_H^* \leq \bar{q}_H^*$ holds because $\bar{r}'(\bar{q}_H^*) > 0$ and v_H' is decreasing.

If the sender's IC constraint is not binding, then the sender's IC can be ignored and thus, the optimal contract is the same as in the main model, and in particular, $\bar{q}_H^* = q_H^*$. Let us denote the externalities received if the high type's contract is purchased by $r_h := \bar{r}(q_H^*)$.

Here we consider how the conditions for offering free contracts change. In the absence of free contracts, expected externalities are given by αr_H, while in the presence of free contracts, expected externalities are given by $\alpha r_H + (1-\alpha) r_L$. Now, consider part 2 of Theorem 4. It says that, for free contracts to be used in the optimal scheme, two conditions have to be met: $r(1-\alpha) \geq CF^*$ and $\xi - \alpha r \geq CF^*$. The first inequality says that the cost of free contracts has to be no more than the increment of the expected externalities. The second says that it has to be no more than the rewards necessary to be paid to compensate for the difference between the cost of talking and the externalities that are generated anyway by high types, in the absence of free contracts. Since the first inequality automatically holds when the sender's IC constraint does not bind, and the second inequality automatically holds when the sender's IC constraint binds, these conditions can be rewritten as:

$$r_L(1-\alpha) \geq CF^* \text{ and } \xi - \alpha r_h \geq CF^*.$$

Since CF^* is unchanged, these conditions imply that low externalities for low types and high externalities for high types both reduce the set of parameters for which free contracts are optimally offered. Thus, free contracts can be optimal only if the dependence of the magnitude of externalities does not vary too much with the quantity consumed by the receivers. Our main analysis corresponds to the (extreme) case with constant $\bar{r}$ functions, and hence best captures the role of free contracts.

D.5 Informed Senders

To simplify the analysis, in the main analysis we assume that each sender has the same information about the type of his receiver as the firm. However, in some markets one can imagine that senders have better information about their friends' willingness to pay than the firm. The objective of this section is to consider a model that accommodates this possibility, and to discuss robustness of and difference from the results of the main analysis. Specifically, let us assume that each sender independently observes a signal $s \in \{s_L, s_H\}$ about his receiver. If the receiver's type is $\theta = H$, the sender sees a signal $s = s_H$ with probability $\beta \in \left(\frac{1}{2}, 1\right)$, and if the receiver's type is $\theta = L$, the sender sees a signal $s = s_H$ with probability $1 - \beta$.[6] Thus, by Bayes rule, a sender who has received a signal s_H believes that the probability of facing a H-type receiver is $\alpha_H = \frac{\alpha\beta}{\alpha\beta + (1-\alpha)(1-\beta)} (> \alpha)$, while a sender who has received a signal s_L instead believes that the probability of facing a H-type receiver is $\alpha_L = \frac{\alpha(1-\beta)}{\alpha(1-\beta)+(1-\alpha)\beta} (< \alpha)$.

How does the firm's optimization problem change? The firm's objective function is a weighted sum of the profit generated by WoM of senders who have received a high signal and the profit generated by WoM of senders who have received a low signal. The two profit functions are as in (4) with the fraction of high valuation receivers being α_H and α_L, respectively. More precisely, a fraction $\alpha\beta + (1 - \alpha)(1 - \beta)$ of senders have received a high signal s_H and the expected profits generated by those senders is just (4) with the fraction of H-type receivers being α_H. A fraction $\alpha(1 - \beta) + (1 - \alpha)\beta$ of senders has received a low signal and the profit generated by those senders is (4) with the fraction of H-type receivers being α_L. Note that the receivers' constraints remain unchanged. However, the firm now faces two IC constraints for the senders - one for the senders who observed s_H and one for the senders who observed s_L.

An important difference to the model we consider in the main part is that Lemma 1 is not valid anymore as the firm can utilize the informational differences with the reward scheme.

Proposition 8 (Rewards with informed senders). *1. Suppose that all senders choose "Refer" under the optimal scheme.*

(a) If the firm does not offer free contracts, then the optimal reward scheme $\mathbf{R}$ *satisfies* $\mathbf{R}(H) \leq \mathbf{R}(L)$ *with the inequality being strict if* $r \in (0, \frac{\xi}{\alpha_L})$.[7]

(b) If the firm offers free contracts, then the optimal reward scheme $\mathbf{R}$ *satisfies* $\mathbf{R}(H) = \mathbf{R}(L) = \max\{\xi - r, 0\}$.

[6] If $\beta = \frac{1}{2}$ was the case, then senders and the firm would have exactly the same information about receivers. Our main model corresponds to this case.

[7] $\mathbf{R}(H) = \xi - r < \mathbf{R}(L) = \xi$ for $\xi \geq r$ and $\mathbf{R}(H) = 0 \leq \mathbf{R}(L) = \max\left\{\frac{\xi - \alpha_L r}{1-\alpha_L}, 0\right\}$ for $\xi < r$.

2. *Suppose that senders who received s_H choose "Refer" but other senders choose "Not" under the optimal scheme.*

   (a) *If the firm does not offer free contracts, then there exists an optimal reward scheme* $\mathbf{R}$ *such that* $\mathbf{R}(H) > \mathbf{R}(L) = 0$. *Moreover, any optimal reward scheme* $\mathbf{R}$ *satisfies* $\mathbf{R}(H) > \mathbf{R}(L) - r$.

   (b) *If the firm offers free contracts, then there exists an optimal reward scheme* $\mathbf{R}$ *such that* $\mathbf{R}(H) > \mathbf{R}(L) = 0$. *Moreover, any optimal reward scheme* $\mathbf{R}$ *satisfies* $\mathbf{R}(H) > \mathbf{R}(L)$.

Each of the four cases arises given a nonempty parameter region that we compute in the proof of Proposition 9 presented at the end of this section.[8] An important implication of this proposition is that, if the firm wants to incentivize all senders to talk, then she must pay *more* for referrals of L-type receivers than for H-type receivers because L-type senders' expected externalities are low. In contrast, if the firm is better off excluding senders who received signal s_L, then one optimal scheme only rewards referrals of premium users. Note that if the firm wants to induce s_L-senders to talk, it should also induce s_H-senders to talk because it is cheaper to provide incentives to s_H-senders and they talk to a better pool of receivers.

Solving the full problem is a daunting task because there are multiple cases to analyze depending on which type of senders are encouraged to talk. If the monopolist decides to encourage every sender to talk, the choice between free contracts and referral rewards can be tricky: offering free contracts can be very attractive in a market with fraction α_L of high types but not attractive in a market with fraction α_H of high types. As the firm cannot differentiate between buyers who have generated a high signal versus a low signal, it needs to trade off the benefits in both markets when deciding whether to offer free contracts. One can, however, easily derive the following results for the extreme cases:

Proposition 9 (Signal strength). *1. If $\xi - r < \alpha(p_H^* - cq_H^*)$, then there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, the unique optimal menu of contracts is given by $((0,0),(p_H^*, q_H^*))$, and there exists an optimal reward scheme $\mathbf{R}$, which satisfies $\mathbf{R}(L) = 0$. If $\xi - r \geq \alpha(p_H^* - cq_H^*)$, then for any $\beta \in (\frac{1}{2}, 1)$, the firm cannot make positive profits.*

[8] The proof for Proposition 8 is presented at the end of this section, too.

2. *Suppose that there exists a unique optimal menu of contracts* $((p_L, q_L), (p_H, q_H))$ *in the model without signals. Then, for all* $r \notin \left\{\frac{\xi}{\alpha}, \frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right\}$, *there exists* $\bar{\beta} > \frac{1}{2}$ *such that for all* $\beta \in (\frac{1}{2}, \bar{\beta})$, *there exists a unique optimal menu of contracts and it is* $((p_L, q_L), (p_H, q_H))$.

Part 1 shows that, if the signal strength β is too large, free contracts are not used by the seller. Part 2 then shows that the model we analyze in the main section without signals is reasonable when we think of the introduction of a new product category because in such a case β would be close to $\frac{1}{2}$.

Proof. **(Proposition 8)** 1. If all senders choose Refer, the IC constraints for all senders— those who see s_H and those who see s_L— must be satisfied. (a) Without free contracts, the senders' IC constraints are given by:

$$\xi \leq \alpha_H r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)) \quad \text{and} \quad \xi \leq \alpha_L r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)).$$

The optimal reward conditional on these constraints minimizes referral reward payments by making both senders' IC constraints binding whenever possible. The firm is able to do this if and only if $r \leq \xi$ and in that case the optimal reward scheme is given by $\mathbf{R}(H) = \xi - r$ and $\mathbf{R}(L) = \xi$. If $r > \xi$, it is optimal to set $\mathbf{R}(H) = 0$ and $\mathbf{R}(L) = \max\left\{\frac{\xi - \alpha_L r}{1 - \alpha_L}, 0\right\}$.

(b) With free contracts, the senders' IC constraints are given by:

$$\xi \leq r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)) \quad \text{and} \quad \xi \leq r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)).$$

Thus, it is optimal to set $\mathbf{R}(H) = \mathbf{R}(L) = \max\{\xi - r, 0\}$.

2. If senders who saw s_L do not talk, then only the IC constraint of a sender who sees s_H must be satisfied and the IC constraint of the sender who sees s_L must be violated.

(a) Without free contracts, the firm minimizes reward payments subject to these constraints by minimizing $\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)$ (i.e., making the IC for the sender with s_H binding whenever possible) such that

$$\alpha_L r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)) < \xi \leq \alpha_H r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)).$$

First, note that these inequalities imply $\mathbf{R}(H) > \mathbf{R}(L) - r$. Second, if a referral scheme with $\mathbf{R}(H), \mathbf{R}(L) \geq 0$ that satisfies these inequalities exists (this is the case whenever $\frac{\xi}{\alpha_L} - r \geq 0$), then the referral scheme given by $\mathbf{R}(L) = 0$, $\mathbf{R}(H) = \max\{\frac{\xi}{\alpha_H} - r, 0\}$ must maximize the seller's profits: The seller cannot increase profits by decreasing $\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)$.

(b) With free contracts, the constraints become

$$r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)) < \xi \leq r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)),$$

which imply $\mathbf{R}(H) > \mathbf{R}(L)$. By an analogous argument as in (a), a reward scheme satisfying these constraints exists if and only if $\xi - r \geq 0$ and in that case the scheme given by $\mathbf{R}(H) = \frac{\xi - r}{\alpha_H}$, $\mathbf{R}(L) = 0$ maximizes profits. □

Proof. **(Proposition 9)** 1. First, note that any optimal scheme results in one of the following three types of behaviors by the senders: Either (i) no senders talks, or (ii) all senders talk, or (iii) only senders who have received a s_H signal talk.[9]

If $\xi - r \geq \alpha(p_H^* - cq_H^*)$, then for all $\beta \in (\frac{1}{2}, 1)$ the firm cannot make positive profits. We assume from now on $\xi - r < \alpha(p_H^* - cq_H^*)$. We will show that for sufficiently large β, the firm can make positive profits, i.e., that we are in case (ii) or (iii).

Fix $\beta \in (\frac{1}{2}, 1)$. If $\xi - r\alpha_L \leq 0$, then all senders talk even without any reward payments as long as H-type receivers consume a positive quantity. Thus, we are in case (ii), and so for any optimal scheme $((p_H, q_H), (p_L, q_L), \mathbf{R})$, $\mathbf{R}(L) = 0$ and $q_L = 0$ hold.

We assume from now on that $r\alpha_L < \xi < \alpha(p_H^* - cq_H^*) + r$. Under a reward scheme $\mathbf{R}$ with $\mathbf{R}(L) = 0$ (as specified in Proposition 8) and $\mathbf{R}(H) = \frac{\max\{\xi - \alpha_H r, 0\}}{\alpha_H}$, the senders who have seen s_H talk, while senders who have seen s_L do not talk.

Next we show that, there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, it is not optimal to offer free contracts and the firm always chooses to be in case (iii). For this purpose, we compute the profits from cases (ii) and (iii).

- **Case (iii):** Since $\alpha_H \to 1$ as $\beta \to 1$, there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, it

[9] Note that there is no optimal scheme in which s_L-senders talk while s_H-senders do not talk. This is because $\alpha_H > \alpha_L$ and thus, given a scheme $((p_H, q_H), (p_L, q_L), \mathbf{R})$ where only s_L-senders talk, the seller can strictly increase profits by choosing a reward scheme $\mathbf{R}'$ with $\mathbf{R}'(H) = \mathbf{R}'(L) = \alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)$ while holding the menu of contracts fixed. Under this scheme, both sender types talk.

is not optimal to offer free contracts by the analysis in Section B. Thus, the profits are given by $\alpha\beta(p_H^* - cq_H^*) - (\alpha\beta + (1-\alpha)(1-\beta))\max\{\xi - \alpha_H r, 0\}$, which is greater than zero for sufficiently large β because it converges to $\overline{\Pi}_H^* \equiv \alpha(p_H^* - cq_H^*) - \alpha\max\{\xi - r, 0\} \geq \max\{\alpha(p_H^* - cq_H^*) - (\xi - r), \alpha(p_H^* - cq_H^*)\} > 0$ as $\beta \to 1$.

- **Case (ii):** We consider two cases: $\xi \geq r$ and $\xi < r$.
  - $\xi \geq r$**:** By Proposition 8, without free contracts, profits are given by $\alpha(p_H^* - cq_H^*) - (\xi - \alpha r)$ and with free contracts they are given by $\alpha(p_H^* - cq_H^*) - CF^* - (\xi - r)$. Both profits are strictly smaller than $\overline{\Pi}_H^*$.
  - $\xi < r$**:** Without free contracts, profits are given by $\alpha(p_H^* - cq_H^*) - (1-\alpha)\max\left\{\frac{\xi - \alpha_L r}{1-\alpha_L}, 0\right\}$ and with free contracts, they are $\alpha(p_H^* - cq_H^*) - CF^*$. Both profits converge to numbers that are smaller than $\overline{\Pi}_H^*$ as $\beta \to 1$.

Hence, there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, it is not optimal to offer free contracts and the firm always chooses to be in case (iii). This concludes the proof.

2. If $\beta = \frac{1}{2}$, then one can immediately see from the expressions above that profits coincide with the ones in the main section. Thus, by continuity, for any $r < \frac{\xi}{\alpha}$, there exists a $\bar{\beta} > \frac{1}{2}$ such that for all $\beta \in (\frac{1}{2}, \bar{\beta})$, $r < \frac{\xi}{\alpha_L}$ and $r < \frac{\xi}{\alpha_H}$. Similarly, for any $r \in \left(\frac{\xi}{\alpha_L}, \frac{CF^*}{1-\alpha_L}\right)$, there exists a $\bar{\beta} > \frac{1}{2}$ such that for all $\beta \in (\frac{1}{2}, \bar{\beta})$, $r \in \left(\frac{\xi}{\alpha_L}, \frac{CF^*}{1-\alpha_L}\right)$ and $r \in \left(\frac{\xi}{\alpha_H}, \frac{CF^*}{1-\alpha_H}\right)$. Analogous conclusions hold for intervals $\left(\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right)$ and $\left(\frac{\xi - CF^*}{\alpha}, \infty\right)$. Thus, there exists a $\bar{\beta} > \frac{1}{2}$ such that for all $\beta \in (\frac{1}{2}, \bar{\beta})$, the same analysis as in the main section applies for β. □

D.6 Multiple Senders per Receiver

In the main model, we consider a stylized network structure between senders and receivers, i.e., receiver i is connected only to sender i, and *vice versa*. In reality, however, it is possible that a receiver is connected to multiple potential senders of the same information. Similarly to the discussion in the Online Appendix where the receiver can learn from an advertisement, a receiver has multiple sources of information if there are multiple senders. Such a situation can arise when senders and receivers are connected through a general network structure.

In this section we discuss how the predictions change when there are multiple senders per receiver. To make our point as clear as possible, let us assume that once a receiver adopts a product, each sender who talked to the receiver experiences the same externalities of r. That is, if there are m senders for a given receiver, then the total externalities generated by the receiver are mr. The reward can be conditioned on the set of senders who talked.

Let $m > 1$ be the number of senders connected to a given receiver. Suppose that, when there is only one sender, R is the optimal expected referral reward. The conclusion in Lemma 1 (or the analysis in the Online Appendix on advertising) entails that, by paying R in expectation to each sender, the firm can give the same incentive of talking to the senders. However, such an adjustment changes the firm's total payment. This is because the expected payment of referral reward is no longer R, but mR.

This implies that the firm becomes reluctant to use referral rewards. More precisely, if the optimal reward level is zero in the model with one sender per receiver, then it is still zero in the model with multiple senders per receiver. At the same time, free contracts become relatively more attractive as it incentivizes senders in the same way as with only one sender. Thus, when there are multiple senders per receiver, the range of parameter values such that only free contracts are used becomes wider because free contracts can substitute referral rewards.

D.7 Social Optimum

In order to understand the monopolist's strategy better, we consider the social planner's solution and compare it with the solution obtained in the main section. Specifically, we consider a social planner who has control over the senders' actions $a_i \in \{\text{Refer}, \text{Not}\}$ and the quantities q_L and q_H offered to receivers, while she does not have control over receivers' choice of whether to actually use the product after it is allocated.[10] Rewards and prices do not show up in the social planner's problem because they are only transfers between agents.

We start with two basic observations. First, whenever WoM takes place under the monopolist's solution, there is a surplus from WoM. Hence, it is also in the social planner's interest to encourage WoM. Second, under the monopolist's optimal scheme, free contracts always make senders weakly better off by increasing the probability of receiving externalities, high-type receivers better off

[10] In the classic setup of Maskin and Riley (1984), all buyers get positive utility from using the product, and thus, they always use the product after purchase. If we were to allow the social planner to have control over the use of the product and $v'_L(q) < c$ for all $q > 0$, then she would have low types use just a little bit of the quantity and generate the externalities r, which we view as implausible.

by reducing the price due to the information rent, and low-type receivers indifferent because their participation constraint is always binding. This implies that, if the monopolist firm optimally offers free contracts, then it is also socially optimal to offer it. We summarize these two observations in the following proposition:

Proposition 10. *1. If there exists a monopolist's solution under which a_i = Refer for all i, then there exists a social planner's solution that entails a_i = Refer for all i.*

2. If there exists $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ for some R under the monopolist's solution, then there exists a social planner's solution that entails $q_L = \underline{q}$.

The converse of each part of the above proposition is not necessarily true, i.e., the monopolist may be less willing to encourage WoM than the social planner or not offer free contracts despite it being socially optimal. To see this clearly, we further investigate the social planner's problem in what follows.

Conditional on free contracts being offered, the welfare-maximizing menu of quantities (q_H, q_L) is exactly the same as the menu offered by the monopolist in the main section. To see why, first note that, as in the classic screening problem in Maskin and Riley (1984), the monopolist's solution results in no distortions at the top, i.e., $v'(q_H) = c$. Conditional on selling to the low types, the low-type quantity q_L under the second best in Maskin and Riley (1984) is distorted to deter high types to switch to the contract offered to low types. This means that the social planner's solution dictates that low types receive more quantity in the first best than in the second best. In our problem, however, the welfare-maximizing quantity cannot be strictly higher than $\underline{q}$ because the marginal cost c is higher than the marginal benefit $v_L'(q)$ for all $q \geq \underline{q}$ (Assumption 2), and the incentive-compatible quantity cannot be strictly lower than $\underline{q}$ because the low types would not use the product for $q_L < \underline{q}$.

Finally, whether or not the sender talks under the social planner's solution depends on the comparison between the total benefit from talking and the cost of talking, ξ: In total, WoM is efficient if and only if

$$\alpha(v_H(q_H^*) - cq_H^* + r) + (1 - \alpha)\max\{r - c\underline{q}, 0\} \geq \xi. \tag{19}$$

Note that there are two social benefits of WoM. First, WoM creates network externalities because the senders and receivers become aware of each other using the product. Second, it creates gains from trade because some high-valuation buyers learn about the product.

In the monopolist's solution, free contracts are not used if $r < \frac{CF^*}{1-\alpha}$. Substituting the definition of CF^* shows that this is equivalent to $r - c\underline{q} < \frac{\alpha}{1-\alpha} v_H(\underline{q})$. Since the social planner uses free contracts if $0 < r - c\underline{q}$, the monopolist uses free contracts too little from the social planner's point of view conditional on it being socially optimal to encourage WoM if r is high, and α or $v_H(\underline{q})$ is high. The reason is as follows. On the one hand, high externalities r imply a high additional benefit r from having a receiver using the product, so that the social planner wants all receivers to use the product. However, such r pertains to the senders and the monopolist cannot extract the entire corresponding surplus. On the other hand, the monopolist is reluctant to use free contracts if the information rent necessary to induce high types to purchase a premium contracts is high relative to the number of low types who choose the free contracts. The "per low-type" information rent $\frac{\alpha}{1-\alpha} v_H(\underline{q})$ is high if α or $v_H(\underline{q})$ is high.

D.8 Effect of Advertising

In this section, we investigate how the optimal incentive scheme changes if the firm can also engage in classic advertising. Formally, consider the situation in which the firm has an option to conduct costly advertising before WoM takes place. The firm spends $a \in \mathbb{R}_+$ for advertising and this is observed by all senders but not by any receivers. Then, each receiver independently becomes aware of the product prior to the communication stage with probability $p(a)$, where $p(0) = 0$ and $p(a) > 0$ for $a > 0$. The firm simultaneously chooses a menu of contracts, a reward scheme, and advertising spending. We assume that the sender does not observe whether the receiver is already aware of the product and only enjoys externalities if the receiver starts using the product *and* she engages in WoM (independently of whether the receiver learns through advertising and/or WoM) since otherwise she cannot know whether the receiver uses the product or not. The reward scheme is now a function $\mathbf{R} : \{L, H\} \times \{A, N\} \to \mathbb{R}_+$. Here, $\mathbf{R}(\theta, A)$ denotes the reward paid to the sender whose receiver purchases the contract offered to θ-types and becomes aware of the product through advertising. Similarly, $\mathbf{R}(\theta, N)$ denotes the reward paid to the sender whose receiver purchases the contract offered to θ-types and does not become aware of the product through advertising.[11]

Having completely specified the model with advertising, let us now analyze it. Note first that Lemma 2 again holds without any modification. Suppose now that the reward scheme $\mathbf{R}$ and the advertising level a is part of the optimal scheme, and all senders choose Refer under such an optimal scheme. We assume $a > 0$ and derive a contradiction. To show this, consider the following modification of the scheme. First, let $R \equiv \alpha\,(p(a)\mathbf{R}(H, A) + (1 - p(a))\mathbf{R}(H, N)) + (1 - \alpha)\,(p(a)\mathbf{R}(L, A) + (1 - p(a))\mathbf{R}(L, N))$ be the expected reward, and construct a new reward scheme $\mathbf{R}'$ such that $\mathbf{R}'(\theta, x) = R$ for all $\theta = H, L$ and $x = A, N$. As in Lemma 1, this new scheme also satisfies the constraints and gives rise to the same expected profit, so it is optimal, too. Now, consider changing $a > 0$ to a new advertising level $a' = 0$. With the new scheme $(\mathbf{R}', a')$, the constraints are still satisfied; in particular all the senders choose Refer. Also, the expected profit to the monopolist increases by $a > 0$. This contradicts the assumption that the original scheme with $(\mathbf{R}, a)$ is optimal. All in all, this argument implies that either (i) the firm chooses a positive advertising level and no WoM takes place or (ii) WoM takes place and $a = 0$. Note that, in case (i), compared to the model in Section 2, advertising either substitutes WoM or allows the firm to inform some receivers if encouragement of WoM was too expensive.

D.9 Dynamic Extension

Our base model assumes a static environment, in which the receiver does not become a sender. A full analysis of a dynamic extension of the model is beyond the scope of this paper, but here we offer a simple dynamic model in a stationary environment to demonstrate the robustness of our results to dynamic extensions. Specifically, our objective is to show that coexistence of a free contract and referral rewards in the optimal scheme.

Specifically, suppose that time is discrete and double infinite, $t = \dots, -2, -1, 0, 1, 2, \dots$. Before the entire dynamic process starts, the seller offers a scheme $((p_L, q_L), (p_H, q_H), R) \in \mathbb{R}^5_+$. At each time t, there are a continuum of customers who know the product and consume a positive amount, and their measure is denoted by $\mu_t > 0$. Each of them talks to another new customer, so that measure μ_t of new customers are informed. Among the customers who are informed, a fraction ρ

[11] We assume that the externalities r do not depend on a. Such dependence may arise if WoM is conducted with self-enhancement motive as in Campbell et al. (2015). In such a model, r would be decreasing in a, and advertising becomes an even less attractive option for the firm than in the current model.

of them drop out from the market for some exogenous reason. Then, depending on the menu of contracts offered by the seller, each new customer makes a purchase decision. Finally, there is also an inflow of customers of size S, who know the product and make purchase decisions depending on the menu of the contract. The sum of the measures of the customers knowing the product and consuming a positive amount is then denoted μ_{t+1}. In total, each customer lives for two periods unless the customer drops out of the market with probability ρ.

We are interested in the steady state of this process, i.e. $\mu_{t+1} = \mu_t$. The seller's objective is to maximize the per-period profit, which we define to be the per-customer profit times stationary μ_t.

There are two differences from the static (full) model. First, the total population size is larger if a contract is offered to the L-type compared to if the L-type does not purchase in equilibrium. To see this, suppose first that the menu of contracts is such that the L-type customers do not make a purchase. Let $\mu^H := \mu_t$ for each t in this case. Then,

$$(1-\rho)\alpha\mu^H + \alpha S = \mu^H, \text{ or } \mu^H = \frac{S}{\rho + (\frac{1}{\alpha} - 1)}.$$

Second, let $\mu^{HL} := \mu_t$ for each t be the population size at each period when the menu of contracts is such that the both types make a purchase. Then,

$$(1-\rho)\mu^{HL} + S = \mu^{HL}, \text{ or } \mu^{HL} = \frac{S}{\rho},$$

hence $\mu^H < \mu^{HL}$.

The second difference is the participation constraint of the receiver. When deciding between purchasing and not, the receiver has to take into account the surplus from talking in the next period. Especially, if a free contract is used, this surplus may be strictly positive.

One can solve this model analytically for each parameter combination. In particular, for parameter combinations specifying a niche market and a not-too private product, one can show that both a free contract and referral rewards are used in an optimal scheme (e.g., any parameter combinations around $\alpha = 0.05$, $r = 58$, and $\rho = 0.8$). Although we do not present the full characterization for the entire parameter space as it is beyond the scope of this paper, this demonstrates that the key insights and tradeoffs are also present in a dynamic environment, showing the robustness of our results in a generalization to a dynamic model.

E Heterogeneous Costs of WoM

In this Online Appendix, we consider the case with heterogeneous costs of talking. Specifically, we assume that, after each sender i sees the menu of contrast, he privately observes his cost of talking ξ_i, drawn from an independent and identical distribution with a cumulative distribution function $G : \mathbb{R}_+ \to [0,1]$. The firm maximizes the expected profits where the expectation is taken with respect to G. We restrict attention to twice differentiable G with $G' = g$ satisfying $g(\xi) > 0$ for all $\xi \in \mathbb{R}_+$ and

Assumption 4. *G is strictly log-concave, i.e., $\frac{g}{G}$ is strictly decreasing.*

This condition is satisfied by a wide range of distributions such as exponential distributions, a class of gamma, Weibull, and chi-square distributions, among others. Note that those restrictions are sufficient to imply the conditions for the existence result (Proposition 5) which are stated in the proof of Proposition 5.

Section E.1 characterizes the optimal scheme. Section E.2 conducts comparative statics of the optimal scheme. Section E.3 contains all the proofs for these results. Section E.4 discusses how the main model with homogeneous costs can be viewed as a limit of models with heterogeneous costs.

E.1 Properties of Optimal Contracts

First, we characterize the optimal reward. If free contracts are offered, it acts as a substitute for reward payments, which results in higher optimal rewards absent free contracts. The following proposition provides conditions under which a positive reward is optimally offered.

Lemma 5 (Optimal Reward)**.** *In the model with heterogeneous costs, there exists r^{free} and $r^{not\ free}$ with $r^{not\ free} > r^{free}$ such that the following are true:*

1. *If $r < r^{free}$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies $R > 0$.*
2. *If $r^{free} \le r < r^{not\ free}$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies either $R > 0$ and $q_L = 0$, or $R = 0$ and $q_L = \underline{q}$.*

3. *If* $r^{\textit{not free}} \leq r$, *then* $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ *implies* $R = 0$.

In order to prove this, we fix a menu of contracts with and without free contracts satisfying the conditions in Lemma 2 and solve for the optimal reward scheme. That is, conditional on offering free contracts ($q_L = \underline{q}$), we define the maximal profit under (r, α) by

$$\Pi^{\text{free}}(r, \alpha) = \max_{R \geq 0} \left(\left[\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - R \right] \cdot G(r + R) \right)$$

and conditional on offering no free contracts ($q_L = 0$), define the maximal profit under (r, α) by

$$\Pi^{\text{not free}}(r, \alpha) = \max_{R \geq 0} \left([\pi((0, 0), (p_H^*, q_H^*)) - R] \cdot G(\alpha r + R) \right).$$

Let us also define the unique optimal reward given that free contracts are offered and that no free contracts are offered by $R^{\text{free}}(r, \alpha)$ and $R^{\text{not free}}(r, \alpha)$, respectively.

There are three reasons why $r^{\text{not free}} > r^{\text{free}}$ holds. As opposed to a situation without free contracts, with free contracts, (i) positive quantity is offered to low types, (ii) information rent is provided to high types, and (iii) the sender receives full externalities conditional on talking. All these effects reduce the incentive to provide referral rewards. Note that $r^{\text{not free}}$ corresponds to $\frac{\xi}{\alpha}$ in the homogeneous model, while r^{free} corresponds to ξ. In the homogeneous-cost setting, only reason (iii) affected the comparison of r^{free} and $r^{\text{not free}}$. The effects (i) and (ii) were present, but they only determined whether offering free contracts generates nonnegative profits.

The following theorem summarizes some general properties of optimal contracts. Unlike Theorem 4, it is not a full characterization, but it shows that many features of the optimal scheme with homogeneous cost carries over to the ones for heterogeneous costs.

Theorem 5 (Optimal Contracts)**.** *The following claims hold in the model with heterogeneous costs:*

1. **(Positive profits)** $\Pi^{\textit{not free}}(r, \alpha) > 0$ *for all* $r \in [0, \infty)$ *and* $\alpha \in (0, 1)$.

2. **(Using both rewards and free contracts)** *There exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *such that* $R > 0$ *(i.e., it is optimal to provide both free contracts and rewards) if and only if*

$$r^{\textit{free}} > r \geq \frac{CF^*}{1 - \alpha}. \tag{20}$$

3. *Suppose that* $\frac{G(\xi)}{g(\xi)}$ *is convex.*

   (a) **(Free vs. no free contracts)** *There exist* $\underline{r}, \overline{r} \in [\frac{CF^*}{1-\alpha}, \infty)$ *such that there exists* $((0, \underline{q}), (\tilde{p}^*_H, q^*_H), R) \in \mathcal{S}$ *for some* $R \geq 0$ *(i.e., it is optimal to provide free contracts) if and only if* $r \in [\underline{r}, \overline{r}]$*.*

   (b) **(Never free contracts)** *If* $\frac{CF^*}{1-\alpha} > r^{not\ free}$*, then* $[\underline{r}, \overline{r}] = \emptyset$*.*

First, unlike in the homogeneous-cost model, profits without offering free contracts are always positive: With homogeneous costs, profits without free contracts are negative when the share of high types are low, so the expected externalities are low. This is because low expected externalities imply that a sufficient size of reward is necessary to encourage WoM, but such a cost cannot be compensated by the profits generated by only a small fraction of high types. With heterogenous costs, there always exists some fraction of customers with sufficiently small WoM costs, who do not need to be rewarded to initiate referrals.

Part 2 of the proposition shows that even with heterogeneous costs we can derive necessary and sufficient conditions for a combination of free contracts and rewards programs to be offered. As with homogeneous cost, free contracts are only optimal for sufficiently large externalities r and rewards are only offered for sufficiently small externalities.

For a full characterization of the optimal menu of contracts, it is useful to impose the additional assumption that $\frac{G}{g}$ is convex. This condition is, for example, satisfied by the exponential distribution. Given this assumption, free contracts are only offered for an intermediate connected range of externalities r. We can extend these results qualitatively as follows.

Remark 4. If we do not impose $\frac{G}{g}$ to be convex, one can still show that $\lim_{r\to 0} \Pi^{\text{not free}}(r, \alpha) > \lim_{r\to 0} \Pi^{\text{free}}(r, \alpha)$ and $\lim_{r\to\infty} \Pi^{\text{not free}}(r, \alpha) > \lim_{r\to\infty} \Pi^{\text{free}}(r, \alpha)$, i.e., free contracts can only be optimal if r is not too large and not too small.

Remark 5. With homogeneous cost $\xi > 0$, $\underline{r}$, r^{free}, $\overline{r}$ and $r^{\text{not free}}$ correspond to $\frac{CF^*}{1-\alpha}$, ξ, $\frac{\xi - CF^*}{\alpha}$, and $\frac{\bar{\xi}}{\alpha}$, respectively. In Section E.4, we formalize this correspondence by considering a limit of models with heterogeneous costs converging to the one with the homogeneous cost.

Table 2 summarizes the results of Lemma 5 and Theorem 5 for the case when $\frac{G(\xi)}{g(\xi)}$ is convex.

Externalities	$r < r^{\text{free}}$	$r^{\text{free}} < r < r^{\text{not free}}$			$r^{\text{not free}} < r$
Referral rewards	Yes	No	or	Yes	No
Free contracts	No $\Leftrightarrow r < \frac{CF^*}{1-\alpha}$	Yes		No	Yes $\Leftrightarrow r$ is small

Table 2: Comparative Statics with respect to r with heterogeneous WoM costs

E.2 Comparative Statics

Deriving precise comparative statics in the heterogeneous setup is daunting. While it is straightforward to show that $\Pi^{\text{not free}}(r, \alpha)$ and $\Pi^{\text{free}}(r, \alpha)$ are increasing in the size of externalities (r) and the fraction of the high types (α), it is hard to pin down how the comparison between these two values are affected as we change parameters (r and α). Nevertheless, using the partial characterization of the optimal contracts we can make comparative statics to understand robustness and changes of our results with the introduction of heterogeneity of WoM costs.

Proposition 11 (Market Structure and Free Contracts)**.** *The following claims hold in the model with heterogeneous costs for any fixed* $r \in [0, \infty)$*.* $\lim_{\alpha\to 0} \Pi^{\textit{not free}}(r, \alpha) > \lim_{\alpha\to 0} \Pi^{\textit{free}}(r, \alpha)$ *and* $\lim_{\alpha\to 1} \Pi^{\textit{not free}}(r, \alpha) > \lim_{\alpha\to 1} \Pi^{\textit{free}}(r, \alpha)$*.*[12]

The intuition for Proposition 11 is as follows. The only reason to offer free contracts is to boost up the expected externalities by $(1 - \alpha)r$, and such boosting is not significant if α is high, hence offering free contracts is suboptimal in those cases. With homogeneous costs, we showed in Section B that free contracts are optimal only when α is small. Similarly, with heterogeneous costs, a free contract cannot be optimal for high α. Moreover, if α is too small, $\Pi^{\text{free}}(r, \alpha) < 0$ holds because there are too few high types to compensate for the high cost of free contracts, and $\Pi^{\text{not free}}(r, \alpha) > 0$ holds because a strictly positive share of senders with very small WoM cost talk by part 1 of Theorem 5. This effect was not present with homogeneous costs, where the seller does not incentivize WoM at all, resulting in $\Pi^* = 0$.

The previous arguments imply that if there exists a set of parameters such that free contracts are optimal, then the choice of free versus non-free contracts is non-monotonic with respect to both r and α.

The comparative statics of the optimal reward scheme is more intricate with heterogeneous costs of WoM as the sender can fine-tune the number of senders that she wants to incentivize to

[12] These limits exist because of the monotonicity in α.

engage in WoM.

Proposition 12 (Optimal Reward Scheme)**.** *Let $r < r^{free}$. Then, the following hold in the model with heterogeneous costs:*

(i) $R^{free}(r,\alpha)$ is increasing in α. $R^{not\ free}(r,\alpha)$ is increasing in α if and only if $\alpha r \hat{G}'(\alpha r + R^{not\ free}(r,\alpha)) < \Pi^{classic}$, where we define $\hat{G}(\xi) \equiv \frac{G(\xi)}{g(\xi)}$ for all $\xi \in \mathbb{R}_+$.

(ii) $R^{free}(r,\alpha)$ and $R^{not\ free}(r,\alpha)$ are decreasing in r.

(iii) Referrals and free contracts are strategic substitutes, i.e. $R^{free}(r,\alpha) < R^{not\ free}(r,\alpha)$ for all $r \in (0, r^{not\ free})$ and $\alpha \in (0,1)$.

Although part (ii) has the same prediction as in the case with homogeneous WoM costs, the prediction in part (i) is different. We first explain the comparative statics regarding $R^{\text{free}}(r,\alpha)$. Under homogeneous costs, every sender talks and every receiver buys anyway under the usage of free contracts, so α does not affect the optimal reward level. With heterogeneous costs, however, the firm needs to tradeoff the gain and loss of increasing the rewards. The gain is the additional receivers who hear from the senders who start talking due to the increase of the rewards. The loss is the additional payments. The gain is increasing in α, so the firm has more incentive to raise the rewards.

The relationship of the optimal reward and α conditional on no free contracts being offered is ambiguous because two forces are present. First, higher α means more benefit from the receivers, and this contributes to the incentive to raise the rewards. On the other hand, higher α means more expected externalities, so there is less need to bribe a given sender. This contributes to lowering the rewards. Naturally, the second effect dominates when senders are relatively homogeneous, and indeed the optimal reward is strictly decreasing when G is completely homogeneous as in the main analysis. To formalize this idea, define

$$HMG \equiv \sup_x \left(\frac{G}{g}\right)'(x)$$

which can be interpreted as a measure of homogeneity of costs. If HMG is large, it means that there is a small range of costs of WoM that are held by many senders and HMG goes to infinity in the limit as G converges to the completely homogeneous one. An implication of the condition in part (i) of Proposition 12 is that there exists $\overline{HMG} > 0$ such that if $HMG < \overline{HMG}$, then

$R^{\text{not free}}(r, \alpha)$ is increasing in α.

Recall that both free contracts and positive rewards are used if and only if $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}})$.

Proposition 13 (Market Structure and Using Both Rewards and Free Contracts)**.** *The following claims hold in the model with heterogeneous costs:*

1. $\frac{CF^*}{1-\alpha}$ *and* r^{free} *are strictly increasing in* α*.*

2. $\frac{CF^*}{1-\alpha}$ *is strictly increasing and* r^{free} *is strictly decreasing in* c*.*

As in the homogeneous-cost model, free contracts can only be optimal if the size of externalities r is larger than $\frac{CF^*}{1-\alpha}$. Since this number is increasing in α, free contracts are optimal for small r in niche markets with small α. Thus, free contracts and referral rewards should be jointly used in niche markets (small α) if externalities are rather small, while they should be used in mass (larger α) markets if externalities are comparably larger.

With homogeneous costs, all receivers use the product under free contracts. Thus, what corresponds to r^{free} (which is ξ) does not vary with α or c. With heterogenous costs, however, it varies with these parameters. This is because the increase in α or decrease in c contributes to an increase of the expected profit per receiver, which increases the firm's incentive to offer referral rewards.

E.3 Proofs

Proof. **(Lemma 5)** First, we show the existence of unique cutoffs r^{free} and $r^{\text{not free}}$. The first-order condition of $\Pi^{\text{free}}(r, \alpha)$ with respect to R is that (i) $R = 0$ or (ii) $R > 0$ and

$$g(r+R) \cdot \left[\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - R - \frac{G(r+R)}{g(r+R)}\right] = 0.$$

Note that the expression in the bracket on the left-hand side is strictly decreasing given Assumption 4 and varies continuously from ∞ to $-\infty$ as R varies from $-\infty$ to ∞. Hence, the optimal reward is always unique in $\mathbb{R}$. Also, the same argument implies that there exists a unique r such that $\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - \frac{G(r)}{g(r)} = 0$. Let this unique r be r^{free}. That is, the left-hand side of the first-order condition is nonpositive and thus $R^{\text{free}}(r, \alpha) = 0$ if and only if $r \geq r^{\text{free}}$.

Analogously, conditional on offering no free contracts ($q_L = 0$), the optimal reward is unique in $\mathbb{R}$ and there exists a unique r such that $\pi((0, 0), (p_H^*, q_H^*)) - \frac{G(\alpha r)}{g(\alpha r)} = 0$. We denote this r by

$r^{\text{not free}}$. As before, we have that $R^{\text{not free}}(r,\alpha) = 0$ if and only if $r \geq r^{\text{not free}}$.

Finally, we show that $r^{\text{free}} < r^{\text{not free}}$. To see this, note that Assumption 4 implies $\frac{G(\alpha r)}{\alpha r} < \frac{G(r)}{r}$ for $r > 0$ and $\alpha \in (0,1)$. Together with $\pi((0,0),(p_H^*,q_H^*)) > \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))$, $r^{\text{free}} < \alpha r^{\text{not free}}$ follows by Assumption 4 and the definitions of r^{free} and $r^{\text{not free}}$. Since $\alpha < 1$, this implies $r^{\text{free}} < r^{\text{not free}}$. □

Proof. **(Theorem 5)**

1. By Assumption 3, $\pi((0,0),(p_H^*,q_H^*)) > 0$ holds. Also, since $g(\xi) > 0$ for all $\xi \in \mathbb{R}_+$, $G(\xi) > 0$ for all $\xi > 0$. Hence, for any $r \in [0,\infty)$ and $\alpha \in (0,1)$, $[\pi((0,0),(p_H^*,q_H^*)) - R] \cdot G(\alpha r + R) > 0$ holds if $R \in (0, \pi((0,0),(p_H^*,q_H^*)))$. Thus, $\Pi^{\text{not free}}(r,\alpha) > 0$.

2. Note that the use of both, free contracts and positive rewards, is optimal only if $r < r^{\text{free}}$. Also, $r < r^{\text{free}}$ implies that rewards are positive. Furthermore, in that case the maximization problems defining $\Pi^{\text{free}}(r,\alpha)$ and $\Pi^{\text{not free}}(r,\alpha)$ both have inner solutions, so the two maximization problems can be rewritten as:

$$\begin{aligned} \Pi^{\text{free}}(r,\alpha) &= \max\nolimits_{x\in\mathbb{R}}(A^{\text{free}} - x)\cdot G(x) \\ \Pi^{\text{not free}}(r,\alpha) &= \max\nolimits_{x\in\mathbb{R}}(A^{\text{not free}} - x)\cdot G(x) \end{aligned} \tag{21}$$

   where $A^{\text{free}} = \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) + r$ and $A^{\text{not free}} = \pi((0,0),(p_H^*,q_H^*)) + \alpha r$. Thus, $\Pi^{\text{free}}(r,\alpha) \geq \Pi^{\text{not free}}(r,\alpha)$ if and only if

$$\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) + r \geq \pi((0,0),(p_H^*,q_H^*)) + \alpha r.$$

   This is equivalent to $r \geq \frac{CF^*}{1-\alpha}$. Also, by part 1 of the current theorem, $\Pi^{\text{free}}(r,\alpha) \geq \Pi^{\text{not free}}(r,\alpha)$ implies $\Pi^{\text{free}}(r,\alpha) > 0$. Overall, there exists an optimal scheme such that both free contracts and positive rewards are used if and only if $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}})$.

3. Consider a variable

$$\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)}. \tag{22}$$

   This variable is well-defined because the denominator is always strictly positive by part 1 of the current theorem.

Step 1: Note that for $r \geq r^{\text{not free}}$, Lemma 5 shows that the rewards are zero in any optimal scheme. Hence, $\Pi^{\text{free}} = \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot G(r)$ and $\Pi^{\text{not free}} = \pi((0,0),(p_H^*, q_H^*)) \cdot G(\alpha r)$ hold, and thus (22) is differentiable with respect to r. If $\frac{G}{g}$ is convex, then

$$\frac{\partial}{\partial r}\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)} = \frac{\partial}{\partial r}\frac{\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot G(r)}{\pi((0,0),(p_H^*, q_H^*)) \cdot G(\alpha r)} = \frac{\left(\frac{G(\alpha r)}{g(\alpha r)} - \alpha\frac{G(r)}{g(r)}\right) \cdot \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot g(r) \cdot g(\alpha r)}{[\pi((0,0),(p_H^*, q_H^*))] \cdot G(\alpha r)^2} < 0.$$

Thus, when $r \geq r^{\text{not free}}$, either (i) free contracts are not optimal for any $r \in [r^{\text{not free}}, \infty)$, or (ii) there exists a $\overline{r}' \geq r^{\text{not free}}$ such that there exists an optimal scheme in which free contracts are offered for $r \in [r^{\text{not free}}, \overline{r}']$, and no free contracts are offered under any optimal scheme for $r > \overline{r}'$. It must be the case that $\overline{r}' < \infty$ because

$$\lim_{r\to\infty}\frac{\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot G(r)}{\pi((0,0),(p_H^*, q_H^*)) \cdot G(\alpha r)} = \frac{\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*))}{\pi((0,0),(p_H^*, q_H^*))} < 1.$$

We let $\bar{r} = \bar{r}'$ in case (ii).

Step 2: Next, we consider the following three different cases: $\frac{CF^*}{1-\alpha} < r^{\text{free}}$, $\frac{CF^*}{1-\alpha} \in [r^{\text{free}}, r^{\text{not free}}]$, and $\frac{CF^*}{1-\alpha} > r^{\text{not free}}$.

- Let $\frac{CF^*}{1-\alpha} < r^{\text{free}}$. Then, it follows from part 2 of the current theorem that no free contracts are offered for $r < \frac{CF^*}{1-\alpha}$ and free contracts are offered for $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}}]$. For $r \in [r^{\text{free}}, r^{\text{not free}}]$,

$$\frac{\partial}{\partial r}\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)} = \frac{\partial}{\partial r}\frac{[\pi((0,\underline{q}),(p_H^*, q_H^*))] \cdot G(r)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*, q_H^*)) - R] \cdot G(\alpha r + R)} = \frac{\left(\frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} - \alpha\frac{G(r)}{g(r)}\right) \cdot \pi((0,\underline{q}),(p_H^*, q_H^*)) \cdot g(r) \cdot g(\alpha r + R^{\text{not free}}(r,\alpha))}{[\pi((0,0),(p_H^*, q_H^*)) - R^{\text{not free}}(r,\alpha)] \cdot G(\alpha r + R^{\text{not free}}(r,\alpha))^2}.$$

  Note that $\Pi^{\text{not free}}(r,\alpha)$ is differentiable in r by the Envelope Theorem. Moreover, if $\frac{G(\alpha r+R)}{g(\alpha r+R)} - \alpha\frac{G(r)}{g(r)} < 0$, then $\alpha r + R < r$ because $\frac{G(\xi)}{g(\xi)}$ is increasing in ξ by Assumption 4. Moreover, $R^{\text{not free}}(r,\alpha)$ is differentiable in r by the implicit function theorem applied to the first-order condition of $\Pi^{\text{not free}}$ and, letting $\hat{G}(\xi) := \frac{G(\xi)}{g(\xi)}$ for all ξ,

$$\frac{\partial}{\partial r}R^{\text{not free}}(r,\alpha) = -\frac{\alpha\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))}{1 + \hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))} < 0.$$

Thus,

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{G(\alpha r+R^{\text{not free}}(r,\alpha))}{g(\alpha r+R^{\text{not free}}(r,\alpha))}-\alpha\frac{G(r)}{g(r)}\right) &= \\
\alpha\left(\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))-\hat{G}'(r)\right)+\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))\frac{\partial}{\partial r}R^{\text{not free}}(r,\alpha) &= \\
\alpha\left(\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))-\hat{G}'(r)-\frac{\hat{G}'(\alpha r+R^{\text{not free}})^2}{1+\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))}\right) &< 0.
\end{aligned}
$$

Thus, if the derivative of (22) is negative at $r'\in[r^{\text{free}},r^{\text{not free}}]$ then (22) is decreasing for all $r\in[r',r^{\text{not free}}]$. Together with Step 1, this implies the following. In case (i), there exists $\overline{r}\in[r^{\text{free}},r^{\text{not free}})$ such that free contracts are offered in an optimal scheme if and only if $r\in[\frac{CF^*}{1-\alpha},\overline{r}]$. In case (ii), the current analysis shows that it is optimal to offer free contracts for all $r\in[r^{\text{free}},r^{\text{not free}}]$, so free contracts are offered if and only if $r\in[\frac{CF^*}{1-\alpha},\overline{r}]$, where $\overline{r}$ is the variable that we defined in Step 1.

- Let $\frac{CF^*}{1-\alpha}\in\left[r^{\text{free}},r^{\text{not free}}\right]$. In that case, offering free contracts is not optimal for any $r<r^{\text{not free}}$. Then, either free contracts are not optimal for any r or by the same argument as above, if free contracts are not used in an optimal scheme for $r=r'$ then they are not used in any optimal scheme for any $r>r'$. This proves the desired claim for this case.
- If $\frac{CF^*}{1-\alpha}>r^{\text{not free}}$, then offering free contracts is not optimal for any $r<r^{\text{free}}$. For $r\in\left[r^{\text{free}},r^{\text{not free}}\right]$ free contracts are also not optimal because

$$
1>\frac{\max_{R\in\mathbb{R}}[\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))-R]\cdot G(r+R)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*))-R]\cdot G(\alpha r+R)}\geq\frac{[\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))]\cdot G(r)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*))-R]\cdot G(\alpha r+R)}.
$$

The first inequality follows from the proof of part 2 of the current theorem. For $r\geq r^{\text{not free}}$, offering free contracts is never optimal by Step 1.

This concludes the proof.

$\square$

Proof. **(Proposition 11)** First, note that we can write the limiting profits as

$$\begin{aligned}
\lim_{\alpha \to 1} \Pi^{\text{free}}(r,\alpha) &= \max_{x \geq r} (\tilde{p}_H^* - cq_H^* + r - x)G(x) < \\
\lim_{\alpha \to 1} \Pi^{\text{not free}}(r,\alpha) &= \max_{x \geq r} (p_H^* - cq_H^* + r - x)G(x).
\end{aligned}$$

It follows immediately from part 1 of Theorem 5 that there exist $\alpha' > 0$ and $\epsilon > 0$ such that $\Pi^{\text{free}}(r,\alpha) + \epsilon < \Pi^{\text{not free}}(r,\alpha)$ for any $\alpha \in (0,\alpha')$, hence the limit result as $\alpha \to 0$ holds. ☐

Proof. **(Proposition 12)** Applying the implicit function theorem to the first-order conditions of Π^{free} and $\Pi^{\text{not free}}$ gives us:

(i) $\frac{R^{\text{free}}(r,\alpha)}{\partial \alpha} = -\frac{p_H^* - q_H^* c - v_H(\underline{q}) + c\underline{q}}{-1-\hat{G}'(r+R)} > 0$ and

$$\frac{\partial R^{\text{not free}}(r,\alpha)}{\partial \alpha} = -\frac{p_H^* - q_H^* c - r\hat{G}'(\alpha r + R)}{-1 - \hat{G}'(r+R)}$$

which is strictly greater than zero if and only if $r\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) < p_H^* - q_H^* c$, or $\alpha r \hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) < \Pi^{\text{classic}}$.

(ii) $\frac{R^{\text{free}}(r,\alpha)}{\partial r} = -\frac{-\hat{G}'(R+r)}{-1-\hat{G}'(R+r)} < 0$ and $\frac{R^{\text{not free}}(r,\alpha)}{\partial r} = -\frac{-\alpha\hat{G}'(R+\alpha r)}{-1-\hat{G}'(R+\alpha r)} < 0$ because $\hat{G}'(x) > 0$ for all $x > 0$, so $-1 - \hat{G}'(x) < 0$.

(iii) First, note that for $r > r^{\text{free}}$, referral rewards are always zero when free contracts are offered, i.e., the statement is trivially true. If $r \leq r^{\text{free}}$, then the optimal reward with free contracts $R^{\text{free}}(r,\alpha)$ satisfies the first-order condition:

$$R^{\text{free}}(r,\alpha) = \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) - \frac{G(r + R^{\text{free}}(r,\alpha))}{g(r + R^{\text{free}}(r,\alpha))}. \tag{23}$$

By the first-order condition for the maximization problem for the case with no free contracts with respect to the reward, the solution $R^{\text{not free}}(r,\alpha)$ must satisfy:

$$g(\alpha r + R^{\text{not free}}(r,\alpha)) \cdot \left(\pi((0,0),(p_H^*, q_H^*)) - R^{\text{not free}}(r,\alpha) - \frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} \right) = 0.$$

Since $g(\cdot) > 0$, this implies that

$$\pi((0,0),(p_H^*,q_H^*)) - R^{\text{not free}}(r,\alpha) - \frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} = 0. \tag{24}$$

Now, substitute $R^{\text{not free}}(r,\alpha)$ by the expression for $R^{\text{free}}(r,\alpha)$ given by (23) on the left hand side of (24), to obtain:

$$\pi((0,0),(p_H^*,q_H^*)) - \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) + \frac{G(r + R^{\text{free}}(r,\alpha))}{g(r + R^{\text{free}}(r,\alpha))} - \frac{G(\alpha r + R^{\text{free}}(r,\alpha))}{g(\alpha r + R^{\text{free}}(r,\alpha))}.$$

This is strictly positive by log-concavity of G (Assumption 4) and because $\pi((0,0),(p_H^*,q_H^*)) > \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))$. Noting that the left hand side of (24) is strictly decreasing in referral rewards, the optimal reward without free contracts $R^{\text{not free}}(r,\alpha)$ is strictly greater than $R^{\text{free}}(r,\alpha)$. □

Proof. **(Proposition 13)** The comparative statics with respect to $\frac{CF^*}{1-\alpha}$ are straightforward from the formula of CF^*. The ones for r^{free} follow from the first-order condition with respect to rewards that appears in the proof of Lemma 5 and Assumption 4. □

E.4 Homogeneous Costs as the Limit of Heterogeneous Costs

Consider a sequence $\{G^n\}_1^\infty$ that converges pointwise to the G defined by $G = \mathbf{1}_{\{\bar{\xi} \leq \xi\}}$ such that for each n, G^n is twice differentiable with $(G^n)'(\xi) = g^n(\xi) > 0$ for all ξ, and Assumption 4 holds. Let the set of all such sequences be $\mathcal{G}$. The set $\mathcal{G}$ is nonempty. For example, consider $\{G^n\}_1^\infty$ such that for each $n \in \mathbb{N}$, G^n is a normal distribution with mean $\bar{\xi} \geq 0$ and variance $\frac{1}{n}$ truncated at $\xi = 0$. By inspection one can check that $\{G^n\}_1^\infty \in \mathcal{G}$. For any given G^n, we can define $\underline{r}^n$, $r^{\text{free},n}$, $\bar{r}^n$, and $r^{\text{not free},n}$. Then, the following statement can be shown: For any $\{G^n\}_1^\infty \in \mathcal{G}$,

$$\lim_{n\to\infty} \underline{r}^n = \frac{CF^*}{1-\alpha}, \quad \lim_{n\to\infty} r^{\text{free},n} = \bar{\xi}, \quad \lim_{n\to\infty} \bar{r}^n = \frac{\bar{\xi} - CF^*}{\alpha}, \quad \text{and} \quad \lim_{n\to\infty} r^{\text{not free},n} = \frac{\bar{\xi}}{\alpha}.$$