Contracting with Word-of-Mouth Management

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Abstract

We propose a model for word of mouth (WoM) management where a firm has two tools at hand: referral rewards and offering a free contract. Current customers' incentives to engage in WoM can affect the contracting problem of a firm in the presence of positive externalities of users. Formally, we consider a classic Maskin-Riley contracting problem for the receiver of WoM where the firm can pay the senders referral rewards and a sender experiences positive externalities if the receiver adopts. A free contract can incentivize WoM because the higher adoption probability increases the expected externalities that the sender receives. We characterize the optimal incentive scheme and show when the two tools serve as substitutes and complements to each other depending on whether the market is niche and whether the product is social. We show that offering a free contract is optimal only if the fraction of premium users in the population is small, which is consistent with the observation that companies that successfully offer freemium contracts oftentimes have a high percentage of free users.

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"Cost per acquisition: \$233-\$388. For a \$99 product. Fail."

—Drew Houston, founder of Dropbox

1 Introduction

In April 2010, Dropbox announced that it would start a referral program, increase visibility of its free 2 GB option, and introduce a sharing option. All in all, this led to 2.8 million direct referral invites within 30 days.[1] Before the change, the costs per acquisition had been more than 200 dollars for a 99 dollar product, so Dropbox was not even able to survive in the market without word of mouth (WoM). The introduction of the sharing option makes Dropbox a "social product," with which users experience positive externalities from friends using the product. Similarly, WoM was essential for the growth of another social product, Skype. The company founded in 2003 spent nothing on marketing until it was acquired by eBay when it already had 54 million registered users.[2] Both Dropbox and Skype use the so-called "freemium" (a free contract + premium contracts) strategy. However, the former combines it with a referral program, which Houston (2010) emphasizes as a way to encourage WoM, while the latter only relies on a freemium strategy.

The objective of this paper is to develop a simple model that highlights when offering a free contract and referral rewards can optimally incentivize WoM. Specifically, we model the incentive for old customers (senders of WoM) to talk to new customers (receivers) who are offered a menu of contracts as in Maskin and Riley (1984). The firm can reward senders directly through referral rewards. A reward to the receivers via a free contract increases the likelihood of them using the product. This in turn raises the size of the expected externalities the senders receive from talking, and thus encourages WoM. All in all, the model highlights a fundamental difference between referral rewards and a "freemium" strategy when it comes to encouraging WoM. Figure 1 offers a schematic presentation of the main logic.

We provide a characterization of the optimal incentive scheme for WoM, which we use to discuss *substitution* and *complementation* between the two strategies. This analysis allows us to understand two fundamentally different situations that a company might be in. First, a company might already incentivize WoM with only one of the tools, but it could save costs by substituting the tool with

[1] See Houston (2010).
[2] See Eisenmann (2006).

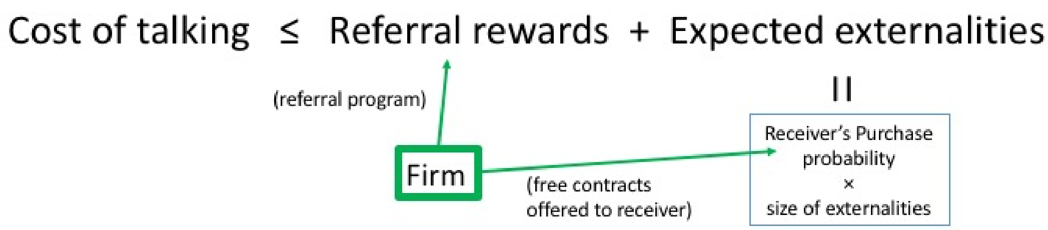


Figure 1: Schematic presentation of the sender's trade off

the other one. Second, the company might not be able to incentivize WoM with any one of the two tools alone, but if it used both tools together it could successfully encourage talking as seems to be the case with Dropbox.

We focus on two market characteristics which turn out to determine whether substitution and complementation occur. First, a market can be either a *niche market* or a *mass market*, depending on the fraction of premium customers. Second, the product can be a *social product*, or a *private product*, depending on the positive externalities it generates among customers. We show that substitution occurs for social products, while complementation occurs for rather private products. For referral rewards to substitute a free contract, the market needs to be mass, while a free contract substitutes referral rewards if the market is niche. This difference arises because the benefit of using a free contract is to expand the expected externalities that the senders receive. The "jump" of the expected externalities is large (and thus effective in incentivizing WoM) only when the fraction of users who would otherwise not use the product is high. For a related reason, a free contract complements referral rewards only in a very niche market, while referral rewards can complement a free contract also in a slightly more mass market.

The optimal scheme that we characterize exhibits a rich pattern of the use of referral rewards and a free contract, and the prediction is roughly consistent with what we observe in the real world. Most notably, our findings are consistent with a paradoxical feature of the customer base of the aforementioned companies: While profits kept increasing, Dropbox faced consistently only 4% of customers actually paying for the product.[3] Similarly, only 8% of the customers who are served by Skype actually pay. As we discussed, our model predicts that a free contract is useful in niche markets because it is used to boost up the expected externalities. Conditional on the fraction of "premium users" being low, referral rewards are not used for sufficiently social products, which is consistent with Skype's strategy. When the externalities are not too low or too high, referral rewards are used in conjunction with a free contract, which is consistent with Dropbox's strategy. For ride share companies like Uber, externalities would be low (it is a private product) and the share of users who would be willing to pay is high (the market is mass). Our model shows that the optimal scheme is not to use a free contract but to offer referral rewards, being consistent with Uber's strategy.

Finally, we note that the key components of the model can be interpreted in many ways. An externality can be a real value of social usage or psychological benefit from having convinced a friend to use the same product (Campbell et al., 2015). The sender may also benefit from the continuation value in a repeated relationship with the receiver. The argument also requires that there is an exogenous cost of talking for the sender. There are many reasons why talking may be costly: senders incur opportunity costs of talking (Lee et al., 2013), and/or they may feel psychological barriers. We assume that each sender wants to talk if and only if the cost of talking is smaller than the benefit.

The paper is structured as follows. Section 2 introduces the model. Section 3 characterizes the optimal scheme when only a free contract can be used, when only referral rewards can be used, and when both tools are available (the full model). Section 4 compares the optimal schemes in those three models to identify when substitution and complementation occur. Section 5 discusses comparative statics for the full model. Finally, Section 6 concludes. The Appendix generalizes the full model in a number of ways, and provides proofs for the results. The Online Appendix discusses various extensions, robustness checks, and welfare considerations.

1.1 Related Literature

Our contribution is to show that a firm can encourage WoM indirectly through free contracts given to the receiver. This explanation complements the existing explanations in the literature on how

[3] See *Economist* (2012).

to encourage WoM:[4] Biyalogorsky et al. (2001) compare the benefits of price reduction and referral programs in the presence of WoM. In their model, a reduced price offered to the sender of WoM is beneficial because it makes the sender "delighted" and thereby encourages him to talk. Depending on the delight threshold, the seller should use one of the two strategies or both. In contrast, our focus is on WoM in the presence of positive externalities of talking and our model accommodates menus of contracts. In Campbell et al. (2015), senders talk in order to affect how they are perceived by the receiver of the information. The perception is better if the information is more exclusive. Thus, a firm can improve overall awareness of the product by restricting access to information (i.e., by advertising less). One could interpret the positive externalities in our model also as a reduced form of a "self-enhancement motive" as in their model. Although we discuss advertising in the Online Appendix, we focus on the relative effectiveness of free contracts and referral rewards instead of advertising. Kornish and Li (2010) consider the tradeoff between referral rewards and pricing in a model where the sender cares about the receiver's surplus and the firm offers a single price. Due to the assumption of a single price, any price set by the firm generating strictly positive profits is necessarily strictly positive, so it cannot accommodate free contracts. Our model, on the other hand, allows for screening by the firm and we analyze how that interacts with referral rewards in the presence of externalities. This enables us to give predictions consistent with the strategies used by various companies.

Most of the other theoretical literature on WoM has focused on mechanical processes of communication in networks. This literature mostly focuses on how characteristics of the social network affect a firm's optimal advertising and pricing strategy. Campbell (2012) analyzes the interaction of advertising and pricing.[5] Galeotti (2010) is concerned with optimal pricing when agents without information search for those with information. Galeotti and Goyal (2009) show that advertising can become more effective in the presence of WoM (i.e., WoM and advertising are complements) as well as that it can be less effective (i.e., WoM and advertising are substitutes). All of these papers consider information transmission processes in which once a link is formed between two agents, they automatically share information.

Costly communication has been studied in the context of working in teams where moral hazard

[4] See also Godes et al. (2005) for a survey of the literature.

[5] On the empirical side, Godes and Mayzlin (2009) analyze the roles of loyal customers and opinion leaders in the context of WoM. Schmitt et al. (2011) study how valuable referred customers are in the data.

problems are present between the sender and receiver, as introduced by Dewatripont and Tirole (2005). Dewatripont (2006), for example, applies their model to study firms as communication networks. Instead, our model does not involve moral hazard but a screening problem, and externalities (which are absent in Dewatripont (2006)) play a key role in formulating the optimal contracting scheme.[6]

There is also a literature on contracting models in the presence of network effects. Besides the critical difference that our focus is on how the firm can optimally affect incentives to talk, there is a subtle difference in the optimal contracts. Csorba (2008) analyzes a contracting model in which the more the other buyers use the product, the higher the utility from using the product is.[7] He shows that an optimal contract scheme introduces a distortion at the top because a reduction of the quantity offered to low types should decrease the value of the product to high types. Unlike in his model, we have no distortions at the top in the optimal contract scheme. The reason is that receivers do not receive externalities from each other, and that we consider quantity-independent externalities rather than assuming that the total quantity consumed generates externalities. We discuss the implications of quantity-dependent externalities in Section D.4. We do not consider the case of externalities between receivers themselves given our focus on the sender's incentives to talk. Introducing such a feature would not change the qualitative results on the optimal incentive schemes to encourage WoM. The modeling difference leads to the difference in terms of applications. When the focus is on receivers generating externalities to each other, the model would be suitable for the analysis of, for example, social networks such as LinkedIn or Facebook. In such a context, a recent working paper by Shi et al. (2017) considers a static model of product line design without WoM when free users generate positive externalities on all premium users. When the firm can manipulate the amount of externalities enjoyed by customers conditional on the user type, freemium contracts can arise as an optimal strategy. In contrast, in our model, there is no manipulation of the size of externalities and the price of the low-type contracts must be zero because the surplus from selling to the low types is negative. Even so, the monopolist sells contracts with positive quantities for free to the low types because those free contracts encourage WoM which attracts premium users.

While the focus of this paper is not to add another rationale for freemium strategies, it is impor-

[6]Lobel et al. (2016, Forthcoming) also consider a model with costly referrals (with no screening), focusing on a referral game played by customers on a network.

[7]See Segal (2003) for a seminal work on this literature. See also Hahn (2003).

tant to note the connection to the literature on "freemium" strategies. The literature has identified various other reasons: (i) free contracts may be useful in penetration of customers or information transmission about the quality of the product to them, which can induce their upgrade,[8] (ii) the firm may hope that the free users will refer someone who will end up using the premium version,[9,10] (iii) free products attract attention of customers and prevent them from purchasing the competitors' products, and (iv) the increased number of customers due to free contracts raises the advertising revenue or sales of data.[11] None of these reasons pertains to the senders' incentives. Instead, our focus (with regards to free contracts) is on how free contracts help firms to manage senders' incentives. Thus, instead of convoluting our model with these other aspects of free contracts, we aim to *isolate* the effect of the tradeoff that the senders of information face. Similarly, we do not intend to create a "complete" model that incorporates all conceivable features that are relevant for firms' decision making. Instead, the goal of this paper is to understand how the incentives for WoM can be managed. Our simplification allows us to isolate the factors pertaining to the encouragement of WoM and to examine the tradeoffs involved.

2 Model

We present a simple model using a specific functional form to illustrate our main points. The model and the results are generalized in the Appendix in a number of ways. Proofs of the results presented in the main text follow from the general results presented in the Appendix. In the main text of the paper we focus on explaining the implications of the results without going into technical details.

A monopolist seller produces a product at marginal cost $c = 0.2$ and zero fixed cost. There are two customers, the *sender* (he) and the *receiver* (she). The sender already knows about the existence of the product, while the receiver does not. The receiver is either a high type (H) with probability α or a low type (L) with probability $1 - \alpha$.[12] The H-type receives $10\sqrt{q}$ for consuming

[8] Formally, we rule out this effect by assuming that, after learning about the existence of the product, each customer has a fixed valuation to (and information about) the product that does not change over time.

[9] A recent working paper by Ajorlou et al. (2015) builds a social-network model that highlights this effect.

[10] Lee et al. (2015) empirically analyze the trade-off between growth and monetization under the use of freemium strategies. In their paper, the value of a free customer is determined by upgrade as in (i) and the free users' referrals as in (ii).

[11] See Shapiro and Varian (1998) for (i)-(iii) and Lambrecht and Misra (2016) for (iv).

[12] Although the Introduction discussed the "share" of each type, here we consider probability because there is only

quantity/quality $q \geq 0$ of the product. Meanwhile, the L-type receives $\sqrt{q}$ for consuming q. We can also interpret q as the quality of the product.[13] Each receiver incurs a fixed installation cost of $I = 3$ if the consumed quantity is strictly positive ($q > 0$). Thus, the net benefit of consuming a quantity $q > 0$ is $10\sqrt{q} - 3$ and $\sqrt{q} - 3$ for the H- and L-type receiver, respectively. The type is private information to the receiver.[14]

The game consists of three stages. First, the seller offers a *scheme* $((p_L, q_L), (p_H, q_H), R)$, which consists of a menu of contracts $((p_L, q_L), (p_H, q_H)) \in (\mathbb{R}_+ \times \mathbb{R}_+)^2$ as well as the referral rewards $R \in \mathbb{R}_+$.[15] Here, for each $\theta = H, L$, p_θ is a price offered to the θ-type buyer, and q_θ is a quantity offered to that buyer. The referral rewards are a payment from the seller to the sender that are made if the sender talks to the receiver. The rewards are assumed to be paid irrespective of the subsequent purchase behavior by the receiver. In the Appendix, we allow for the possibility that the seller makes the referral rewards conditional on the receiver's purchase behavior and show that such conditioning does not increase the seller's profit. In the second stage, observing the menu offered by the seller, the sender decides whether to talk to the receiver or not.[16] Third, the receiver makes a purchase decision if and only if the sender has talked to the receiver.

The objective of the receiver is to choose the contract that maximizes her surplus (as in Maskin and Riley (1984)). The sender incurs a constant cost $\xi = 10$ of talking.[17] The benefit of taking is the sum of two components. The first component is the referral rewards paid by the seller. The second is the *expected externalities*, which can be calculated as follows: If the receiver purchases and uses the product, then the sender experiences the externalities of level $r \geq 0$.[18]

Hence, if the sender expects that both types buy the product, then the expected externalities are r. If instead he expects that only the H-type uses the product, then the expected externalities are αr.[19] The seller's objective is to maximize the expected profit from the receiver net of the

one receiver in this simple setup. This assumption is generalized in the Appendix.

[13] Interpreting q as quality would make a difference if we had learning about quality in the model, where using different contracts may result in different ex-post valuations.

[14] It is not crucial for our results that the sender does not know the type of the receiver, while it is important that the seller knows less about the type of the receiver than the sender does, which we view as a reasonable assumption.

[15] In the Appendix, we allow for negative prices as well.

[16] The Online Appendix examines the case in which multiple senders talk to a single receiver.

[17] We consider the possibility in which the cost is drawn from some distribution over $\mathbb{R}_+$ in the Online Appendix.

[18] This level r does not depend on the type of the receiver. In the Online Appendix, we consider an extension in which it does depend on the type of the receiver. The Online Appendix also discusses the case when the receiver receives externalities r as well, and shows that it would not change the essence of our analysis.

[19] The assumed functional form of the payoff functions implies that there is no possibility of only the L-type using the product.

referral rewards, subject to the following participation constraints (PC)

$$(10\sqrt{q_H} - 3) - p_H \geq 0 \quad \text{and} \quad (\sqrt{q_L} - 3) - p_L \geq 0,^{20}$$

and incentive compatibility (IC) conditions for the two types

$$(10\sqrt{q_H} - 3) - p_H \geq (10\sqrt{q_L} - 3) - p_L \quad \text{and} \quad (\sqrt{q_L} - 3) - p_L \geq (\sqrt{q_H} - 3) - p_H,$$

as well as the incentive compatibility for the sender[21]

$$\xi \leq R + \begin{cases} r & \text{if the sender expects that both types buy} \\ \alpha r & \text{if the sender expects that only the } H\text{-type buys} \end{cases}.$$

In order to be able to formally state our results, we denote the (non-empty) set of optimal schemes (i.e., maximizing the seller's profit) given parameters (α, r) to this problem by

$$\mathcal{S}(\alpha, r) \subseteq (\mathbb{R}_+ \times \mathbb{R}_+)^2 \times \mathbb{R}_+.^{22}$$

3 Optimal Scheme

In this section, we characterize the optimal scheme for the model described in Section 2. Before doing so, we first consider two benchmark models, in which either referral rewards or a free contract is not allowed. We analyze these benchmark models in order to later compare them with the full model. This helps us understand the role as substitutes or complements of referral rewards and a free contract in the optimal scheme. Note that these cases are also interesting in themselves to understand 1) for which parameters a firm can incentivize WoM solely with referral rewards and 2) for which parameters it can incentivize WoM solely with free contracts.

Section 3.1 considers a benchmark model in which using referral rewards is prohibited and

[20] An implicit assumption in the participation constraints is that the outside option generates zero surplus. The result that the price for the L-type buyer is 0 still holds (although the quantity offered is adjusted accordingly) even if the outside option generates a positive surplus.

[21] We assume the sender has already purchased the product so there is no additional revenue from the sender.

[22] Existence is proven in a more general environment in the Appendix. We will also introduce notations $\mathcal{S}^{\mathrm{NR}}(\alpha, r)$ and $\mathcal{S}^{NF}(\alpha, r)$, and one can show by analogous proofs that those are also nonempty.

characterizes the optimal scheme. Section 3.2 then characterizes the optimal scheme for the model in which using a free contract is prohibited. Finally, Section 3.3 characterizes the optimal scheme for the full model. Although we will not be detailed about the derivation in Section 3.3, we give rather detailed explanation in Sections 3.1 and 3.2 as they provide some relevant intuition in a very simple setting.

3.1 Benchmark without Referral Reward

First, we consider the situation where the referral rewards R are exogenously set to be equal to 0. We call this model the *no rewards model.* The set of optimal schemes in the no rewards model given parameters (α, r) is denoted by

$$\mathcal{S}^{\mathrm{NR}}(\alpha, r) \subseteq (\mathbb{R}_+ \times \mathbb{R}_+)^2 \times \{0\}.$$

Fix an optimal scheme $((p_L^*, q_L^*), (p_H^*, q_H^*), 0) \in \mathcal{S}^{\mathrm{NR}}$. Notice that, if the L-type uses quantity q of the product, then her value is nonnegative only if $\sqrt{q} - 3 \geq 0$, and the L-type's marginal benefit from using the product is $\frac{1}{2\sqrt{q}}$. This implies that the marginal benefit is at most $\frac{1}{6}$ when the value is nonnegative. Since the marginal cost of production is $c = 0.2 > \frac{1}{6}$, the only reason that the seller would offer a positive quantity of the product to the L-type in an optimal scheme is to induce the sender to talk.[23] Since the sender's cost of talking is 10, the L-type is offered a product only if $10 > \alpha r$ (assuming that doing so results in a nonnegative profit). Moreover, when the L-type is offered a product under the optimal scheme, q_L^* must be the lowest quantity in order for the L-type to use the product. Hence, we must have $\sqrt{q_L^*} - 3 = 0$, or $q_L^* = 9$.

If the seller offers a contract to the H-type only, then as in the standard model of screening, the price is set to extract the entire surplus from the H-type, and q_H^* is a solution of the first-order condition of the seller's problem, $\frac{10}{2\sqrt{q_H^*}} - 0.2 = 0$, i.e., $q_H^* = 625$. The sender's IC constraint is $\xi = 10 \leq \alpha r$.

If both types are offered a contract, then only the H-type's IC and L-type's PC are binding as

[23] This conclusion can be different if the L-type buyer generates other revenues such as advertising revenue. In our applications (Skype, Dropbox, Uber, etc.), however, advertising revenue seems not to play an important role.

in the standard screening models:

$$10\sqrt{q_H^*} - p_H^* = 10\sqrt{q_L^*} - p_L^* \quad \text{and} \quad \sqrt{q_L^*} - 3 - p_L^* = 0.$$

Since we know $q_L^* = 9$, we have $p_L^* = 0$. Also, $q_H^* = 625$ implies that $p_H^* = 220$. The profit is thus

$$\alpha(220 - 0.2 \cdot 625) + (1 - \alpha)(0 - 0.2 \cdot 9) = 96.8\alpha - 1.8.$$

This is strictly positive if and only if $\alpha > \frac{1.8}{96.8}$. The sender's IC constraint is simply $\xi = 10 \leq r$.

Given this, the following theorem characterize the optimal scheme.

Proposition 1 (Characterization for the No Rewards Model). *1.* **(Positive profits)** *There exists an optimal scheme generating a strictly positive profit if and only if*

$$\alpha > \frac{1.8}{96.8} \text{ and } r > 10. \tag{1}$$

If (1) *is satisfied, then* $\mathcal{S}^{NR} \subseteq \{((0,0),(250,625),0),((0,9),(220,625),0)\}$.

2. **(Free vs. no free contracts)** $((0,9),(220,625),0) \in \mathcal{S}^{NR}$ *if and only if* $r \leq \frac{10}{\alpha}$.

Figure 2 depicts the optimal scheme for each (α, r) pair. The figure labels each region with a description of an optimal scheme in that region (including the boundaries of the region) whenever its interior generates a strictly positive profit. It also shows a region in which the maximized profit is zero (including the boundaries of the region). In the interior of each region with a name of a scheme, Theorem 1 implies that the scheme achieves the unique optimum.[24] The theorem implies that a free contract is used in an optimal scheme if $10 \leq r \leq \frac{10}{\alpha}$ and $\alpha \geq \frac{1.8}{96.8}$. The reason is that, if the externality r is too low ($r < 10$), then the sender cannot be incentivized to talk even if a free contract is offered and if it is too high ($r > \frac{10}{\alpha}$), the sender talks anyway to receive externalities from the H-type even absent a free contract. If the probability α is too low ($\alpha < \frac{1.8}{96.8}$), the revenue from the H-type is not enough to cover the cost of a free contract (we will be more explicit about what this cost is in Section 3.3).

[24] The same remark applies to other figures in this paper, too.

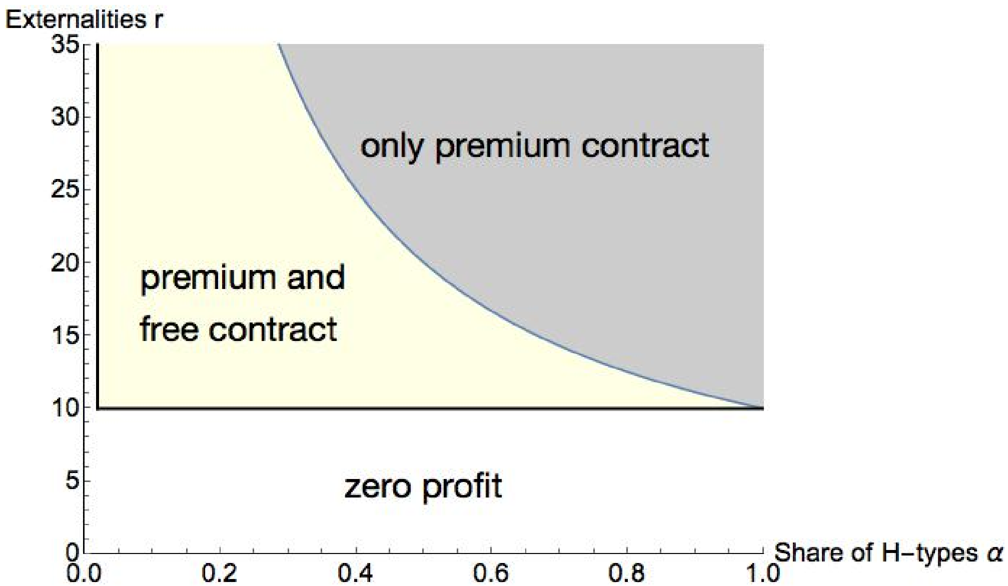


Figure 2: No Rewards Model

Remark 1. Note that a free contract arises endogenously in our model. If the firm serves the L-type customer to incentivize WoM, it is optimal to offer just enough to make her use the product, making zero the only possible price to the L-type. Although this might seem like an artifact of having only two types, we show in Online Appendix D.2 that free contracts arise endogenously even with a continuous type space. Thus, in a sense, the L-type in the two-type model can be interpreted as the customer who the firm should not serve absent of the need to encourage WoM. In the extension with a continuous type space, we also show that only the marginal type who buys a free contract is made indifferent between using the product and not using the product while other "higher" low types enjoy some surplus from using the free product.

3.2 Benchmark without a Free Contract

Second, we consider a model in which the seller is restricted to offer only one contract to the receiver. We call this model the *no free-contract model*. The set of optimal schemes in the no free-contract model given parameters (α, r) is denoted by

$$\mathcal{S}^{NF}(\alpha, r) \subseteq (\{0\} \times \{0\}) \times (\mathbb{R}_+ \times \mathbb{R}_+) \times \mathbb{R}_+.$$

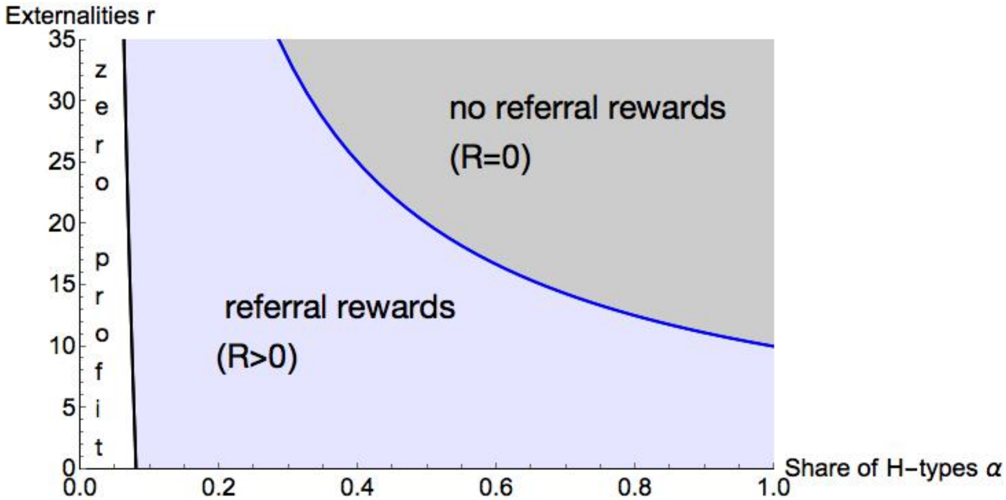


Figure 3: No Free-Contract Model

In this model, in a reasoning similar to the one in the no rewards model, the firm only offers one contract that only the H-type buys, which means that if $((0,0),(p_H^*,q_H^*),R^*) \in \mathcal{S}^{NF}(\alpha,r)$, then we must have $(p_H^*,q_H^*) = (250,625)$. The sender is incentivized to talk even with $R^* = 0$ if $\alpha r \geq 10$, while referral rewards of the amount $10 - \alpha r$ need to be paid to incentivize WoM otherwise. However, if the revenue from the H-type, which is 125α $(= \alpha(250 - 0.2 \times 625))$, is less than the reward payment $10 - \alpha r$, then WoM cannot be incentivized under the optimal scheme and the maximized profit is zero. This leads to the following characterization.

Proposition 2 (Characterization for the No Free-Contract Model)**.** *1.* **(Positive profits)** *There exists an optimal scheme generating a strictly positive profit if and only if*

$$10 < \alpha r + 125\alpha. \tag{2}$$

If (2) is satisfied, then $\mathcal{S}^{NF}(\alpha,r) = \{((0,0),(250,625),R)\}$ for some $R \geq 0$.

2. **(Rewards vs. no rewards)** *$\mathcal{S}^{NF} = \{((0,0),(250,625),R)\}$ with $R > 0$ if and only if $r < \frac{10}{\alpha}$.*

The result is illustrated in Figure 3. The intuition is simple: Referral rewards are useful if the size of the expected externalities is not enough to cover the cost of talking ($\alpha r < 10$), while covering the cost of talking by paying the referral rewards is not too expensive relative to the revenue from the receiver. Since the rewards payment and the revenue from the receiver (conditional on the receiver buying) are both decreasing in α, the region for which referral rewards are used in a optimal scheme requires α not to be too low.

3.3 The Full Model

Now we consider the full model. Conditional on each case, the set of optimal menus of the contracts is the same as in the no rewards model for each (α, r). One can completely characterize the optimal scheme, which we present below. To state the result formally, it is useful to define the following "cost of a free contract," denoted by CF^*:

$$CF^* = 30\alpha + 1.8(1-\alpha).$$

To understand this, note that there are two disadvantage of providing a free contract. The first is that the seller has to pay the cost of production when the buyer is of L-type. The quantity provided to the L-type is 9, and the firm incurs the marginal cost $c = 0.2$ for each unit. Since there is a $1-\alpha$ probability of the buyer being the L-type, this part of the cost amounts to $0.2 \times 9 \times (1-\alpha)$, which is the second term of CF^*. Second, the fact that the L-type is offered a positive quantity implies that the H-type must be incentivized to choose the contract offered to her over the one offered to the L-type. For this purpose, the seller needs to reduce the price by the amount of information rent, which is the valuation difference between the two types for the quantity that the L-type is offered, which is given by $10\sqrt{9} - (\sqrt{9} - 3) = 30$. Since the probability of the receiver being an H-type is α, this part of the cost amounts to $30 \times \alpha$, which is the first term of CF^*.

Furthermore, it is useful to note that the profit for a hypothetical case in which, as in a classic screening model, the cost of talking is zero and hence the sender always informs the buyer of the existence of the product is given by $\alpha(10\sqrt{625} - 0.2 \cdot 625) = 125\alpha$, where 625 is the quantity that we solved for in the previous section in analyzing the no rewards model.

Using these two values, we can now characterize the optimal scheme for the full model.

Proposition 3 (Characterization for the Full Model)**.** *1.* **(Positive profits)** *There exists an*

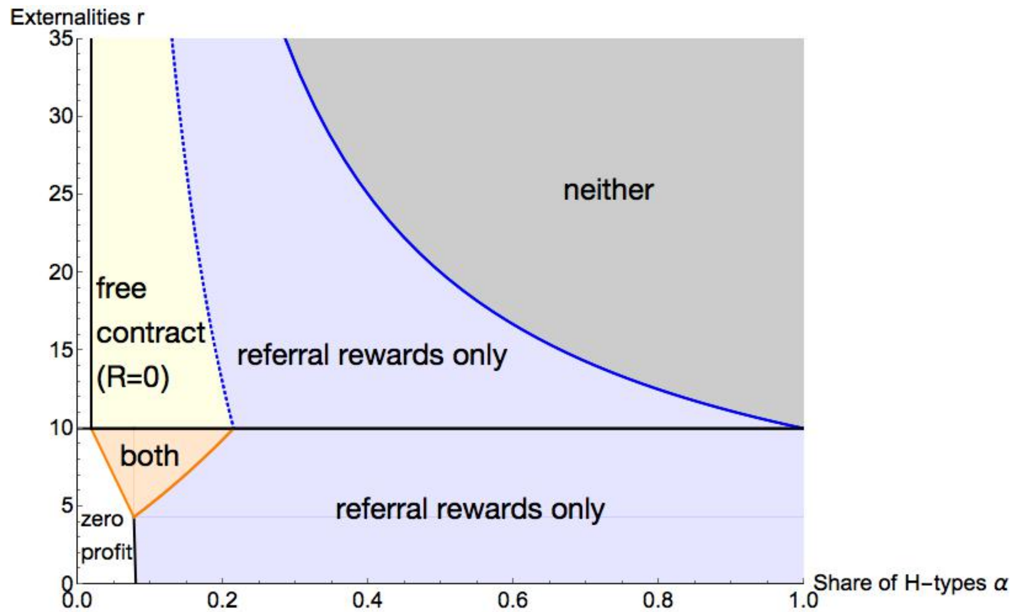


Figure 4: Full Model

optimal scheme generating a strictly positive profit if and only if[25]

$$10 < \max\{125\alpha - CF^* + \min\{r, 10\},\ 125\alpha + \alpha r\}. \tag{3}$$

If (3) *is satisfied, then* $\mathcal{S} \subseteq \{((0,0),(250,625),R)|R \in \mathbb{R}_+\} \cup \{((0,9),(220,625),R)|R \in \mathbb{R}_+\}$.

2. **(Free vs. no free contracts)** *There exists* $((0,9),(220,625),R) \in \mathcal{S}$ *for some* R *if and only if* $r \in \left[\frac{CF^*}{1-\alpha}, \frac{10-CF^*}{\alpha}\right]$.[26]

3. **(Rewards vs. no rewards)**

   (a) **(With free contracts)** *If* $r \in [\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}]$, *then* $((0,9),(220,625),R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < 10$, *and*

   (b) **(With no free contracts)** *If* $r \notin [\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}]$, *then* $((0,0),(250,625),R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < \frac{10}{\alpha}$.

The optimal scheme is illustrated in Figure 4. As one can see, the characterization of the optimal scheme in the full model entails a rich pattern. In particular, there are five different regions, a region in which the profit is zero, only a free contract is used, only referral rearwards are

[25] $10 < 125 - CF^* + 10$ is equivalent to $\alpha > \frac{1.8}{98.6}$.

[26] If $\frac{CF^*}{1-\alpha} > \frac{10-CF^*}{\alpha}$, then $[\frac{CF^*}{1-\alpha}, \frac{10-CF^*}{\alpha}] = \emptyset$.

used, both are used, and none is used while the profit is strictly positive. The detailed intuition for this rich pattern will be investigated in the next section by comparing the full model with the no rewards and no free-contract models. We only note here that if the probability of the H-type is too small (i.e., $\alpha < \frac{1.8}{96.8}$), then profits generated become too small to make it worthwhile to encourage WoM (i.e., the maximized profit is zero).[27] With small externalities r, the sender have little innate benefit from WoM, so the lower bound of α above which the profit is positive is large.

4 Substitution and Complementation

In Section 3 we show that the optimal scheme can take many different forms depending on the relevant parameters in the model. Using the characterizations in the previous section, this section aims to shed light on the interaction of referral rewards and free contracts. There are two fundamentally different situations that a company might be in. First, a company might only use one of the tools and be successfully incentivizing WoM, but substituting the tool with the other one could be cost-saving. Second, the company might not be able to generate WoM with any one of the two tools alone, but if it used both tools together it could successfully encourage talking. For example, in the leading example of Dropbox that we discussed in the Introduction, it seems that it was important to use both tools together. To understand the profit-maximization problem faced by such companies, we address the following research questions:

1. When are referral rewards better substitutes for free contracts and vice versa?

2. When can referral rewards and free contracts complement each other?

To answer those questions, we first clarify what we mean by substitution and complementation. First, the introduction of the possibility of referral rewards can make it unnecessary to use a free contract in incentivizing the sender to talk in the optimal scheme. In such a case, we say that *referral rewards substitute a free contract*.[28] In contrast, the introduction of the possibility of referral rewards may make a free contract useful in incentivizing the sender to talk in the optimal

[27] This region disappears with heterogeneous priors as we show in the Online Appendix.

[28] Formally, it corresponds to the case where it is uniquely optimal for the seller to offer a free contract under the no rewards model, while it is uniquely optimal not to offer it while offering referral rewards in the full model.

scheme. In such a case, we say that *referral rewards complements a free contract.*[29] A free contract substituting and complementing referral rewards is defined analogously, by comparing the full model with the no free-contract model.

We will discuss for which parameters α and r substitution and complementation occur. We can interpret markets with high α as *mass markets* and markets with small α as *niche markets*. We can also interpret products with high degrees of positive externalities r as *social products*, and those with low degrees as *private products*. In the following, we identify for which α and r one of the two tools substitutes or complements the other and explain the intuition behind it (Sections 4.1 and 4.2). We show that there is a subtle difference between referral rewards substituting and complementing a free contract, and a free contract substituting and complementing referral rewards (Section 4.3).

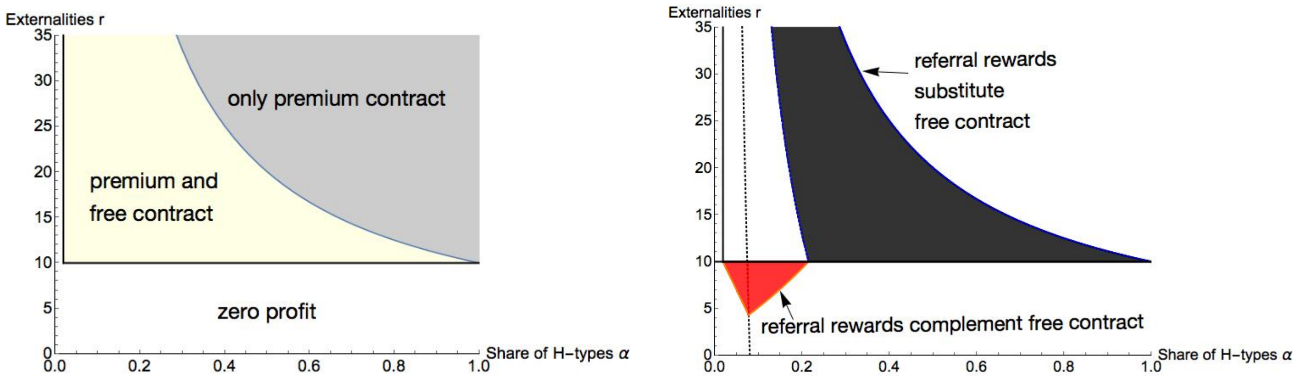
4.1 Referral Rewards Substituting and Complementing a Free Contract

We first compare the full model with the no rewards model. By doing so, we aim to understand when referral rewards can substitute as well as complement a free contract. To this end, the left panel of Figure 5 reproduces Figure 2, while the right panel shows the regions where substitution and complementation occur due to the introduction of referral rewards. Specifically, the interior of the black region in the right panel of Figure 5 corresponds to the parameter combinations under which referral rewards substitute a free contract, and the interior of the red region of the same panel shows the parameter combinations under which referral rewards complement a free contract. Substitution and complementation occur in the regions shown in Figure 5 for the following reasons.

- **Substitution:** Substituting a free contract with referral rewards is an effective strategy if it is cheap enough to do so. Notice that offering a free contract boosts up the benefit of talking by $(1-\alpha)r$. By using referral rewards instead of a free contract, the seller must pay referral rewards up to that amount. This payment is small if α is high (mass market). Hence, substitution occurs when α is high.

- **Complementation:** The reward payment required to induce the sender to talk can be kept low enough if a free contract alone could have already covered most of the cost of talking.

[29] Formally, it corresponds to the case where it is uniquely optimal for the seller not to offer a free contract under the no rewards model, while it is uniquely optimal to offer it with also offering referral rewards.



(a) No Rewards Model (b) Regions for substitution and complementation

Figure 5: Referral Rewards Substituting and Complementing a Free Contract

This is the case when the externality level r is not too low (social product). Moreover, for a free contract to be offered, α cannot be too low as then the revenue from the receiver is too low, while it cannot be too high as then $(1-\alpha)r$ is too small so offering a free contract is not worth the cost and it is better to use only referral rewards.

Before closing this subsection, we formalize our findings. Recall that each of $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ and $((p_L, q_L), (p_H, q_H), 0) \in \mathcal{S}^{\text{NR}}$ implies $p_L = 0$ and $q_H = 625$, and either (i) $q_L = 0$ and $p_H = 250$ or (ii) $q_L = 9$ and $p_H = 220$. Say that $(\alpha, r) \in \text{SUB}^{\text{rewards}}$ if $((0,9),(220,625),0) \in \mathcal{S}^{\text{NR}}(\alpha, r)$ while there is no $R \geq 0$ such that $((0,9),(220,625),R) \in \mathcal{S}(\alpha, r)$. Similarly, we say that $(\alpha, r) \in \text{COM}^{\text{rewards}}$ if $((0,9),(220,625),0) \notin \mathcal{S}^{\text{NR}}(\alpha, r)$ while there exists an $R \geq 0$ such that $((0,9),(220,625),R) \in \mathcal{S}(\alpha, r)$. That is, $\text{SUB}^{\text{rewards}}$ and $\text{COM}^{\text{rewards}}$ correspond to the parameter regions such that referral rewards substitute and complement, respectively, a free contract. Finally, let $\Pi^{NR}(\alpha, r)$ be the maximized profit under parameters (α, r) in the no rewards model.

Theorem 1 (The Effect of Referral Rewards)**.**

1. *(Substitution) Suppose that* $(\alpha, r) \in \text{SUB}^{\text{rewards}}$*. Then, for any* α' *such that* $((0,9),(220,625),0) \in \mathcal{S}^{NR}(\alpha', r)$*,* $(\alpha', r) \notin \text{SUB}^{\text{rewards}}$ *implies* $\alpha' < \alpha$*.*

2. *(Complementation)*

   (a) *Suppose that* $(\alpha, r) \in \text{COM}^{\text{rewards}}$*. For any* r' *such that* $\Pi^{NR}(\alpha, r') = 0$*,* $(\alpha, r') \notin \text{COM}^{\text{rewards}}$ *implies* $r' < r$*.*

(b) Fix r. There are $\underline{\alpha} > 0$ and $\bar{\alpha} < \infty$ with $\underline{\alpha} \leq \bar{\alpha}$ such that the following holds. If $(\alpha, r) \in \mathrm{COM}^{\mathrm{rewards}}$, then, $\alpha \in (\underline{\alpha}, \bar{\alpha})$.

The proof immediately follows from Propositions 1 and 3. The first part of this theorem states that it is cost efficient to completely substitute referral rewards with a free contract in mass markets, but not in niche markets. The second part states that if a firm cannot incentivize word of mouth only with a free contract alone, referral rewards can complement a free contract and help to incentivize word of mouth in markets that are niche but not too niche to guarantee a positive profit, while having a sufficiently high level of externalities.

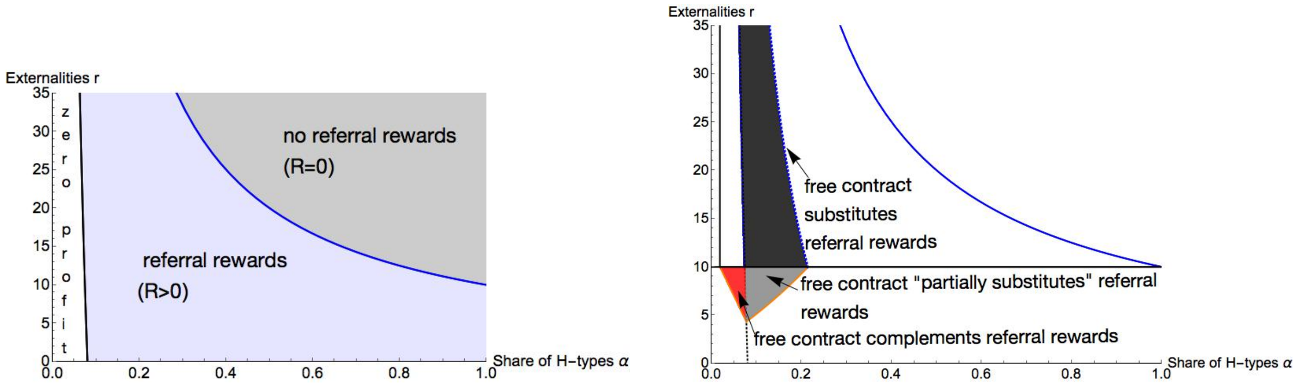
Remark 2. We note that a parameter combination (α, r) is in the region in which neither tool is needed to incentivize WoM in the no rewards model if and only if it is in such a region in the full model since the sender talks anyway without any additional incentives in either model in such a region. Also, the region in which the profit is zero under the full model is a subset of such a region in the no rewards model because the profit is always weakly greater in the full model than in the no rewards model. An analogous set of comments applies to the comparison between the full model and the no free-contract model.

4.2 A Free Contract Substituting and Complementing Referral Rewards

Now we compare the full model with the no free-contract model. Analogous to Section 4.1, we aim to understand when a free contract can substitute as well as complement referral rewards. The comparison is displayed in the right panel of Figure 6, together with a reproduction of Figure 3 in the left panel. Substitution and complementation occur in the respective parameter regions for the following reasons:

- **Substitution:** The interior of the black region in Figure 6 is such that it is uniquely optimal for the seller to offer referral rewards under the no free-contract model, while it is uniquely optimal not to pay referral rewards when offering a free contract. This region has the feature that r is not too low and α is not too high so that the size of the additional expected externalities, $(1-\alpha)r$, is high enough.

- **Complementation:** The interior of the red region in the right panel of Figure 3 is such that



(a) No Free-Contract Model
(b) Regions for substitution and complementation

Figure 6: A Free Contract Substituting and Complementing Referral Rewards

it is uniquely optimal for the seller not to offer referral rewards under the no free-contract model, while it is uniquely optimal to offer both referral rewards and a free contract under the full model. In this region, r is not too high but not too low. On the one hand, r cannot be too high because high r implies high additional expected externalities $(1-\alpha)r$, so a free contract would rather substitute, not complement, the referral rewards. On the other hand, r cannot be too low because even in the presence of a free contract, the referral rewards of $10-r$ must be paid to incentivize WoM. As a result, the reduction in the referral rewards due to the introduction of a free contract is not enough to cover the cost of a free contract CF^*. In addition, α needs to be high enough in this region because otherwise the revenue from the H-type is too low and thus incentivizing WoM would not generate a positive profit.

As before, we say $(\alpha, r) \in \mathrm{SUB}^{\mathrm{free}}$ if there is $R > 0$ such that $((0,0),(250,625),R) \in \mathcal{S}^{\mathrm{NF}}(\alpha, r)$ while there are no (q'_L, p'_H) and $R' > 0$ such that $((0,q'_L),(p'_H,625),R') \in \mathcal{S}(\alpha, r)$. Similarly, we say $(\alpha, r) \in \mathrm{COM}^{\mathrm{free}}$ if there is no $R > 0$ such that $((0,0),(250,625),R) \in \mathcal{S}^{\mathrm{NF}}(\alpha, r)$ while there is $R' > 0$ such that $((0,0),(250,625),R') \in \mathcal{S}(\alpha, r)$.

Theorem 2 (The Effect of a Free Contract)**.**

1. *(Substitution) Suppose that* $(\alpha, r) \in \mathrm{SUB}^{\mathrm{free}}$.

   (a) *For any* α' *such that there is* $R' > 0$ *satisfying* $((0,0),(250,625),R') \in \mathcal{S}^{\mathrm{NF}}(\alpha', r)$, $(\alpha', r) \notin \mathrm{SUB}^{\mathrm{free}}$ *implies* $\alpha < \alpha'$.

   (b) $r \geq 10$.

2. *(Complementation)*

   (a) *Fix α. There are $\underline{r} > 0$ and $\bar{r} < \infty$ with $\underline{r} \leq \bar{r}$ such that the following holds. If $(\alpha, r) \in \mathrm{COM}^{\mathrm{free}}$, then, $r \in (\underline{r}, \bar{r})$.*

   (b) *Suppose that $(\alpha, r) \in \mathrm{COM}^{\mathrm{free}}$. Then, for any α' such that $\Pi^{\mathrm{NF}}(\alpha', r) = 0$, $(\alpha', r) \notin \mathrm{COM}^{\mathrm{free}}$ implies $\alpha' < \alpha$.*

The proof immediately follows from Propositions 2 and 3. The first part of this theorem states that it is cost efficient to completely substitute a free contract with referral rewards in niche markets, but not in mass markets. It also shows that substitution can only occur if the product is "sufficiently social" compared to the cost of talking. The second part states that if a firm cannot incentivize word of mouth only with referral rewards alone, a free contract can complement referral rewards and help to incentivize word of mouth in markets that are "sufficiently mass" and with intermediate positive externalities.

Remark 3. One can think of the grey region as describing parameters under which a free contract "partially substitutes" referral rewards. In the interior of this region, the unique optimal scheme under the full model entails positive payment of referral rewards, but its amount is strictly less than under the no free-contract model. The substitution is only "partial" because the benefit from a free contract $(1 - \alpha)r$ is not large enough to incentivize WoM due to small r. Note that an analogous region (referral rewards "partially substituting" a free contract) does not exist in Figure 5 because a free contract can either do or do not exit, being different from referral rewards that can take a value from $\mathbb{R}_+$.

4.3 Comparing the Two Directions of Substitution and Complementation

Having completed the analysis of substitution and complementation in both directions, we are now in a position to make a few remarks on the similarities and differences between referral rewards substituting/complementing a free contract and vice versa.

First, a company that can induce WoM with one of the tools alone should think about substituting it with the other tool based on whether the company expects a high level of externalities between customers and whether the market is niche or not. In general, substitution occurs when the externalities are expected to be high. This is because a strictly positive profit needs to be generated in the no reward model for substitution to occur, and for that, r must be high enough to cover the cost of talking. If the market is niche, the company should use a free contract, while if it is mass, referral rewards.

A company that cannot incentivize WoM with one of the tools, can incentivize WoM by adding the other tool and offering both tools together when externalities are not too small but not too high either. Even if referral rewards complements a free contract, the latter may not complement the former under the same parameter values. The reason for the difference is that a free contract is useful in incentivizing WoM only in markets that are not too mass, so there is a region in which referral rewards are used in the no free-contract model while a free contract is not used in the no rewards model. Thus, it is not sufficient to test the effectiveness of the two tools separately, but it is essential to consider the combined effect. More precisely, the intersections of the red areas in panels (b) of Figures 5 and 6 (which is just the red region of panel (b) in Figure 6) is the region where both tools are absolutely necessary in order to incentivize WoM. Dropbox seems to be in this region given the history of how it grew. The grey region in panel (b) of Figure 6, in turn, is a region where a free contract only is not effective to incentivize WoM, but referral rewards alone are. However, combining the two is the most cost-effective. Thus, as long as the profit is positive, the more "niche" the market, the more important it becomes to combine both strategies.

We formalize the findings about the difference between the two-way substitution and complementation follows.

Theorem 3 (Difference between the Two-Way Substitution and Complementation)**.**

1. *(Substitution) Fix* r*. Suppose that* $(\alpha, r) \in \mathrm{SUB}^{\mathrm{free}}$ *and* $(\alpha', r) \in \mathrm{SUB}^{\mathrm{rewards}}$*. Then,* $\alpha < \alpha'$ *holds.*

2. *(Complementation) Fix* r*. Suppose that* $(\alpha, r) \in \mathrm{COM}^{\mathrm{free}}$ *and* $(\alpha', r) \in \mathrm{COM}^{\mathrm{rewards}} \setminus \mathrm{COM}^{\mathrm{free}}$*. Then,* $\alpha < \alpha'$ *holds.*

5 Comparative Statics for the Full Model

The characterization in Section 3 also allows us to conduct comparative statics of the full model. We first analyze the different implications for the optimal scheme as the market size α varies.

Proposition 4 (Market Structure and Free Contracts)**.**
(i) Consider two markets that are identical to each other except for the probability of the buyer being the H-type, denoted α_1 and α_2. Suppose that a free contract is offered under an optimal scheme in the market with α_1, the maximized profit is strictly positive in the market with α_2, and $\alpha_2 < \alpha_1$. Then, a free contract is offered under any optimal scheme in the market with α_2.
(ii) If $\alpha > \frac{r-1.8}{r+28.2}$ ($\Leftrightarrow r < \frac{CF^*}{1-\alpha}$), then a free contract is never offered under any optimal scheme.

This proposition shows that the monopolist should encourage WoM using free contracts in a niche markets with a small fraction α of H-type buyers as long as the market is profitable enough. Intuitively, if the probability of the H-type is low, the seller is better off using a free contract because a free contract significantly increases the probability of purchase. The exact trade-off is determined by the comparison of the information rent and the per-low-type surplus $r - 1.8$ that the seller can extract. The cutoff for α is increasing in this surplus.

These findings are consistent with the observation that digital service providers with small production costs who successfully offer free contracts (e.g., Dropbox or Skype), have a large number of free users. Moreover, free contracts are combined with a reward program, if the externalities are not large (as in Dropbox: one may use it for oneself to store files and access them from multiple computers, or share files with others), while only free contracts are offered if the externalities are large (as in Skype: any usage generates externalities). In contrast, transportation services such as Amtrak or Uber that solely rely on referral rewards programs would correspond to monopolists facing high α and low r, as many customers would be willing to pay for such services and those services would not be subject to significant externalities.[30]

We next consider how the optimal scheme depends on r. One might think that the smaller the

[30]Note that the fraction of the consumers purchasing free contracts is an endogenous variable, and one might think that our association of observable fractions for these real products to the exogenous parameter α is not justifiable. However, such association is justified because the map from consumer types to the choices of contracts is one-to-one given that free contracts are used. That is, if a positive fraction of consumers purchases free contracts, then within our model, such a fraction is exactly equal to $1 - \alpha$. Yet, it may be hard to empirically test our predictions for firms that do not offer free contracts because we do not observe α when free contracts are absent.

Externalities	$r < \frac{CF^*}{1-\alpha}$	$\frac{CF^*}{1-\alpha} < r < 10$	$10 < r < \frac{10-CF^*}{\alpha}$	$\frac{10-CF^*}{\alpha} < r < \frac{10}{\alpha}$	$\frac{10}{\alpha} < r$
Referral rewards	Yes	Yes	No	Yes	No
Free contract	No	Yes	Yes	No	No
Profit	Positive or zero	Positive or zero	Positive	Positive	Positive

Table 1: Comparative Statics with respect to r when $10 < \frac{CF}{1-\alpha}$. The use of referral rewards and free contracts is conditional on the firm generating positive profits.

externalities are, the more likely rewards are used. Figure 4 illustrates that this type of comparative statics fails for externalities. For example, at $\alpha = 0.15$, referrals are used when $r = 30$ but not when $r = 15$. The reason is that (i) when r is high, only one of a free contract and referral rewards suffices to incentivize the sender, i.e., these two are substitutes, and (ii) the cost of offering a free product CF^* is constant across r's while the optimal reward monotonically decreases with r. Thus, conditional on offering a free contract being sufficient to encourage WoM (i.e., $r \geq \xi$), offering a free contract is more cost-saving for smaller r while rewards are more cost-saving for larger r. Table 1 summarizes the different regions as functions of r for the case in which $\xi < \frac{CF}{1-\alpha}$.[31]

6 Conclusion

In this paper we propose a model that shows how referral rewards and offering a free contract can be effective tools to incentivize WoM for new products. They can be used separately, but substitution can result in cost savings while the two tools can also complement each other and encourage WoM in markets in which one tool alone is not effective. The main take-aways can be summarized as follows:

1. In general, substitution and complementation may or may not occur depending on whether the market is niche and whether the product is social.

2. Substitution occurs when the product is social, while complementation occurs when it is not too social but not too private either.

3. For social products, it is better to substitute a free contract with referral rewards when the market is mass, while it is better to substitute referral rewards with a free contract in niche markets.

[31] If this condition is not satisfied, some regions cease existing.

4. For less social products, a free contract can complement referral rewards in niche markets. Referral rewards complement a free contracts for the same markets, but also for markets that have more premium customers (i.e., are “more mass”). Those markets, however, cannot be too niche as the revenue from the product needs to be sufficiently high to make it worthwhile to incentivize WoM.

5. The pattern of the optimal scheme is consistent with the strategies we observe for companies such as Dropbox, Skype, Uber, and Amtrak.

In the Appendix and the Online Appendix we employ several robustness checks in order to show that our insights are not an artifact of the assumptions we impose in the model and analyze a few extensions. First, we generalize the functional forms and also allow for the possibility that the seller makes the referral rewards conditional on the receiver's purchase behavior. We show that such conditioning does not increase the seller's profit. We also prove that introducing heterogeneity in the costs of WoM does not change the qualitative results. Moreover, for a continuous type space of receivers (rather than only allowing for low-valuation and high-valuation receivers), we show that free contracts correspond to bunching at the bottom, i.e., among the customers who purchase positive quantities, customers buying the free contract correspond to a positive mass at the bottom of the type distribution. Importantly, all receivers who buy the free contract (except for the very lowest type) receive positive surplus. We also consider a model in which a receiver can be reached by many senders, and illustrate qualitative robustness of our results.

A few results change if we allow for externalities both on the sender and receiver side, as well as when externalities depend on the quantity consumed. In yet another extension, we let the senders be better informed than the firm, and conclude that in general the optimal reward must additionally depend on the type of receiver being acquired. We then discuss what the socially optimal contract scheme would look like if the social planner had control over the sender's actions. It turns out that free contracts are underutilized under the optimal scheme relative to the social optimum because the firm does not fully internalize the benefits from externalities and gains from trade with the receivers (corresponding to the information rent).

There are many direction of future research that are beyond the scope of this paper. For example, we have enumerated potential reasons for the use of free products in Section 1.1, and it would be interesting include those effects in the analysis. One possibility is to enrich our model by having the receiver take the role of the sender once she is informed. This can be done in either a diffusion-type model in which the penetration takes place over time, or in a stationary environment in which the population size is constant through time. Possible challenges in such models are that, when a customer decides whether to adopt the product, she not only considers the price and quantity (as in the receiver in our model), but also the future benefit from talking as a sender. In turn, the sender has to take into account this tradeoff of the receiver.

In another interesting extension, the receiver could be uncertain about the quality of the product, and the sender might have a higher incentive to talk when he knows the quality is higher. In such a model, if the receiver knows that the sender would receive referral rewards, then she may adjust their belief about the quality downwards. This requires a significant divergence from the Maskin-Riley model, but may be a worthwhile direction for future research.

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APPENDIX

This Appendix generalizes the "full model" provided in the main text. Section A presents such a generalized model and Section B analyzes the model. Section C provides proofs of the general results, which also prove the results presented in the main text.

A Generalized Model and Results

Basics. We consider a monopolist producing a single product at constant marginal cost $c > 0$. *Senders* (male) $\{1, \ldots, N\}$ can inform *receivers*, (female) $\{1, \ldots, N\}$ about the existence of the product. The monopolist's goal is to maximize the expected profit generated by receivers by offering them a menu of contracts and, in addition, offering a referral scheme to senders.

Receivers' preferences. Each receiver privately observes her type $\theta \in \{L, H\}$ that determines her valuation of the product. It is drawn independently such that a receiver is of type H with probability $\alpha \in (0, 1)$ and of type L otherwise. A type-θ receiver is associated with a valuation function $v_\theta : \mathbb{R}_+ \to \mathbb{R}$ that assigns to each quantity (or quality) q her valuation $v_\theta(q)$. Over the strictly positive domain, i.e., $q \in (0, \infty)$, we assume that v_θ is continuously differentiable, strictly increasing, strictly concave, $v_H(q) > v_L(q)$, $v'_H(q) > v'_L(q)$ for all q and $\lim_{q\to\infty} v'_H(q) < c$. We assume that $v_H(0) = v_L(0) = 0$, which can be interpreted as the utility of the outside option of not using the product at all. We make the following additional assumptions:

Assumptions.

1. **(Minimum quantity for low types)** $\exists \underline{q} > 0$ such that $v_L(\underline{q}) = 0$.
2. **(No gains from trade with low types)** $v'_L(q) < c$ for all $q \geq \underline{q}$.
3. **(Gains from trade with high types)** There exists a $q > 0$ such that $v_H(q) > q \cdot c$.

The first assumption can be interpreted as low types incurring some fixed installation cost of the product, and the low valuation buyer only wanting to start using the product if a minimum quantity of $\underline{q} > 0$ is consumed.[32] This assumption together with the normalization that $v_L(0) = 0$ and the assumption that v_L is strictly increasing in the strictly positive domain implies that the function v_L is necessarily discontinuous at $q = 0$ because v_L is strictly increasing on the strictly

[32]Note that this does not preclude the possibility of positive fixed installation costs for high types.

positive domain.[33] The second assumption captures that there are some consumers who would never use the product if they were not needed to incentivize WoM. Without the third assumption, the monopolist would not be able to earn positive profits, so the problem becomes trivial.

Senders' preferences and WoM technology. First, each sender i observes the monopolist's choice of menu of contracts and referral scheme (specified below). Each sender i then decides whether to inform receiver i at a cost $\xi \geq 0$ or not. We denote sender i's action by $a_i \in \{\text{Refer}, \text{Not}\}$, where $a_i = \text{Refer}$ if sender i refers receiver i and $a_i = \text{Not}$ otherwise. If (and only if) receiver i learns about the product, she decides whether to purchase a contract or not, and whether to consume the product or not upon purchasing. If receiver i consumes a positive quantity, sender i receives *externalities* $r \geq 0$.

Monopolist's problem. As in Maskin and Riley (1984), the monopolist offers a menu of contracts given by $((p_L, q_L), (p_H, q_H)) \in (\mathbb{R} \times \mathbb{R}_+)^2$ to receivers, where q_θ is the quantity type θ is supposed to buy at a price p_θ. Furthermore, she offers a reward scheme $\mathbf{R} : \{L, H\} \to \mathbb{R}_+$ such that a sender receives $\mathbf{R}(\theta)$ if he has referred a receiver who purchases the θ-contract. Rewards are assumed to be nonnegative because otherwise senders would be able to secretly invite new customers. We assume that the monopolist only receives revenue from new customers who do not know about the product unless a sender talks to them. In order to exclusively focus on the senders' incentive to talk, we assume that the monopolist receives no revenue from senders. Thus, the monopolist solves

$$\max_{p_L,\, p_H \in \mathbb{R},\, q_L, q_H \geq 0,\, \mathbf{R} \in \mathbb{R}_+^{\{L,H\}}} \quad \sum_{i=1}^{N} \mathbf{1}_{\{a_i = \text{Refer}\}}] \cdot \Big(\underbrace{\alpha \cdot (p_H - q_H \cdot c)) + (1-\alpha) \cdot (p_L - q_L \cdot c)}_{\text{total average profit per referred receiver}} \\ - (\alpha \mathbf{R}(H) + (1-\alpha) \mathbf{R}(L)) \Big) \tag{4}$$

subject to the incentive compatibility and participation constraints given by

$$\left. \begin{array}{rcll} \max\{v_H(q_H), 0\} - p_H & \geq & \max\{v_H(q_L), 0\} - p_L & \text{(H-type's IC)} \\ \max\{v_L(q_L), 0\} - p_L & \geq & \max\{v_L(q_H), 0\} - p_H & \text{(L-type's IC)} \\ \max\{v_H(q_H), 0\} - p_H & \geq & 0 & \text{(H-type's PC)} \\ \max\{v_L(q_L), 0\} - p_L & \geq & 0 & \text{(L-type's PC)} \end{array} \right\} \tag{5}$$

[33] Recall also that continuous differentiability of v_L is assumed only on the strictly positive domain.

and for all i, $a_i =$ Refer if and only if

$$\xi \leq r\left(\alpha + (1-\alpha)\cdot \mathbf{1}_{\{q_L>0, v_L(q_L)\geq 0\}}\right) + (\alpha \mathbf{R}(H) + (1-\alpha)\mathbf{R}(L)) \quad \text{(Senders' IC)}$$

Let Π^* denote the value of this problem. The monopolist chooses contracts given by quantities and prices, while *managing WoM.* The management of WoM appears as the senders' incentive compatibility (IC) constraint. On the left hand side is the cost of talking, ξ, which we assume to be homogeneous across senders. This simple case allows us to illustrate the main trade-offs. As a robustness check, the Online Appendix analyzes the case of heterogeneous costs in detail. On the right hand side, the quantity sold to L-type receivers q_L affects WoM by controlling the *expected externalities* given by $r\left(\alpha + (1-\alpha)\cdot \mathbf{1}_{\{q_L>0, v_L(q_L)\geq 0\}}\right)$. The senders' optimal decision determines the value of the indicator function in the objective function and thereby controls the number of informed receivers.

Let us explain a few assumptions implicit in this formulation. First, as standard in contract theory, we assume tie-breaking conditions for senders and receivers that are most favorable for the monopolist. Senders who are indifferent between referring and not will refer, and receivers that are indifferent between buying and not buying always buy. Second, we assume that if the buyer purchases a contract (p, q) such that $v_\theta(q) < 0$, then the monopolist cannot "force" the receiver to consume even if she pays the buyer a negative price. Thus, a type-θ receiver who purchases such a contract enjoys utility $\max\{v_\theta(q), 0\}$. There is no such max operation in the constraints in the main text. This is because, in the model in the main text, we assume for simplicity that prices are nonnegative. Under such an assumption it is straightforward to see that $v_\theta(q)$ is always the maximum under any optimal scheme.

A.1 Benchmark with free WoM

We first consider a benchmark case where $\bar{\xi} = 0$, i.e., WoM is costless and customers are automatically informed about the product. Then, the monopolist simply solves the classic problem as in Maskin and Riley (1984):

$$\Pi^{\text{classic}} \equiv \max_{p_H,\, p_L \in \mathbb{R}\; q_H, q_L \geq 0} \quad \alpha \cdot (p_H - q_H \cdot c) + (1-\alpha)\cdot (p_L - q_L \cdot c)$$

subject to the constraints (5). It is always optimal for the seller not to sell to L-type buyers such that $q_L^* = 0$ and the optimal quantity q_H^* sold to H-type buyers satisfies $v_H'(q_H^*) = c$. Assumption 3, strict concavity, continuous differentiability of v_H and $\lim_{q\to\infty} v_H'(q) < c$ ensure that there is a unique such q_H^*. The price for high types is given by $p_H^* = v_H(q_H^*)$ and the maximal static profit is $\Pi^{\text{classic}} = \alpha \cdot (p_H^* - q_H^* \cdot c)$. All in all, we can summarize our findings as follows:

$$v_H'(q_H^*) = c, \quad p_H^* = v_H(q_H^*), \quad \text{and } \Pi^{\text{classic}} = \alpha \cdot (p_H^* - q_H^* \cdot c).$$

A.2 Preliminaries

Before proceeding to the main analysis, we present several preliminary results. First, observe that $\mathbf{R}(\cdot)$ affects the monopolist's optimization problem only through the ex ante expected reward $R \equiv \alpha\mathbf{R}(H) + (1-\alpha)\mathbf{R}(L)$. Thus, profits are identical for all reward schemes $\mathbf{R}(\cdot)$ that share the same expected value. Formally, this means:

Lemma 1 (Reward Reduction)**.** *If a menu of contracts* $((p_L, q_L), (p_H, q_H)) \in (\mathbb{R} \times \mathbb{R}_+)^2$ *and a reward scheme* $\mathbf{R}^{**} : \{L, H\} \to \mathbb{R}_+$ *solve* (4)*, then the same menu of contracts* $((p_L, q_L), (p_H, q_H))$ *and any reward scheme* $\mathbf{R} : \{L, H\} \to \mathbb{R}_+$ *with* $\mathbb{E}[\mathbf{R}] = \mathbb{E}[\mathbf{R}^{**}]$ *solve* (4)*.*

Despite being a simple observation, this result implies an important feature of the optimization problem faced by the firm. As long as the firm and the senders have the same expectation about the receivers' types, there is no reason for the firm to condition their payment on the purchased contract. Indeed, in the Online Appendix, we show that if the senders have more accurate information about the receivers' types than the firm, the conclusion of Lemma 1 no longer holds. Thus, the detail of the optimal reward scheme crucially depends on the senders' knowledge. We relegate the analysis of this detail to the Online Appendix, while here we consider senders who have the same information about the receiver's types as the firm does. Note also that Lemma 1 does not imply that the sender receives referral rewards when the receiver does not end up using the product, for example when the low types are offered zero quantity.[34]

Plugging the sender's IC constraint into the objective function and noting that all senders share

[34]We can set $\mathbf{R}(\mathbf{L}) = \mathbf{0}$ and $\mathbf{R}(H) = R/\alpha$, so that senders who refer low types receive zero referral rewards.

the same IC constraint, Lemma 1 allows us to simplify the problem as follows:

$$\Pi^* = \max_{p_L, p_H \in \mathbb{R},\ q_L, q_H \geq 0,\ R \in \mathbb{R}_+} \quad \begin{array}{l} N \cdot \mathbf{1}_{\{\xi \leq r\left(\alpha + (1-\alpha)\cdot \mathbf{1}_{\{q_L > 0, v_L(q_L) \geq 0\}}\right) + R\}} \cdot \\ \left[\alpha \cdot (p_H - q_H \cdot c) + (1-\alpha) \cdot (p_L - q_L \cdot c) - R\right] \end{array} \tag{6}$$

subject to the constraints (5). We prove the existence of a solution to this problem. It is not immediate as the objective function is not necessarily continuous, but right-continuity of those functions and the fact that the number of discontinuous points is finite suffices to establish existence.[35]

Proposition 5 (Existence)**.** *The maximization problem (6) subject to* (5) *has a solution.*

Given parameters (α, r), we denote the (non-empty) *set of solutions* to this problem by

$$\mathcal{S} \subseteq (\mathbb{R} \times \mathbb{R}_+)^2 \times \mathbb{R}_+.$$

Moreover, for any menu of contracts $((p_L, q_L), (p_H, q_H))$ satisfying (5), we denote the firm's expected profits obtained from a receiver conditional on being informed by

$$\pi((p_L, q_L), (p_H, q_H)) \quad = \quad \alpha(p_H - q_H \cdot c) + (1-\alpha)(p_L - q_L \cdot c).$$

The monopolist can always choose not to sell to anyone and attain zero profits, i.e., $\Pi^* \geq 0$. Furthermore, whenever $\Pi^* = 0$ the seller can attain the maximum by inducing no sender to talk. This can be done by offering unacceptable contracts to receivers and no rewards.[36] We, thus, focus the characterization of optimal menu of contracts and rewards programs on the case when $\Pi^* > 0$.[37] The following lemma summarizes some basic properties of optimal menus of contracts.

Lemma 2. *If* $\Pi^* > 0$ *and* $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$*, then:*

(i) **Low types don't pay:** $q_L \in \{0, \underline{q}\}$ *and* $p_L = 0$.[38]

[35] The proof is done in a more general context, in which after each sender i sees the menu of contrast, he privately observes his cost of talking drawn from an independent and identical distribution with a cumulative distribution function that has at most finitely many jumps.

[36] Note that if there is a positive mass of senders with $\xi = 0$, then by Assumption 3 the seller can attain strictly positive profits by only selling to H-receivers and offering no reward.

[37] In part 1 of Theorem 4, we give a necessary and sufficient condition for $\Pi^* > 0$ to hold.

[38] The proof shows that we do not need to restrict prices to be nonnegative in order to obtain this result.

(ii) **No distortions at the top:** $q_H = q_H^*$.

(iii) **No free contracts:** *If* $q_L = 0$, *then* $p_H = p_H^*$.

(iv) **Free contracts:** *If* $q_L = \underline{q}$, *then* $p_H = p_H^* - \underbrace{v_H(\underline{q})}_{\text{information rent}} \equiv \tilde{p}_H^*$.

We have illustrated these results in the context of the no rewards model in the main text. Note that parts (iii) and (iv) follow because the incentive compatibility constraint of H-type receivers must be binding.

Lemma 2 restricts the set of possible optimal contracts significantly. In particular, it uniquely pins down the price offered to low types and the quantity offered to high types whenever $\Pi^* > 0$. At a price of zero for low types, the seller either chooses $q_L = 0$ (*no free contracts*) or $q_L = \underline{q}$ (*free contracts*). A full characterization of optimal contracts requires us to characterize the optimal reward scheme R and whether free contracts are optimal for the monopolist. These choices depend on the parameters that have not been used so far: the cost structure, the magnitude of externalities, and the composition of different types of buyers.

B Analysis of the Generalized Model

Here we aim to characterize the optimal schemes. Technically, the full characterization is involved for two reasons. First, there is a non-monotonicity of the use of rewards with respect to the size of externalities. That is, it is possible that the optimal reward changes from positive to zero and back to positive when externalities are increased because free contracts substitutes rewards. Second, the total cost of offering free contracts is determined by two factors, that is, the production cost (which is low for products such as Skype and Dropbox) of the free products and informational asymmetry, which forces the firm to pay an information rent to high-valuation buyers. This total cost of offering free contracts plays a key role in fully characterizing the optimal incentive scheme.

B.1 Characterization of Optimal Scheme

We characterize the optimal contracts in steps. First, we characterize the optimal referral reward scheme given a menu of contracts satisfying (5) (Lemma 3). Then, we solve for the optimal menu of contracts (Lemma 4) and finally, use these optimal contracts to derive the optimal reward using

Lemma 3 (Theorem 4).

With homogeneous costs of talking, if $r\left(\alpha + (1-\alpha)\cdot \mathbf{1}_{\{q_L>0, v_L(q_L)\geq 0\}}\right) + R \geq \xi$, then for any menu of contracts satisfying the constraints (5), profits are given by $\pi((p_L, q_L), (p_H, q_H)) - R$. Otherwise, profits are zero. Thus, if incentivizing WoM is not more expensive than the expected profits, the monopolist would like to pay senders just enough to make them talk. The following lemma formalizes this intuition. Let

$$R^{**}((p_L, q_L), (p_H, q_H)) \quad = \quad \max\left\{\xi - \underbrace{r\cdot\left[\alpha + (1-\alpha)\cdot \mathbf{1}_{\{q_L>0, q_L\geq \underline{q}\}}\right]}_{\text{expected externalities}}, 0\right\}. \tag{7}$$

Lemma 3 (Referral Program)**.** *Given contracts* (p_L, q_L) *and* (p_H, q_H) *satisfying* (5) *and* $v_H(q_H) \geq 0$*, the optimal referral reward is unique as long as* $R^{**}((p_L, q_L), (p_H, q_H)) < \pi((p_L, q_L), (p_H, q_H))$ *and is given by* $R^{**}((p_L, q_L), (p_H, q_H))$*.*

Using Lemma 2 and the formula of the optimal reward function R^{**} in Lemma 3, we can determine whether it is optimal to offer free contracts or not, which then pins down the full optimal menu of contracts. As in the main text, we define the cost of free contracts:

$$CF^* \quad \equiv \quad \alpha \underbrace{\cdot v_H(\underline{q})}_{\text{information rent}} + (1-\alpha)\cdot \underbrace{c\cdot \underline{q}}_{\text{production cost of free product}}. \tag{8}$$

Using this variable, let us first provide a heuristic argument: In order for free contracts to be optimal, this cost has to be outweighed by the benefit generated by providing the product to low types, i.e.,

$$CF^* \leq (1-\alpha)r, \tag{9}$$

or equivalently $\frac{CF^*}{1-\alpha} \leq r$. Notice that $\frac{CF^*}{1-\alpha}$ represents the "break-even externalities" necessary to compensate for the cost of free contracts. Moreover, $\frac{CF^*}{1-\alpha}$ is increasing in α. The average profit generated by a receiver if free contracts are offered can be written as

$$\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) = \Pi^{\text{classic}} - CF^*$$

The following result shows that, with additional boundary conditions, (9) is also sufficient to guarantee optimality of free contracts. We denote the set of optimal q_L by Q_L^{**}.

Lemma 4 (Free Contract). *Whenever $\Pi^* > 0$, an optimal contract to the type-L receiver must satisfy the following:*

(i) Let $r \in [\frac{\bar{\xi}}{\alpha}, \infty)$. Then, $Q_L^{**} = \{0\}$ (i.e., it is not optimal to provide free contracts).

(ii) Let $r \in [\bar{\xi}, \frac{\bar{\xi}}{\alpha})$.

1. **(Free contracts)** $\underline{q} \in Q_L^{**}$ *if and only if*

$$\underbrace{\xi - \alpha r}_{\text{reward w/o free contract}} \geq CF^*. \tag{10}$$

2. **(No free contracts)** $0 \in Q_L^{**}$ *if and only if* $\xi - \alpha r \leq CF^*$.

(iii) Let $r \in [0, \bar{\xi})$.

1. **(Free contracts)** $\underline{q} \in Q_L^{**}$ *if and only if* $r \geq \frac{CF^*}{1-\alpha}$.
2. **(No free contracts)** $0 \in Q_L^{**}$ *if and only if* $r \leq \frac{CF^*}{1-\alpha}$.

The intuition for this lemma is the following. First, there is no need for the seller to provide any incentives for WoM (i.e., $q_L = 0$) if the cost of talking ξ is smaller than the lowest expected externalities αr because in that case people talk anyway (Lemma 4 (i)). If $r \in [\bar{\xi}, \frac{\bar{\xi}}{\alpha})$ (Lemma 4 (ii)), then the cost of talking is larger than αr, but free contracts can boost the expected externalities to $r \geq \xi$. Then, free contracts are used whenever the referral reward that the seller had to pay without free contracts $\xi - \alpha r$ is larger than the cost of offering a free contract CF^* which is the sum of the information rent and cost of producing $\underline{q}$. Note that in this case, whenever free contracts are offered, the optimal reward is zero by Lemma 3. Finally, for high costs of talking $\xi > r$ (Lemma 4 (iii)), by Lemma 3 the seller pays a reward as long as the optimal reward does not exceed expected profits. If free contracts are offered, the expected externalities can be increased by $(1-\alpha)r$. Hence, free contracts are offered only if this benefit exceeds the cost of production and the information rent so that $r \geq \frac{CF^*}{1-\alpha}$ as explained above.

Lemmas 2, 3 and 4 pave the way for a full characterization of the optimal menu of contracts and reward scheme summarized in the following theorem. It shows that the optimal incentive scheme depends on the market structure given by parameters such as the cost of production c, the externalities r, the cost of talking ξ, and the fraction of H-type receivers α. Note that Proposition 3 is a special case of this theorem applied to the model in the main text.

Theorem 4 (Full Characterization)**.** *1.* **(Positive profits)** $\Pi^* > 0$ *if and only if*

$$\xi < \max\left\{\Pi^{classic} - CF^* + \min\{r, \xi\},\ \Pi^{classic} + \alpha r\right\}. \tag{11}$$

For the following cases, assume that (11) *is satisfied:*

2. **(Free vs. no free contracts)** *There exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *for some* R *if and only if* $r \in \left[\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right]$.[39]

3. **(Rewards vs. no rewards)**

(a) **(With free contracts)** *If* $r \in [\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}]$*, then* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < \xi$*, and*

(b) **(With no free contracts)** *If* $r \notin [\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}]$*, then* $((0, 0), (p_H^*, q_H^*), R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < \frac{\xi}{\alpha}$*.*

First, it is straightforward that the monopolist should provide no incentives for WoM either if senders talk anyway because the cost of talking is small (i.e., $\xi < \alpha r$) or if it is too expensive because the cost of talking ξ is too large relative to its benefits given in (11). A necessary condition for free contracts to be optimal is that r is large enough (i.e., $r > \frac{CF^*}{1-\alpha}$). An immediate implication is that without any externalities, free contracts are of no value to the seller. At the same time, free contracts are more effective to encourage WoM than rewards only if the cost of talking ξ is sufficiently large relative to r (i.e., $\xi > CF^* + \alpha r$ which is derived from the upper bound of r in part 2 of Theorem 4). Otherwise, it is cheaper to pay a small reward for talking.

We can also generalize the comparative statics in Proposition 4 as follows.

Proposition 6 (Market Structure and Free Contracts)**.**

(i) Consider two markets that are identical to each other except for the share of H-types, denoted α_1 *and* α_2*. Suppose that free contracts are offered under an optimal scheme in the market with* α_1*,*

[39] If $\frac{CF^*}{1-\alpha} > \frac{\xi - CF^*}{\alpha}$, then $[\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}] = \emptyset$.

$\Pi^* > 0$ *in the market with* α_2*, and* $\alpha_2 < \alpha_1$*. Then, free contracts are offered under any optimal scheme in the market with* α_2*.*

(ii) Suppose $v_H(\underline{q}) + r > c\underline{q}$*. Then,* $\alpha > \frac{r - c\underline{q}}{v_H(\underline{q}) + r - c\underline{q}}$ $(\Leftrightarrow r < \frac{CF^*}{1-\alpha})$ *implies that free contracts are never offered under any optimal scheme.*

In the interest of brevity, we do not generalize Theorems 1 and 2 here, but one can show that analogous results hold in the generalize model, too.

C Proofs of the General Results

Proof. **(Proposition 5)** As discussed in footnote 35, we prove the result for a general environment in which, after each sender i sees the menu of contrast, he privately observes his cost of talking ξ_i, drawn from an independent and identical distribution with a cumulative distribution function $G : \mathbb{R}_+ \to [0, 1]$ that has at most finitely many jumps. With this formulation, the present proof shows that the existence result is also valid for the general setup discussed in the Online Appendix. First, we show that it is without loss of generality to restrict attention to choice variables in a compact set. To see this, first note that, as we will show in the proof of Lemma 2, a scheme $((p_L, q_L), (p_H, q_H), R)$ with $q_L \in (0, \underline{q})$ generates a strictly lower profit than a scheme $((p_L, 0), (p_H, q_H), R)$. The same proof also shows that a scheme $((p_L, q_L), (p_H, q_H), R)$ with $q_L > \underline{q}$ generates a strictly lower profit than a scheme $((p_L, \underline{q}), (p_H, q_H), R)$. Thus it is without loss of generality to restrict attention to $\{0, \underline{q}\}$ as the space from which q_L is chosen. This and the participation constraint for low types imply that if a scheme $((p_L, q_L), (p_H, q_H), R)$ satisfies the constraints then $p_L \leq 0$. Also, the proof for Lemma 2 shows that for any scheme $((p_L, q_L), (p_H, q_H), R)$, $p_L < 0$ implies that the participation constraints for both types are non-binding, hence there exists $\epsilon > 0$ such that there exists a scheme $((p_L + \epsilon, q_L), (p_H + \epsilon, q_H), R)$ that satisfies the constraints and generates a higher profit than the original scheme. Consequently, it is without loss of generality to restrict attention to a scheme $((p_L, q_L), (p_H, q_H), R)$ with $p_L = 0$.

Also, since $\lim_{q \to \infty} v'_H(q) < c$, there exists q' such that any scheme $((p_L, q_L), (p_H, q_H), R)$ with $q_H > q'$ generates a strictly negative profit. Thus it is without loss of generality to restrict attention to $[0, q']$ for the space for q_H, where q' is any number satisfying $v'_H(q') < c$. Fix such $q' < \infty$ arbitrarily. Then, any scheme $((p_L, q_L), (p_H, q_H), R)$ with $R > v_H(q')$ generates a strictly negative profit, so again it is without loss to restrict attention to $[0, v_H(q')]$ as the space for R.

These bounds for q_H and q_L together with the PC constraints imply that it is without loss of generality to consider $p_H \leq v_H(q')$. The incentive compatibility condition for low types implies that $0 = \max\{v_L(q_L), 0\} - p_L \geq \max\{v_L(q_H), 0\} - p_H$, which implies $p_H \geq \max\{v_L(q_H), 0\} \geq 0$. Thus, it is without loss of generality to consider $p_H \in [0, v_H(q')]$.

These facts and the fact that all constraints are weak inequalities with continuous functions imply that the optimal scheme is chosen from a compact set. Now, note that the objective function is right-continuous in each choice variable because G is a cumulative distribution function, and all jumps are upwards.

These facts and the assumption that G has only finitely many discontinuities imply that there exists a partition of the compact space of the choice variables C with a finite number of cells $(P_1, \ldots, P_K)$ for some integer $K \in \mathbb{N}$, such that over each cell, the objective function is continuous.

Let $\hat{\pi}$ be the supremum of the objective function over C. Then there exists a sequence $(y^k)_{k=1,2,\ldots}$ with $y^k \in C$ for all k such that the value of the objective function under y^k converges to $\hat{\pi}$. Since $K < \infty$, this implies that there exists a cell of the partition, denoted P_{i^*} (choose one arbitrarily if there are multiples of such cells), and a subsequence $(z^k)_{k=1,2,\ldots}$ of $(y^k)_{k=1,2,\ldots}$ such that $z^k \in P_{i^*}$ for all k.

Since P_{i^*} is a bounded set, $(z^k)_{k=1,2,\ldots}$ has an accumulation point. Let an arbitrary choice of an accumulation point be z^*. If $z^* \in P_{i^*}$, then by continuity the objective function attains the value $\hat{\pi}$ at z^*. If $z^* \notin P_{i^*}$, then by the assumption of the upward jumps, the objective function attains the value strictly greater than $\hat{\pi}$ at z^*, which is a contradiction. This completes the proof. $\square$

Proof. **(Lemma 2)** Let $((p_L, q_L), (p_H, q_H), R)$ be an optimal scheme.

(i) Given a menu of contracts with $q_L > \underline{q}$ that satisfy (5), continuity of v_L implies that the monopolist can decrease q_L and p_L slightly, such that $\max\{v_L(q_L), 0\} - p_L$ remains constant (by Assumption 1) without violating (5) because $v_H(q_L) - p_L$ decreases with such a change (as $v'_H > v'_L$). This strictly increases profits by Assumption 2. Similarly, given a menu of contracts with $0 < q_L < \underline{q}$ that satisfy (5) and such that $\Pi^* > 0$, the monopolist can decrease q_L to zero and increase profits without violating (5).

The equation $p_L = 0$ can be shown by noting that type L's participation constraint must be binding: Assume $p_L < \max\{v_L(q_L), 0\} = 0$. First, note that then type H's participation constraint cannot be binding: If it was, then

$$0 = \max\{v_H(q_H), 0\} - p_H \geq \max\{v_H(q_L), 0\} - p_L \geq \max\{v_L(q_L), 0\} - p_L > 0$$

which is a contradiction. Thus, the monopolist can strictly increase profits by increasing p_L and p_H by the same small amount such that (5) remains to be satisfied. Consequently, $p_L = \max\{v_L(q_L), 0\} = 0$.

(ii) Given a R, $p_L = 0$ and fixing $q_L \in \{0, \underline{q}\}$, H-type's contract (p_H, q_H) must solve $\max_{p_H, q_H} \alpha(p_H - q_H c)$ subject to $\max\{v_H(q_H), 0\} - p_H \geq \max\{v_H(q_L), 0\}$ and $\max\{v_H(q_H), 0\} - p_H \geq 0$. If we ignored the participation constraint, and solved a relaxed problem, the incentive compatibility constraint must be binding and it follows that $q_H = q_H^*$ and $p_H = \max\{v_H(q_H^*), 0\} - \max\{v_H(q_L), 0\}$. This automatically satisfies the participation constraint:

$$\max\{v_H(q_H^*), 0\} - [\max\{v_H(q_H^*), 0\} - \max\{v_H(q_L), 0\}] = \max\{v_H(q_L), 0\} > \max\{v_L(q_L), 0\} = 0.$$

The above proof shows that IC constraint of the H-type is binding. Using this fact, parts (iii) and (iv) follow by plugging q_L into type-H's incentive compatibility constraint. □

Proof. **(Lemma 3, Referral Program)** A sender talks if and only if

$$\xi \leq r\left(\alpha + (1-\alpha) \cdot \mathbf{1}_{\{q_L > 0, v_L(q_L) \geq 0\}}\right) + R.$$

As a result, the monopolist must pay at least (7) in order to assure that senders talk and thus, the monopolist pays exactly this as long as it is profitable to inform receivers, i.e., as long as $R^{**}((p_L, q_L), (p_H, q_H)) < \pi((p_L, q_L), (p_H, q_H))$ holds. □

Proof. **(Lemma 4, Free Contracts)** (i) If $\xi \leq \alpha r$, then the senders' IC constraint is always satisfied, so that the seller's problem collapses to

$$\max_{p_L, p_H \in \mathbb{R},\, q_L, q_H \geq 0} N \cdot \left[\alpha \cdot (p_H - q_H \cdot c) + (1-\alpha) \cdot (p_L - q_L \cdot c) - R\right]$$

which is equivalent to the maximization problem in the benchmark case with free WoM. Thus, no free contracts are offered under any optimal scheme.

(ii) First, note that if $\Pi^* > 0$, it suffices to show when profits with free contracts (and the optimal reward scheme given by Lemma 3) are greater than profits without free contracts.

Let $\alpha r < \xi \leq r$. First, if $\xi - \alpha r > \Pi^{\text{classic}}$, then by Lemma 3, not offering free contracts yields negative profits and cannot be optimal. If $\xi - \alpha r \leq \Pi^{\text{classic}}$, then by Lemma 3, the optimal reward is $R = 0$ whenever $q_L = \underline{q}$ and is $R = \xi - \alpha r$ whenever $q_L = 0$. With $p_L = 0$ and (p_H, q_H) as in Lemma 2, it follows immediately that offering free contracts generates weakly higher profits than offering $q_L = 0$ if and only if $\Pi^{\text{classic}} - \alpha v_H(\underline{q}) - (1-\alpha) \cdot \underline{q} \cdot c \geq \Pi^{\text{classic}} - (\xi - \alpha r)$, which is equivalent to (10).

(iii) Let $\xi > r$. Then, by Lemma 3 if the monopolist chooses $q_L = \underline{q}$, then profits are given by $\Pi^{\text{classic}} - CF^* - (\xi - r)$ and if $q_L = 0$, then profits are given by $\Pi^{\text{classic}} - (\xi - \alpha r)$. Thus, offering free contracts generates a weakly higher profit than offering no free contracts if and only if $\Pi^{\text{classic}} - CF^* - (\xi - r) \geq \Pi^{\text{classic}} - (\xi - \alpha r)$, which is equivalent to $CF^* \leq (1-\alpha) r$. □

Proof. **(Proposition 3 and Theorem 4, Full Characterization)** Since Proposition 3 is a corollary of Theorem 4, we only prove the latter.

1. By Lemmas 2 and 3, $\Pi^* > 0$ if and only if $\Pi^{\text{classic}} - CF^* - \max\{\xi - r, 0\} > 0$ or $\Pi^{\text{classic}} - \max\{\xi - \alpha r, 0\} > 0$. Since $\Pi^{\text{classic}} > 0$, this can be rewritten as $\Pi^{\text{classic}} - CF^* - \max\{\xi - r, 0\} > 0$ or $\Pi^{\text{classic}} - (\xi - \alpha r) > 0$.

2. This follows immediately from Lemma 4.

3. (a) By Lemma 3, in the presence of free contracts, a reward must only be paid if $r > \xi$.

(b) Similarly, if no free contracts are offered, positive rewards are only being paid if $\alpha r < \xi$. □

Proof. **(Propositions 4 and 6)**

Since Proposition 4 is a corollary of Proposition 6, we only prove the latter.

(i) Denote the maximal expected profit without free contracts (i.e., $q_L = 0$ is offered to low types) under α by $\Pi^{\text{not free}}(\alpha)$. Similarly, denote the maximal expected profit with free contracts under α by $\Pi^{\text{free}}(\alpha)$.[40] The function $\Pi^{\text{not free}}(\alpha)$ is concave as long as $\Pi^{\text{not free}}(\alpha) > 0$, and $\Pi^{\text{free}}(\alpha)$

[40] Existence of these maxima follows from an analogous proof to the one for Proposition 5.

is linear in α as long as $\Pi^{\text{free}}(\alpha) > 0$. Moreover, we have that

$$\begin{aligned}
\lim_{\alpha \to 1} \Pi^{\text{free}}(\alpha) &= \lim_{\alpha \to 1} \alpha(p_H^* - q_H^* c - v_H(\underline{q})) - (1-\alpha)\underline{q}c - \max\{\xi - r, 0\} \\
&< \lim_{\alpha \to 1} \alpha(p_H^* - q_H^* c) - \max\{\xi - \alpha r, 0\} = \Pi^{\text{not free}}(\alpha).
\end{aligned}$$

This implies that $\Pi^{\text{not free}}(\alpha)$ and $\Pi^{\text{free}}(\alpha)$ intersect at most once. Hence, if $\Pi^{\text{free}}(\alpha_1) \geq \Pi^{\text{not free}}(\alpha_1)$, then $\Pi^{\text{free}}(\alpha_2) > \Pi^{\text{not free}}(\alpha_2)$ for all $\alpha_2 < \alpha_1$. This concludes the proof.

(ii) This part follows directly from part 2 of Theorem 4. □

Proof. **(Proposition 6)** (i) Denote the maximal expected profit without free contracts (i.e., $q_L = 0$ is offered to low types) under α by $\Pi^{\text{not free}}(\alpha)$. Similarly, denote the maximal expected profit with free contracts under α by $\Pi^{\text{free}}(\alpha)$.[41] The function $\Pi^{\text{not free}}(\alpha)$ is concave as long as $\Pi^{\text{not free}}(\alpha) > 0$, and $\Pi^{\text{free}}(\alpha)$ is linear in α as long as $\Pi^{\text{free}}(\alpha) > 0$. Moreover, we have that

$$\begin{aligned}
\lim_{\alpha \to 1} \Pi^{\text{free}}(\alpha) &= \lim_{\alpha \to 1} \alpha(p_H^* - q_H^* c - v_H(\underline{q})) - (1-\alpha)\underline{q}c - \max\{\xi - r, 0\} \\
&< \lim_{\alpha \to 1} \alpha(p_H^* - q_H^* c) - \max\{\xi - \alpha r, 0\} = \Pi^{\text{not free}}(\alpha).
\end{aligned}$$

This implies that $\Pi^{\text{not free}}(\alpha)$ and $\Pi^{\text{free}}(\alpha)$ intersect at most once. Hence, if $\Pi^{\text{free}}(\alpha_1) \geq \Pi^{\text{not free}}(\alpha_1)$, then $\Pi^{\text{free}}(\alpha_2) > \Pi^{\text{not free}}(\alpha_2)$ for all $\alpha_2 < \alpha_1$. This concludes the proof.

(ii) This part follows directly from part 2 of Theorem 4. □

[41] Existence of these maxima follows from an analogous proof to the one for Proposition 5.