Online Supplementary Appendix to: Contracting with Word-of-Mouth Management

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B Heterogeneous Costs of WoM

In this Online Appendix, we consider the case with heterogeneous costs of talking. We restrict attention to twice differentiable G with $G' = g$ satisfying $g(\xi) > 0$ for all $\xi \in \mathbb{R}_+$ and

Assumption 4. *G is strictly log-concave, i.e., $\frac{g}{G}$ is strictly decreasing.*

This condition is satisfied by a wide range of distributions such as exponential distributions, a class of gamma, Weibull, and chi-square distributions, among others.

Section B.1 characterizes the optimal scheme. Section B.2 conducts comparative statics of the optimal scheme. Section B.3 contains all the proofs for these results. Section B.4 discusses how the main model with homogeneous costs can be viewed as a limit of models with heterogeneous costs.

B.1 Properties of Optimal Contracts

First, we characterize the optimal reward. If free contracts are offered, it acts as a substitute for reward payments, which results in higher optimal rewards absent free contracts. The following proposition provides conditions under which a positive reward is optimally offered.

Lemma 5 (Optimal Reward)**.** *In the model with heterogeneous costs, there exists r^{free} and $r^{not\ free}$ with $r^{not\ free} > r^{free}$ such that the following are true:*

1. *If $r < r^{free}$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies $R > 0$.*
2. *If $r^{free} \leq r < r^{not\ free}$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies either $R > 0$ and $q_L = 0$, or $R = 0$ and $q_L = \underline{q}$.*
3. *If $r^{not\ free} \leq r$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies $R = 0$.*

In order to prove this, we fix a menu of contracts with and without free contracts satisfying the conditions in Lemma 2 and solve for the optimal reward scheme. That is, conditional on offering free contracts ($q_L = \underline{q}$), we define the maximal profit under (r, α) by

$$\Pi^{\text{free}}(r, \alpha) = \max_{R \geq 0} \left(\left[\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - R \right] \cdot G(r + R) \right)$$

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and conditional on offering no free contracts ($q_L = 0$), define the maximal profit under (r, α) by

$$\Pi^{\text{not free}}(r, \alpha) = \max_{R \geq 0} \left([\pi((0,0),(p_H^*, q_H^*)) - R] \cdot G(\alpha r + R) \right).$$

Let us also define the unique optimal reward given that free contracts are offered and that no free contracts are offered by $R^{\text{free}}(r, \alpha)$ and $R^{\text{not free}}(r, \alpha)$, respectively.

There are three reasons why $r^{\text{not free}} > r^{\text{free}}$ holds. As opposed to a situation without free contracts, with free contracts, (i) positive quantity is offered to low types, (ii) information rent is provided to high types, and (iii) the sender receives full externalities conditional on talking. All these effects reduce the incentive to provide referral rewards. Note that $r^{\text{not free}}$ corresponds to $\frac{\bar{\xi}}{\alpha}$ in the homogeneous model, while r^{free} corresponds to $\bar{\xi}$. In the homogeneous-cost setting, only reason (iii) affected the comparison of r^{free} and $r^{\text{not free}}$. The effects (i) and (ii) were present, but they only determined whether offering free contracts generates nonnegative profits.

The following theorem summarizes some general properties of optimal contracts. Unlike Theorem 1, it is not a full characterization, but it shows that many features of the optimal scheme with homogeneous cost carries over to the ones for heterogeneous costs.

Theorem 2 (Optimal Contracts)**.** *The following claims hold in the model with heterogeneous costs:*

1. **(Positive profits)** $\Pi^{\textit{not free}}(r, \alpha) > 0$ *for all* $r \in [0, \infty)$ *and* $\alpha \in (0, 1)$.

2. **(Using both rewards and free contracts)** *There exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *such that* $R > 0$ *(i.e., it is optimal to provide both free contracts and rewards) if and only if*

$$r^{\textit{free}} > r \geq \frac{CF^*}{1 - \alpha}. \tag{13}$$

3. *Suppose that* $\frac{G(\xi)}{g(\xi)}$ *is convex.*

   (a) **(Free vs. no free contracts)** *There exist* $\underline{r}, \overline{r} \in [\frac{CF^*}{1-\alpha}, \infty)$ *such that there exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *for some* $R \geq 0$ *(i.e., it is optimal to provide free contracts) if and only if* $r \in [\underline{r}, \overline{r}]$.

   (b) **(Never free contracts)** *If* $\frac{CF^*}{1-\alpha} > r^{\textit{not free}}$, *then* $[\underline{r}, \overline{r}] = \emptyset$.

First, unlike in the homogeneous-cost model, profits without offering free contracts are always positive: With homogeneous costs, profits without free contracts are negative when the share of high types are low, so the expected externalities are low. This is because low expected externalities imply that a sufficient size of reward is necessary to encourage WoM, but such a cost cannot be compensated by the profits generated by only a small fraction of high types. With heterogenous costs, there always exists some fraction of customers with sufficiently small WoM costs, who do not need to be rewarded to initiate referrals.

Part 2 of the proposition shows that even with heterogeneous costs we can derive necessary and sufficient conditions for a combination of free contracts and rewards programs to be offered.

Externalities	$r < r^{\text{free}}$	$r^{\text{free}} < r < r^{\text{not free}}$			$r^{\text{not free}} < r$
Referral rewards	Yes	No	or	Yes	No
Free contracts	No $\Leftrightarrow r < \frac{CF^*}{1-\alpha}$	Yes		No	Yes $\Leftrightarrow r$ is small

Table 2: Comparative Statics with respect to r with heterogeneous WoM costs

As with homogeneous cost, free contracts are only optimal for sufficiently large externalities r and rewards are only offered for sufficiently small externalities.

For a full characterization of the optimal menu of contracts, it is useful to impose the additional assumption that $\frac{G}{g}$ is convex. This condition is, for example, satisfied by the exponential distribution. Given this assumption, free contracts are only offered for an intermediate connected range of externalities r. We can extend these results qualitatively as follows.

Remark 1. If we do not impose $\frac{G}{g}$ to be convex, one can still show that $\lim_{r\to 0} \Pi^{\text{not free}}(r, \alpha) > \lim_{r\to 0} \Pi^{\text{free}}(r, \alpha)$ and $\lim_{r\to\infty} \Pi^{\text{not free}}(r, \alpha) > \lim_{r\to\infty} \Pi^{\text{free}}(r, \alpha)$, i.e., free contracts can only be optimal if r is not too large and not too small.

Remark 2. With homogeneous cost $\bar{\xi} > 0$, $\underline{r}$, r^{free}, $\overline{r}$ and $r^{\text{not free}}$ correspond to $\frac{CF^*}{1-\alpha}$, $\bar{\xi}$, $\frac{\bar{\xi}-CF^*}{\alpha}$, and $\frac{\bar{\xi}}{\alpha}$, respectively. In Appendix B.4, we formalize this correspondence by considering a limit of models with heterogeneous costs converging to the one with the homogeneous cost.

Table 2 summarizes the results of Lemma 5 and Theorem 2 for the case when $\frac{G(\xi)}{g(\xi)}$ is convex.

B.2 Comparative Statics

Deriving precise comparative statics in the heterogeneous setup is daunting. While it is straightforward to show that $\Pi^{\text{not free}}(r, \alpha)$ and $\Pi^{\text{free}}(r, \alpha)$ are increasing in the size of externalities (r) and the fraction of the high types (α), it is hard to pin down how the comparison between these two values are affected as we change parameters (r and α). Nevertheless, using the partial characterization of the optimal contracts we can make comparative statics to understand robustness and changes of our results with the introduction of heterogeneity of WoM costs.

Proposition 7 (Market Structure and Free Contracts)**.** *The following claims hold in the model with heterogeneous costs for any fixed* $r \in [0, \infty)$. $\lim_{\alpha\to 0} \Pi^{\textit{not free}}(r, \alpha) > \lim_{\alpha\to 0} \Pi^{\textit{free}}(r, \alpha)$ *and* $\lim_{\alpha\to 1} \Pi^{\textit{not free}}(r, \alpha) > \lim_{\alpha\to 1} \Pi^{\textit{free}}(r, \alpha)$.[28]

The intuition for Proposition 7 is as follows. The only reason to offer free contracts is to boost up the expected externalities by $(1-\alpha)r$, and such boosting is not significant if α is high, hence offering free contracts is suboptimal in those cases. With homogeneous costs, we showed in Section 3 that free contracts are optimal only when α is small. Similarly, with heterogeneous costs, a free contract cannot be optimal for high α. Moreover, if α is too small, $\Pi^{\text{free}}(r, \alpha) < 0$ holds because there are too few high types to compensate for the high cost of free contracts, and $\Pi^{\text{not free}}(r, \alpha) > 0$ holds

[28] These limits exist because of the monotonicity in α.

because a strictly positive share of senders with very small WoM cost talk by part 1 of Theorem 2. This effect was not present with homogeneous costs, where the seller does not incentivize WoM at all, resulting in $\Pi^* = 0$.

The previous arguments imply that if there exists a set of parameters such that free contracts are optimal, then the choice of free versus non-free contracts is non-monotonic with respect to both r and α.

The comparative statics of the optimal reward scheme is more intricate with heterogeneous costs of WoM as the sender can fine-tune the number of senders that she wants to incentivize to engage in WoM.

Proposition 8 (Optimal Reward Scheme)**.** *Let $r < r^{free}$. Then, the following hold in the model with heterogeneous costs:*
(i) $R^{free}(r,\alpha)$ is increasing in α. $R^{not\ free}(r,\alpha)$ is increasing in α if and only if $\alpha r \hat{G}'(\alpha r + R^{not\ free}(r,\alpha)) < \Pi^{classic}$, where we define $\hat{G}(\xi) \equiv \frac{G(\xi)}{g(\xi)}$ for all $\xi \in \mathbb{R}_+$.
(ii) $R^{free}(r,\alpha)$ and $R^{not\ free}(r,\alpha)$ are decreasing in r.
(iii) Referrals and free contracts are strategic substitutes, i.e. $R^{free}(r,\alpha) < R^{not\ free}(r,\alpha)$ for all $r \in (0, r^{not\ free})$ and $\alpha \in (0,1)$.

Although part (ii) has the same prediction as in the case with homogeneous WoM costs, the prediction in part (i) is different. We first explain the comparative statics regarding $R^{\text{free}}(r,\alpha)$. Under homogeneous costs, every sender talks and every receiver buys anyway under the usage of free contracts, so α does not affect the optimal reward level. With heterogeneous costs, however, the firm needs to tradeoff the gain and loss of increasing the rewards. The gain is the additional receivers who hear from the senders who start talking due to the increase of the rewards. The loss is the additional payments. The gain is increasing in α, so the firm has more incentive to raise the rewards.

The relationship of the optimal reward and α conditional on no free contracts being offered is ambiguous because two forces are present. First, higher α means more benefit from the receivers, and this contributes to the incentive to raise the rewards. On the other hand, higher α means more expected externalities, so there is less need to bribe a given sender. This contributes to lowering the rewards. Naturally, the second effect dominates when senders are relatively homogeneous, and indeed the optimal reward is strictly decreasing when G is completely homogeneous as in the main analysis. To formalize this idea, define

$$HMG \equiv \sup_x \left(\frac{G}{g}\right)'(x)$$

which can be interpreted as a measure of homogeneity of costs. If HMG is large, it means that there is a small range of costs of WoM that are held by many senders and HMG goes to infinity in the limit as G converges to the completely homogeneous one in (4). An implication of the condition in part (i) of Proposition 8 is that there exists $\overline{HMG} > 0$ such that if $HMG < \overline{HMG}$, then $R^{\text{not free}}(r,\alpha)$ is increasing in α.

Recall that both free contracts and positive rewards are used if and only if $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}})$.

Proposition 9 (Market Structure and Using Both Rewards and Free Contracts)**.** *The following claims hold in the model with heterogeneous costs:*

1. *$\frac{CF^*}{1-\alpha}$ and r^{free} are strictly increasing in α.*
2. *$\frac{CF^*}{1-\alpha}$ is strictly increasing and r^{free} is strictly decreasing in c.*

As in the homogeneous-cost model, free contracts can only be optimal if the size of externalities r is larger than $\frac{CF^*}{1-\alpha}$. Since this number is increasing in α, free contracts are optimal for small r in niche markets with small α. Thus, free contracts and referral rewards should be jointly used in niche markets (small α) if externalities are rather small, while they should be used in mass (larger α) markets if externalities are comparably larger.

With homogeneous costs, all receivers use the product under free contracts. Thus, what corresponds to r^{free} (which is $\bar{\xi}$) does not vary with α or c. With heterogenous costs, however, it varies with these parameters. This is because the increase in α or decrease in c contributes to an increase of the expected profit per receiver, which increases the firm's incentive to offer referral rewards.

B.3 Proofs

Proof. **(Lemma 5)** First, we show the existence of unique cutoffs r^{free} and $r^{\text{not free}}$. The first-order condition of $\Pi^{\text{free}}(r, \alpha)$ with respect to R is that (i) $R = 0$ or (ii) $R > 0$ and

$$g(r+R) \cdot \left[\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - R - \frac{G(r+R)}{g(r+R)} \right] = 0.$$

Note that the expression in the bracket on the left-hand side is strictly decreasing given Assumption 4 and vary continuously from ∞ to $-\infty$ as R varies from $-\infty$ to ∞. Hence, the optimal reward is always unique in $\mathbb{R}$. Also, the same argument implies that there exists a unique r such that $\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - \frac{G(r)}{g(r)} = 0$. Let this unique r be r^{free}. That is, the left-hand side of the first-order condition is nonpositive and thus $R^{\text{free}}(r, \alpha) = 0$ if and only if $r \geq r^{\text{free}}$.

Analogously, conditional on offering no free contracts ($q_L = 0$), the optimal reward is unique in $\mathbb{R}$ and there exists a unique r such that $\pi((0,0), (p_H^*, q_H^*)) - \frac{G(\alpha r)}{g(\alpha r)} = 0$. We denote this r by $r^{\text{not free}}$. As before, we have that $R^{\text{not free}}(r, \alpha) = 0$ if and only if $r \geq r^{\text{not free}}$.

Finally, we show that $r^{\text{free}} < r^{\text{not free}}$. To see this, note that Assumption 4 implies $\frac{G(\alpha r)}{\alpha r} < \frac{G(r)}{r}$ for $r > 0$ and $\alpha \in (0,1)$. Together with $\pi((0,0), (p_H^*, q_H^*)) > \pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*))$, $r^{\text{free}} < \alpha r^{\text{not free}}$ follows by Assumption 4 and the definitions of r^{free} and $r^{\text{not free}}$. Since $\alpha < 1$, this implies $r^{\text{free}} < r^{\text{not free}}$. $\square$

Proof. **(Theorem 2)**

1. By Assumption 3, $\pi((0,0), (p_H^*, q_H^*)) > 0$ holds. Also, since $g(\xi) > 0$ for all $\xi \in \mathbb{R}_+$, $G(\xi) > 0$ for all $\xi > 0$. Hence, for any $r \in [0, \infty)$ and $\alpha \in (0,1)$, $[\pi((0,0), (p_H^*, q_H^*)) - R] \cdot G(\alpha r + R) > 0$ holds if $R \in (0, \pi((0,0), (p_H^*, q_H^*)))$. Thus, $\Pi^{\text{not free}}(r, \alpha) > 0$.

2. Note that the use of both, free contracts and positive rewards, is optimal only if $r < r^{\text{free}}$. Also, $r < r^{\text{free}}$ implies that rewards are positive. Furthermore, in that case the maximization problems defining $\Pi^{\text{free}}(r,\alpha)$ and $\Pi^{\text{not free}}(r,\alpha)$ both have inner solutions, so the two maximization problems can be rewritten as:

$$\begin{aligned}\Pi^{\text{free}}(r,\alpha) &= \max_{x\in\mathbb{R}}(A^{\text{free}} - x)\cdot G(x) \\ \Pi^{\text{not free}}(r,\alpha) &= \max_{x\in\mathbb{R}}(A^{\text{not free}} - x)\cdot G(x)\end{aligned} \tag{14}$$

where $A^{\text{free}} = \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))+r$ and $A^{\text{not free}} = \pi((0,0),(p_H^*,q_H^*))+\alpha r$. Thus, $\Pi^{\text{free}}(r,\alpha) \geq \Pi^{\text{not free}}(r,\alpha)$ if and only if

$$\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) + r \geq \pi((0,0),(p_H^*,q_H^*)) + \alpha r.$$

This is equivalent to $r \geq \frac{CF^*}{1-\alpha}$. Also, by part 1 of the current theorem, $\Pi^{\text{free}}(r,\alpha) \geq \Pi^{\text{not free}}(r,\alpha)$ implies $\Pi^{\text{free}}(r,\alpha) > 0$. Overall, there exists an optimal scheme such that both free contracts and positive rewards are used if and only if $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}})$.

3. Consider a variable

$$\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)}. \tag{15}$$

This variable is well-defined because the denominator is always strictly positive by part 1 of the current theorem.

Step 1: Note that for $r \geq r^{\text{not free}}$, Lemma 5 shows that the rewards are zero in any optimal scheme. Hence, $\Pi^{\text{free}} = \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))\cdot G(r)$ and $\Pi^{\text{not free}} = \pi((0,0),(p_H^*,q_H^*))\cdot G(\alpha r)$ hold, and thus (15) is differentiable with respect to r. If $\frac{G}{g}$ is convex, then

$$\begin{aligned}\frac{\partial}{\partial r}\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)} = \frac{\partial}{\partial r}\frac{\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))\cdot G(r)}{\pi((0,0),(p_H^*,q_H^*))\cdot G(\alpha r)} &= \\ \frac{\left(\frac{G(\alpha r)}{g(\alpha r)} - \alpha\frac{G(r)}{g(r)}\right)\cdot\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))\cdot g(r)\cdot g(\alpha r)}{[\pi((0,0),(p_H^*,q_H^*))]\cdot G(\alpha r)^2} &< 0.\end{aligned}$$

Thus, when $r \geq r^{\text{not free}}$, either (i) free contracts are not optimal for any $r \in [r^{\text{not free}},\infty)$, or (ii) there exists a $\overline{r}' \geq r^{\text{not free}}$ such that there exists an optimal scheme in which free contracts are offered for $r \in [r^{\text{not free}},\overline{r}']$, and no free contracts are offered under any optimal scheme for $r > \overline{r}'$. It must be the case that $\overline{r}' < \infty$ because

$$\lim_{r\to\infty}\frac{\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))\cdot G(r)}{\pi((0,0),(p_H^*,q_H^*))\cdot G(\alpha r)} = \frac{\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))}{\pi((0,0),(p_H^*,q_H^*))} < 1.$$

We let $\bar{r} = \bar{r}'$ in case (ii).

Step 2: Next, we consider the following three different cases: $\frac{CF^*}{1-\alpha} < r^{\text{free}}$, $\frac{CF^*}{1-\alpha} \in [r^{\text{free}}, r^{\text{not free}}]$, and $\frac{CF^*}{1-\alpha} > r^{\text{not free}}$.

- Let $\frac{CF^*}{1-\alpha} < r^{\text{free}}$. Then, it follows from part 2 of the current theorem that no free contracts are offered for $r < \frac{CF^*}{1-\alpha}$ and free contracts are offered for $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}}]$. For $r \in [r^{\text{free}}, r^{\text{not free}}]$,

$$\frac{\partial}{\partial r}\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)} = \frac{\partial}{\partial r}\frac{[\pi((0,\underline{q}),(p_H^*,q_H^*))]\cdot G(r)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*)) - R]\cdot G(\alpha r + R)} =$$
$$\frac{\left(\frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} - \alpha\frac{G(r)}{g(r)}\right)\cdot \pi((0,\underline{q}),(p_H^*,q_H^*))\cdot g(r)\cdot g(\alpha r + R^{\text{not free}}(r,\alpha))}{[\pi((0,0),(p_H^*,q_H^*)) - R^{\text{not free}}(r,\alpha)]\cdot G(\alpha r + R^{\text{not free}}(r,\alpha))^2}.$$

Note that $\Pi^{\text{not free}}(r,\alpha)$ is differentiable in r by the Envelope Theorem. Moreover, if $\frac{G(\alpha r+R)}{g(\alpha r+R)} - \alpha\frac{G(r)}{g(r)} < 0$, then $\alpha r + R < r$ because $\frac{G(\xi)}{g(\xi)}$ is increasing in ξ by Assumption 4. Moreover, $R^{\text{not free}}(r,\alpha)$ is differentiable in r by the implicit function theorem applied to the first-order condition of $\Pi^{\text{not free}}$ and, letting $\hat{G}(\xi) := \frac{G(\xi)}{g(\xi)}$ for all ξ,

$$\frac{\partial}{\partial r}R^{\text{not free}}(r,\alpha) = -\frac{\alpha \hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))}{1+\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))} < 0.$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial r}\left(\frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} - \alpha\frac{G(r)}{g(r)}\right) &= \\ \alpha\left(\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) - \hat{G}'(r)\right) + \hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))\frac{\partial}{\partial r}R^{\text{not free}}(r,\alpha) &= \\ \alpha\left(\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) - \hat{G}'(r) - \frac{\hat{G}'(\alpha r + R^{\text{not free}})^2}{1+\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))}\right) &< 0. \end{aligned}$$

Thus, if the derivative of (15) is negative at $r' \in [r^{\text{free}}, r^{\text{not free}}]$ then (15) is decreasing for all $r \in [r', r^{\text{not free}}]$. Together with Step 1, this implies the following. In case (i), there exists $\overline{r} \in [r^{\text{free}}, r^{\text{not free}})$ such that free contracts are offered in an optimal scheme if and only if $r \in [\frac{CF^*}{1-\alpha}, \overline{r}]$. In case (ii), the current analysis shows that it is optimal to offer free contracts for all $r \in [r^{\text{free}}, r^{\text{not free}}]$, so free contracts are offered if and only if $r \in [\frac{CF^*}{1-\alpha}, \overline{r}]$, where $\overline{r}$ is the variable that we defined in Step 1.

- Let $\frac{CF^*}{1-\alpha} \in \left[r^{\text{free}}, r^{\text{not free}}\right]$. In that case, offering free contracts is not optimal for any $r < r^{\text{not free}}$. Then, either free contracts are not optimal for any r or by the same argument as above, if free contracts are not used in an optimal scheme for $r = r'$ then they are not used in any optimal scheme for any $r > r'$. This proves the desired claim for this case.
- If $\frac{CF^*}{1-\alpha} > r^{\text{not free}}$, then offering free contracts is not optimal for any $r < r^{\text{free}}$. For $r \in \left[r^{\text{free}}, r^{\text{not free}}\right]$ free contracts are also not optimal because

$$1 > \frac{\max_{R\in\mathbb{R}}[\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) - R]\cdot G(r+R)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*)) - R]\cdot G(\alpha r + R)} \geq \frac{[\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))]\cdot G(r)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*)) - R]\cdot G(\alpha r + R)}.$$

The first inequality follows from the proof of part 2 of the current theorem. For $r \geq$

$r^{\text{not free}}$, offering free contracts is never optimal by Step 1.

This concludes the proof.

$\square$

Proof. **(Proposition 7)** First, note that we can write the limiting profits as

$$\begin{aligned}\lim_{\alpha\to 1}\Pi^{\text{free}}(r,\alpha) &= \max_{x\geq r}(\tilde{p}_H^* - cq_H^* + r - x)G(x) < \\ \lim_{\alpha\to 1}\Pi^{\text{not free}}(r,\alpha) &= \max_{x\geq r}(p_H^* - cq_H^* + r - x)G(x).\end{aligned}$$

It follows immediately from part 1 of Theorem 2 that there exist $\alpha' > 0$ and $\epsilon > 0$ such that $\Pi^{\text{free}}(r,\alpha) + \epsilon < \Pi^{\text{not free}}(r,\alpha)$ for any $\alpha \in (0,\alpha')$, hence the limit result as $\alpha \to 0$ holds. $\square$

Proof. **(Proposition 8)** Applying the implicit function theorem to the first-order conditions of Π^{free} and $\Pi^{\text{not free}}$ gives us:
(i) $\frac{R^{\text{free}}(r,\alpha)}{\partial\alpha} = -\frac{p_H^* - q_H^* c - v_H(\underline{q}) + c\underline{q}}{-1-\hat{G}'(r+R)} > 0$ and

$$\frac{\partial R^{\text{not free}}(r,\alpha)}{\partial\alpha} = -\frac{p_H^* - q_H^* c - r\hat{G}'(\alpha r + R)}{-1-\hat{G}'(r+R)}$$

which is strictly greater than zero if and only if $r\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) < p_H^* - q_H^* c$, or $\alpha r\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) < \Pi^{\text{classic}}$.
(ii) $\frac{R^{\text{free}}(r,\alpha)}{\partial r} = -\frac{-\hat{G}'(R+r)}{-1-\hat{G}'(R+r)} < 0$ and $\frac{R^{\text{not free}}(r,\alpha)}{\partial r} = -\frac{-\alpha\hat{G}'(R+\alpha r)}{-1-\hat{G}'(R+\alpha r)} < 0$ because $\hat{G}'(x) > 0$ for all $x > 0$, so $-1-\hat{G}'(x) < 0$.
(iii) First, note that for $r > r^{\text{free}}$, referral rewards are always zero when free contracts are offered, i.e., the statement is trivially true. If $r \leq r^{\text{free}}$, then the optimal reward with free contracts $R^{\text{free}}(r,\alpha)$ satisfies the first-order condition:

$$R^{\text{free}}(r,\alpha) = \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) - \frac{G(r+R^{\text{free}}(r,\alpha))}{g(r+R^{\text{free}}(r,\alpha))}. \tag{16}$$

By the first-order condition for the maximization problem for the case with no free contracts with respect to the reward, the solution $R^{\text{not free}}(r,\alpha)$ must satisfy:

$$g(\alpha r + R^{\text{not free}}(r,\alpha))\cdot\left(\pi((0,0),(p_H^*,q_H^*)) - R^{\text{not free}}(r,\alpha) - \frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))}\right) = 0.$$

Since $g(\cdot) > 0$, this implies that

$$\pi((0,0),(p_H^*,q_H^*)) - R^{\text{not free}}(r,\alpha) - \frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} = 0. \tag{17}$$

Now, substitute $R^{\text{not free}}(r,\alpha)$ by the expression for $R^{\text{free}}(r,\alpha)$ given by (16) on the left hand side of (17), to obtain:

$$\pi((0,0),(p_H^*,q_H^*)) - \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) + \frac{G(r+R^{\text{free}}(r,\alpha))}{g(r+R^{\text{free}}(r,\alpha))} - \frac{G(\alpha r+R^{\text{free}}(r,\alpha))}{g(\alpha r+R^{\text{free}}(r,\alpha))}.$$

This is strictly positive by log-concavity of G (Assumption 4) and because $\pi((0,0),(p_H^*,q_H^*)) > \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))$. Noting that the left hand side of (17) is strictly decreasing in referral rewards, the optimal reward without free contracts $R^{\text{not free}}(r,\alpha)$ is strictly greater than $R^{\text{free}}(r,\alpha)$. □

Proof. **(Proposition 9)** The comparative statics with respect to $\frac{CF^*}{1-\alpha}$ are straightforward from the formula of CF^*. The ones for r^{free} follow from the first-order condition with respect to rewards that appears in the proof of Lemma 5 and Assumption 4. □

B.4 Homogeneous Costs as the Limit of Heterogeneous Costs

Consider a sequence $\{G^n\}_1^\infty$ that converges pointwise to (4) such that for each n, G^n is twice differentiable with $(G^n)'(\xi) = g^n(\xi) > 0$ for all ξ, and Assumption 4 holds. Let the set of all such sequences be $\mathcal{G}$. The set $\mathcal{G}$ is nonempty. For example, consider $\{G^n\}_1^\infty$ such that or each $n \in \mathbb{N}$, G^n is a normal distribution with mean $\bar{\xi} \geq 0$ and variance $\frac{1}{n}$ truncated at $\xi = 0$. By inspection one can check that $\{G^n\}_1^\infty \in \mathcal{G}$. For any given G^n, we can define $\underline{r}^n$, $r^{\text{free},n}$, $\bar{r}^n$, and $r^{\text{not free},n}$. Then, the following statement can be shown: For any $\{G^n\}_1^\infty \in \mathcal{G}$,

$$\lim_{n\to\infty} \underline{r}^n = \frac{CF^*}{1-\alpha}, \quad \lim_{n\to\infty} r^{\text{free},n} = \bar{\xi}, \quad \lim_{n\to\infty} \bar{r}^n = \frac{\bar{\xi} - CF^*}{\alpha}, \quad \text{and} \quad \lim_{n\to\infty} r^{\text{not free},n} = \frac{\bar{\xi}}{\alpha}.$$