Valence Candidates and Ambiguous Platforms in Policy Announcement Games

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Abstract

Over the course of election campaigns, candidates often use ambiguous language to describe their policies in the early stages of the campaign, and sometimes clarify their policies later on. We explain this phenomenon by constructing a dynamic model of campaigns. In the model, two candidates obtain opportunities to make their policies unambiguous over the course of a campaign period until the predetermined election date. While there is no incentive to keep policies ambiguous if the two candidates are perfectly symmetric, each candidate has a strategic incentive to keep policies ambiguous if one candidate is slightly stronger than the other.

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1 Introduction

In elections, candidates' policy announcements are often ambiguous. In the gubernatorial election in 2014 for Tokyo, Japan, Yoichi Masuzoe and Morihiro Hosokawa fought a close campaign. Although Masuzoe has been seen as the strongest candidate from the outset of the campaign, Hosokawa became popular in the middle of the campaign period when he announced opposition to the restart of nuclear power generation. Then Masuzoe, who originally had not specified his policy about nuclear power generation, clarified his position to aim at a gradual phase-out of nuclear power. As a result, Masuzoe won against Hosokawa.[1]

The reason for such ambiguity has long been discussed in the politics and economics literature. For example, in the context of US presidential election, Shepsle (1972, page 555) quoted Nicholas Biddle, the manager of William Henry Harrison's campaign for the US presidency in 1840-1841, who advised Harrison in these words: "Let him say not a single word about his principles, or his creed - let him say nothing - promise nothing. Let no Committee, no convention - no town meeting ever extract from him a single word, about what he thinks now, or what he will do hereafter."[2]

This paper has two main objectives. The first is to explain the phenomenon just described. That is, we explain why candidates use ambiguous language in campaigns, and why it is sometimes refined subsequently, as in the Tokyo gubernatorial election. The second objective is somewhat ambitious: Despite the apparent importance of election campaigns on the electoral outcome and the fact that the campaigns are dynamic in nature, there seem to be no models of dynamic campaigns in the literature, to the best of our knowledge.[3] One possible reason is that there is no obvious way to model campaigns in a way that would give rise to dynamic strategic considerations. We aim to fill this gap, by proposing a tractable model in which candidates face dynamic strategic considerations.

We propose a "policy announcement game," in which candidates strategically use ambiguous language which, in equilibrium, is sometimes refined subsequently. In our model, each of two candidates obtains opportunities to announce their policies according to a Poisson process over a

[1] Tokyo Shimbun argued on January 8, 2014 that Masuzoe was seen as the strongest candidate, Tokyo Mix News reported on January 14, 2014 that Hosokawa clarified his policy about nuclear power, and Asahi Shimbun reported on January 15, 2014 that Masuzoe declared that he supported a gradual phase-out of nuclear power.

[2] McGrane, Reginald Charles C. (ed.), *The Correspondence of Nicholas Biddle*. Boston, New York: The Houghton Mifflin Company, 1919.

[3] By a model of dynamic election campaigns, we mean a model with a single election; in particular, when we speak of "models of dynamic election campaigns," we are excluding models that have primaries and the general election.

campaign period until a predetermined election date. The assumption of Poisson opportunities is a simple way to represent the situation in which administrative procedures to obtain an internal approval of a change in the policy announcement may not be always successful, or candidates may not always be able to communicate with the voters about such changes even if these procedures go through. Moreover, voters may not be convinced by such announcements.[4] At each opportunity, candidates can either clarify their policies or keep their language ambiguous. Once a candidate has made his or her policy clear, he or she cannot change the specified policy afterwards.[5] We first show that, if two candidates are perfectly symmetric, there are no interesting strategic considerations. Specifically, each candidate makes his or her policy clear as soon as possible. Next we show that, if one candidate is slightly stronger than the other (has more valence), there are rich strategic considerations involved in equilibrium. For example, the weak candidate will not make his policy clear in the early stages of the election campaign.[6] This is because if he does so, then the strong candidate will simply copy that policy afterwards (as Masuzoe did in the Tokyo gubernatorial election), so that the weak candidate will certainly lose. Depending on the environment, the strong candidate may also have an incentive to use ambiguous language, if she expects a sufficient benefit from copying the weak candidate's policy near the election date.

Our work shows that *a candidate's valence* leads to *ambiguous language* in dynamic election campaigns. Let us now position our work in the literature with respect to these two factors.

In the standard simultaneous-move Hotelling-Downs model with valence candidates, there exists no pure strategy equilibrium: the strong candidate always wants to copy the weak candidate's policy, while the weak candidate does not want to be copied, just as in the "matching pennies" game. There are two approaches to addressing this issue. The first approach is to assume that the strong candidate is the incumbent and the weak candidate is the entrant (Bernhardt and Ingberman (1985), Berger et al. (2000)). In this approach, a typical result is that the strong candidate positions her policy close to the median voter and the weak candidate positions his policy at a slight distance from the strong candidate's policy, where the distance between the two policies is determined by

[4] A richer modeling of administrative procedure or dynamics of voter beliefs would generate a more accurate prediction, but we assume these away and try to concentrate on the key effects by investigating what we can say in our simplest framework. As it turns out, the result that comes out from our simple setting is quite rich.

[5] This assumption is also made for simplicity as a first step to tackle this problem. We discuss this issue in the conclusion.

[6] For ease of exposition throughout the paper, we use feminine pronouns to refer to the strong candidate and masculine pronouns to refer to the weaker candidate.

the degree of asymmetry between candidates' valences. The second approach is that of Aragonès and Palfrey (2002), who consider the simultaneous-move game seriously and characterize a mixed equilibrium.[7] They show that the strong candidate assigns high probabilities to the platforms which are close to the location of the median voter with high probabilities while the weak candidate assigns small probabilities to such platforms. Although these two approaches give us an understanding of what the equilibrium behavior looks like in an electoral situation with valence candidates, in both these models the order of policy announcements is exogenously given by the modelers. In contrast, we view our work as *endogenizing* the order of policy announcements.[8]

The mechanism that generates ambiguous policy announcements in our model is starkly different from those presented in the existing literature. For example, Shepsle (1972) and Aragonès and Postlewaite (2002) assume that candidates choose their policy positions simultaneously and once and for all.[9] Ambiguity occurs because voters are assumed to possess convex utility functions and therefore prefer uncertainty, that is, ambiguous policy announcements. On the other hand, ambiguity in our model arises from dynamic strategic interactions in an election campaign: each candidate's strategic concern about the opponent's future play causes ambiguity. In particular, we do not assume convexity; rather, in one of the variants of our model, we show that ambiguity occurs even when voters have concave utility functions.

Before proceeding to the details, we wish to emphasize that we do not aim to provide a model and results that are definitive. Rather, we view them as suggestive. Our whole objective is to formalize ambiguity as a result of valence and dynamic strategic considerations, and perhaps more importantly, to provide a basis for future research on dynamic campaigns by proposing a tractable model to analyze issues arising from the dynamic nature of election campaigns. To this end, we keep our model as simple as possible to highlight the effect of dynamics.

1.1 Literature Review

In the introduction, we have already identified existing work on models with valence candidates, as well as a few papers on ambiguous policy announcements. Ambiguous policy announcements

[7] More specifically, Aragonès and Palfrey (2002) characterize the unique equilibrium in a discrete policy space and consider a limit as the discrete space approximates the standard continuous policy space.

[8] This answers the question posed by Aragonès and Palfrey (2002), who ask "What is the correct sequential model."

[9] We will discuss more papers on ambiguity in the literature review section.

have received much attention in the literature, so in this section we discuss some of these papers and compare them with our work. We also compare our model with recent theoretical models that have features similar to ours.

Ambiguity papers:

Page (1976, 1978) proposes a theory that attributes ambiguity to the fact that candidates have limited resources to make their policy positions precise, and to voters' limited capacity to understand these positions. In our model, however, voters are capable of understanding what the candidates are announcing. Candidates do have a positive probability of not being able to have any chance to make a policy announcement, but we obtain ambiguity even in the limit as this probability shrinks to zero.

Glazer (1990) argues that, if candidates do not have control over which policy to specify when they intend to make a policy announcement, they may prefer being ambiguous.[10,11] Ambiguity occurs when either (i) the policy space consists of unequally dispersed points; (ii) the median voter is assumed to believe that a policy resulting from an ambiguous announcement is close enough to her bliss point; or (iii) in a sequential-announcement model, each candidate has private information about the position of the median so that observing the opponent's position gives new information. None of these assumptions drives the conclusion in our model.

Alesina and Cukierman (1990) and Aragonès and Neeman (2000) show that ambiguity occurs in elections if candidates prefer to keep the freedom to choose their policies after being elected, even though voters would prefer that their candidates commit themselves to precise policies before the election. That is, the driving force of ambiguity is different from office motivation. In contrast, we derive ambiguity from pure office motivation.

When the selection of candidates consists of more than one step, as is true for the US presidential election with its primaries and general elections, Meirowitz (2005) shows that candidates announce ambiguous policies in the earlier stages if voter preferences are unknown at the beginning but

[10]It is true that even in our model, even if a candidate intends to specify his of her policy from the outset, he or she cannot do so with positive probability. However, such a probability goes to zero as the length of the campaign phase goes to infinity.

[11]Glazer (1990) claims that the assumption that candidates have no control over policies represents a situation in which candidates are uncertain about the median voter's preferences. He models the candidate's problem as a binary-choice problem between being ambiguous and being umambiguous. For the median voter's preference to be uncertain, there should be more than one policy positions. With multiple positions, even if the preferences of the median voter are uncertain, a candidate's incentive to deviate from being ambiguous would be higher if he/she could control which policy to deviate to.

are revealed by the result of the earlier stages. In our model, no new information arrives about voter preferences and ambiguous policies are purely the result of the strategic interaction between candidates.[12]

In the base model, we find that ambiguity is likely when the probability distribution of the median voter's position is close to uniform. Although we view our results as only suggestive, this result is consistent with the empirical finding by Campbell (1983), who suggests that opinion dispersion has a strong positive effect on the ambiguity in candidates' language. See Section 5.2 for the discussion of the empirical implications.

Revision games and related models:

To formally model the dynamics of policy announcements, we employ a framework with continuous time, a finite horizon, and a Poisson revision process. This modeling device has been extensively explored recently. The revision games in Kamada and Kandori (2013) and Calcagno, Kamada, Lovo and Sugaya (2014) consider settings in which players obtain opportunities to revise their preparation of actions according to Poisson processes, and the final revised action profile is played at the predetermined deadline.[13] In these papers, revisions of actions are not restricted, in the sense that players can freely choose their actions from a fixed action space at each opportunity to move, as opposed to our assumption that once candidates make their policy platform clear, they cannot change it afterwards. The other difference is in the nature of the game analyzed: these papers analyze games in which cooperation and coordination are at issue, while we analyze a constant-sum game. This leads to, among other things, a difference in the effects of heterogeneity in arrival rates, which we discuss in Section 5.3.

As for the idea of announcing ambiguous language in expectation of future events, Gale's (1995, 2001) model of "monotone games" also considers a similar problem. In his model, at each period, players can only (weakly) increase their actions. In effect, in each period players therefore commit to a smaller and smaller subset of their action spaces, and they will never be able to "expand" that subset (thus, the revisions are called restricted). The main difference is that he analyzes "games with positive spillover" played over an infinite horizon and show that collusive outcomes can be

[12] Alesina and Holden (2008) show that candidates announce ambiguous policies even without primaries if (i) candidates have policy motivation, (ii) the policy motivation is their private information unknown to the voters, and (iii) campaign contributions from the voters to the candidates affect the electoral outcomes. In contrast, we assume that candidates are purely office motivated and this is common knowledge among the candidates and the voters.

[13] Ambrus and Lu (2013) consider a bargaining model in a similar fashion.

achieved, while we analyze a constant-sum game played over a finite horizon and show that there exists an (essentially) unique equilibrium.

The paper proceeds as follows. In Section 2, we introduce the model of a policy announcement game. Section 3 analyzes the case in which there is no campaign phase, i.e., the policy announcement occurs simultaneously and once and for all. Section 4 analyzes the dynamic model. In Section 4.1, we establish that if two candidates are perfectly symmetric, then both candidates would want to be clear as soon as possible. In Section 4.2, we establish that if one candidate is slightly stronger than the other, then there are rich strategic considerations driving the incentive for each candidate to announce ambiguous policies. Therefore, we conclude that the key assumption for ambiguous policies is valence. In Section 5, we compare the dynamic model with the one-shot game and offer the empirical implications. We also discuss the robustness of our ambiguity result to model specifications by analyzing other variants of the model, such as those with heterogeneous arrival rates, generalized payoff structures, and synchronous announcements. Section 6 concludes. The appendix contains a proof of a lemma on continuous-time backward induction that we use repeatedly in the proofs and a proof for the result for the base model (Proposition 3). All the proofs not provided in the main text or in the appendix are provided in the online appendix (starting from page 40).

2 The Model - Policy Announcement Game

There are two candidates, S and W, interpreted as a "strong candidate" and a "weak candidate," respectively.[14] The base model is particularly simple, so as to highlight the complexity introduced by the campaign phase into an election model. Specifically, the policy platform consists of two points $X := \{0, 1\}$. Notice that this is the minimal environment in which we could potentially have strategic ambiguity. Section 5.4 presents a general version of the model that involves many other cases, such as a continuous policy space.

In our policy announcement game, time is continuous and flows from $-T$ to 0 where $T > 0$ is large. Imagine that 0 is the fixed election date and the campaign starts at $-T$. For each $-t \in [-T, 0]$, according to the Poisson process with arrival rate $\lambda > 0$, each candidate obtains

[14] As before, for ease of exposition, we use feminine pronouns to refer to S and masculine pronouns to refer to W.

an opportunity to announce a nonempty subset of X that we call a "policy set." In this section, we assume that the Poisson processes are independent between the candidates. In particular, this implies that policy announcements are asynchronous. The case of synchronous announcements is discussed in Section 5.5.

The set of candidate i's possible announcements at time $-t$ depends on i's own past policy announcement: If i has already announced $\{0\}$ in the past, then i can only announce $\{0\}$. Similarly, if i has already announced $\{1\}$ in the past, then i can only announce $\{1\}$. However, if i's policy set has been only $\{0, 1\}$ in the past, then i can announce either $\{0\}$, $\{1\}$, or $\{0, 1\}$. Thus, once the candidates specify their platforms, they cannot change them later. We let the policy set at time $-T$ be exogenously given to be $\{0, 1\}$.

Here, we interpret announcing $\{0, 1\}$ as announcing the "ambiguous policy" while announcing $\{0\}$ or $\{1\}$ is seen as specifying a policy platform. To simplify the exposition, we will occasionally use "enter" to denote the act of announcing either $\{0\}$ or $\{1\}$.

The result of the election is determined by the voter distribution over the policy space $\{0, 1\}$ and (X_S, X_W), where X_i with $i \in \{S, W\}$ is candidate i's most recently announced policy at time 0 (the election date). During the campaign, the voter distribution is unknown but the distribution of the median voter is known to follow the probability mass function: $f(0) = p$, $f(1) = 1-p$, where $p \in (0, \frac{1}{2})$.[15]

Let us now specify the voters' utility function and behavior rules. If a candidate $i \in \{S, W\}$ wins the election and implements policy $x \in \{0, 1\}$, then a voter with position $y \in \{0, 1\}$ obtains the payoff of

$$u(|x - y|) + \delta \cdot \mathbb{I}_{i=S},$$

where $u(0) > u(1)$ and $0 \leq \delta < (u(0) - u(1))/2$. The voters believe that, if candidate i has specified a policy $x \in \{0, 1\}$, then x will be implemented. If candidate i with the ambiguous policy $X_i = \{0, 1\}$ wins, then the voters believe the policies $\{0\}$ and $\{1\}$ will be implemented with equal probability $\frac{1}{2}$.[16] The voters are sincere, that is, they each vote for the candidate who, if elected, maximizes their expected payoff. The candidate with more votes wins. (In the case of a tie, each

[15] In our two-policy model, given a realization of the voters' positions, the platform with voters with a share more than $\frac{1}{2}$ (which we assume to exist with probability one) is the position of the median voter.

[16] The model is not a knife-edge case with respect to this assumption. For an open set of environments, our main results are unchanged.

$\{X_S, X_W\}$ at the deadline	Voters at 0 vote for	Voters at 1 vote for	S's expected utility	W's expected utility
$\{0,1\},\{0,1\}$	S	S	1	0
$\{0,1\},\{0\}$	W	S	$1-p$	p
$\{0,1\},\{1\}$	S	W	p	$1-p$
$\{0\},\{0,1\}$	S	W	p	$1-p$
$\{0\},\{0\}$	S	S	1	0
$\{0\},\{1\}$	S	W	p	$1-p$
$\{1\},\{0,1\}$	S	S	$1-p$	p
$\{1\},\{0\}$	W	S	$1-p$	p
$\{1\},\{1\}$	W	S	1	0

Figure 1: Voter behaviors and the expected payoffs

candidate wins with probability 1/2.) Note that δ is the utility in having S as a winner, that is, S is stronger than W by nature (valences) for $\delta \geq 0$.

The candidate who obtains more votes wins, and obtains the payoff of 1, while the other candidate obtains the payoff of 0; these are the only payoffs that they receive in this model. Hence we are assuming purely office motivated candidates. Each candidate's objective is to maximize the expected payoff, that is, their objective is to maximize their probability of winning. We summarize in Figure 1 the voters' behaviors and the resulting expected payoffs for the candidates, given these specifications and $\delta > 0$. Note that the environment just specified is the one in which we can apply the median voter theorem; that is, without valence, the candidate who specifies the policy at the position of the median voter wins with positive probability.

In what follows, we will analyze the subgame perfect equilibria of this game. To formally define strategies in our setting, we first define history. A **history** for candidate i is denoted by:

$$\left(\left(t_S^k, x_S^k\right)_{k=0}^{k_S}, \left(t_W^k, x_W^k\right)_{k=0}^{k_W}, t, z_i\right),$$

where $-T < -t_i^1 < ... < -t_i^{k_i} < -t$ for $i = S, W$; $x_i^k \in 2^X \setminus \{\emptyset\}$ for all k and $i = S, W$; and $z_i \in \{yes, no\}$. The interpretation is that $-t_i^k$ is the time at which candidate i receives his or her k'th revision opportunity, and x_i^k is the policy set that i has chosen at time $-t_i^k$. The third element t denotes the current remaining time, and the indicator z_i expresses whether there is an opportunity for candidate i at time $-t$. The set of histories in which candidate i for $i = S, W$ has

$S \setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	$1, 0$	$p, 1-p$	$p, 1-p$
$\{1\}$	$1-p, p$	$1, 0$	$1-p, p$
$\{0,1\}$	$1-p, p$	$p, 1-p$	$1, 0$

Figure 2: The one-shot game

received k_i opportunities in the past is denoted $H_i^{(k_S,k_W)}$. The set of all histories for candidate i is $H_i := \bigcup_{k_S=0}^{\infty} \bigcup_{k_W=0}^{\infty} H_i^{(k_S,k_W)}$.

A **strategy** for candidate i is denoted by $\sigma_i : H_i \to \Delta(\{0\},\{1\},\{0,1\})$, with two restrictions: First, $\sigma_i(h_i) = x_i^{k_i}$ where k_i is specified in the first or second element of h_i if the fourth element in h_i specifies $z_i = no$. That is, if there is no opportunity at $-t$, then for notational convenience, we specify that the candidate takes the same policy as specified in the last opportunity. Second, if $z_i = yes$, then the strategy $\sigma_i(h_i)$ must put probability zero on $x \in 2^X$ if $x \not\subseteq x_i^{k_i}$. This constraint implies that once a candidate specifies a policy, he/she cannot change it later.

Let Σ_i be the set of all strategies of candidate i. Let $u_i(\sigma|h_i)$ be candidate i's continuation payoff given history $h_i \in H_i$ and the continuation strategy profile $\sigma \in \Sigma_S \times \Sigma_W$.[17] A strategy profile (σ_1, σ_2) is a **subgame perfect equilibrium** if, for each $i = S, W$, the strategy σ_i maximizes $u_i(\sigma|h_i)$ for every $h_i \in H_i$.[18]

3 The One-Shot Case

To better understand the incentive problems that candidates face, let us first demonstrate what would happen if our game were the one-shot simultaneous-move game. If $\delta = 0$ (i.e., the candidates are symmetric), then since the median voter is located with a higher probability at $\{1\}$, both candidates take $\{1\}$. Hence, the symmetric candidates do not use ambiguous language.

On the other hand, if $\delta > 0$ (i.e., the candidates are asymmetric with respect to valence), then they use ambiguous language. To see this, note that the game can be represented by the payoff matrix as shown in Figure 2.

[17] This is well-defined because H_i is a countable union of subsets of a finite-dimensional space.

[18] To be precise, since candidate i at time $-t$ does not know if the opponent has received an opportunity at the same time, there is no proper subgame. However, since such an event occurs with probability zero, we simply call it a subgame perfect equilibrium.

By inspection one can show that there is a unique (completely mixed) Nash equilibrium in this game. In this equilibrium, S and W take $\{0\}$, $\{1\}$, and $\{0,1\}$ independently with probabilities

$$\left(\frac{p^2}{1-p+p^2}, \frac{(1-p)^2}{1-p+p^2}, \frac{p(1-p)}{1-p+p^2}\right) \quad \text{and} \quad \left(\frac{(1-p)^2}{1-p+p^2}, \frac{p^2}{1-p+p^2}, \frac{p(1-p)}{1-p+p^2}\right). \tag{1}$$

Here, in the first (or second) parenthesis, the first, second, and the third elements are probabilities that S (or W) takes $\{0\}$, $\{1\}$, and $\{0,1\}$, respectively.

This means that *even without the dynamic considerations, candidates face the incentive to use ambiguous language.* This deserves an explanation. We first note that a simple calculation shows that if $\delta = 0$, there is a unique Nash equilibrium, in which both candidates choose policy $\{1\}$. The situation changes once δ becomes positive. Notice that, in equilibrium with $\delta > 0$, clearly no candidate uses a pure strategy. So suppose that S mixes between the two unambiguous policies while assigning zero probability to the ambiguous policy. In this case, $\{0,1\}$ dominates both $\{0\}$ and $\{1\}$ for W for the following reason. Fix the realization of S's mixture $x_S \in \{0,1\}$. For each $\{x\} = \{0\},\{1\}$ that W specifies, (i) if x is different from x_S, then both $\{x\}$ and $\{0,1\}$ allow W to win if and only if the median voter is at x; and (ii) if x is equal to x_S, then only $\{0,1\}$ allows W to win with positive probability. Hence, $\{0,1\}$ dominates $\{x\} = \{0\},\{1\}$ for W.

If W takes the ambiguous policy $\{0,1\}$ with positive probability, then the ambiguous policy $\{0,1\}$ also becomes attractive for S, because if both candidates take $\{0,1\}$, then S wins for sure. This is the main intuition for why both candidates assign positive probabilities to the ambiguous policy.

Thus, our model predicts ambiguous policy announcements even without a dynamic component.[19] However, this is only a part of our story. What we will show in the main section (Section 4) is that the candidates face complicated dynamic incentive problems in our policy announcement game. Specifically, *the candidates' incentives to announce ambiguous policies change over time.*

4 The Dynamic Case

Now we turn to the dynamic model. In the first subsection we consider the case of $\delta = 0$ as a benchmark case. It turns out that there are no strategic incentives to announce the ambiguous

[19]To the best of our knowledge, this analysis itself is new.

policy $\{0,1\}$. Then, in the second subsection, we consider the case of $\delta > 0$, and demonstrate that candidates face complicated strategic considerations that are absent in the model with $\delta = 0$.

4.1 The Benchmark Case: Perfectly Symmetric Candidates

Suppose that $\delta = 0$. The following proposition gives us a stark result:

Proposition 1 *Suppose $\delta = 0$. Then, each candidate announces $\{1\}$ as soon as possible in equilibrium.*

Proof. Fix t, and suppose that at each time $-s > -t$, if each candidate has an opportunity to enter, then he/she enters at 1. Then, at time $-t$, entering at 1 gives a payoff strictly greater than $\frac{1}{2}$, entering at $\{0\}$ gives $p < \frac{1}{2}$, and not entering gives the payoff of $\frac{1}{2}$ by symmetry. Thus, entering at 1 is the strict best-response. Therefore, by the continuity of expected payoffs in probability, for sufficiently small $\varepsilon > 0$, it is strictly optimal to enter at 1 for all $-\tau \in (-t-\varepsilon, -t]$. By a backward induction argument, we obtain the desired result.[20] ■

This negative result is very general. In particular, it is straightforward to verify that the result holds also in the other versions of our model that we will present in Section 5. Hence, the assumption of $\delta > 0$ is the key for the ambiguous policy announcements. From the next subsection on, we will demonstrate that (i) the above simple argument breaks down once we introduce asymmetry with respect to candidates' valence, and (ii) candidates face complicated dynamic incentive problems.

4.2 The Cases with Valence Candidates

In this section, we demonstrate that if $\delta > 0$, then there are rich strategic considerations involved in equilibrium, which involve ambiguous policy announcements. Therefore, a small valence matters.

Let us start with the following lemma. It states that, if S has an opportunity to enter after W has entered at $x \in \{0,1\}$, then S enters at x and wins for sure.[21] On the other hand, if W has an opportunity to enter after S has entered at $x \in \{0,1\}$, then W is indifferent between announcing $\{0,1\}$ and entering at $x' \in \{0,1\} \setminus \{x\}$. These two conclusions imply that, since the median is more likely to be at $\{1\}$ ($p < \frac{1}{2}$), if a candidate enters before the opponent, he/she enters at $\{1\}$.

[20] A formal backward induction argument in continuous time is given by Lemma 1 of Calcagno, Kamada, Lovo and Sugaya (2013), which is reproduced as Lemma 9 in the appendix for readers' convenience. We use this lemma in proving other propositions too.

[21] Recall that the term "enter" means "clarify the policy" or "announce the policy $\{0\}$ or $\{1\}$."

Lemma 2 *In any subgame perfect equilibria, the following are true:*

1. *Given that W has already entered, S enters at the same platform as soon as possible for all t.*

2. *Given that S has already entered, W is indifferent between announcing $\{0,1\}$ and entering at the platform different from S for all t.*

3. *If a candidate i enters before the opponent, then i enters at $\{1\}$.*

From now on, we assume that after S's entry, W will not enter. The uniqueness results in this paper are up to this assumption.

The above lemma pins down the equilibrium behaviors on and off the equilibrium path except when no candidates have yet entered. It also says that if both are still using ambiguous language and a candidate i enters, then i enters at $\{1\}$. Hence, in the following analysis, we consider the incentives to enter at $\{1\}$ in such a situation.

Before presenting the characterization of the equilibrium, we first provide the basic intuition. For the time being, consider the case with $p = \frac{1}{2}$.[22] Suppose that at time $-t$, both S and W have previously announced $\{0,1\}$. If there is no further revision, W's payoff is 0. So W needs to specify his policy to obtain a positive payoff. Thus, W announces $\{0\}$ or $\{1\}$ at some point, if he can. Since $\{0\}$ and $\{1\}$ are symmetric with $p = \frac{1}{2}$, assume without loss of generality that W announces $\{1\}$ when he clarifies his policy.

On the other hand, S does not have an incentive to specify her policy until W specifies his policy; this is because she gets $\frac{1}{2}$ for sure by specifying her policy, while using ambiguous language gives her either $\frac{1}{2}$ or 1 with the latter taking place with positive probability (when W does not enter afterwards and when W enters and S copies his policy).

If W announces $\{1\}$ in an early stage of the campaign, then the probability with which S enters afterwards is high. So W wants to postpone announcing. But waiting too much is not optimal for W either, since if he does not have a chance to revise his policy set, W gets the payoff of 0. So there should exist a "cutoff," $-t^*$, until which W announces $\{0,1\}$ and after which W announces $\{1\}$ when he gets an opportunity of policy announcement.

[22]Strictly speaking, since $p < \frac{1}{2}$, this is actually outside of the model, but we consider such a case to provide the intuition. The same comment applies to the case $p = 0$ that we consider next.

Recall that we do not have this type of strategic consideration when $\delta = 0$. The reason that the simple argument in the proof of Proposition 1 breaks down is that the continuation payoff after taking each action is different once we introduce valence. For example, W expects a payoff close to zero if he specifies some policy when the deadline is far away, as opposed to the payoff equal to $\frac{1}{2}$ when $\delta = 0$.

Next, consider the case with $p = 0$. In this case, S would want to commit to $\{1\}$ as soon as possible, because she can then obtain the payoff of 1, which is the highest possible payoff. Since S's strategy is stationary and W can win if and only if he enters at $\{1\}$ and S does not have an opportunity, W also enters at $\{1\}$ as soon as possible.

The next proposition fully characterizes the form of the equilibrium, which we prove to be unique, for each $p \in (0, \frac{1}{2}) \setminus \{\frac{1}{1+e}\}$. Suppose that the current policy set of each candidate is $\{0, 1\}$. The equilibrium strategy of W is to wait until a finite cutoff and to enter as soon as possible after that cutoff. In contrast to the case of $p = 0$, the cutoff is finite for any strictly positive p because the probability that the median voter is at 0 is strictly positive. The equilibrium strategy for S depends on the value of p. If p is close to $\frac{1}{2}$ (part 1 of Proposition 3), S does not enter until W enters for the same reason as when $p = \frac{1}{2}$. On the other hand, for small p (part 2 of Proposition 3), S enters when the deadline is far away as when $p = 0$, but does not do so when the deadline is close. The value $p = \frac{1}{1+e}$ corresponds to the cutoff at which S's incentive changes. The intuition for the ambiguity near the deadline when p is close to 0 is as follows: If S obtains an opportunity at $-t$ when the deadline is close, then the probability with which W has a chance to announce his policy afterwards is small. So it is likely that W uses ambiguous language at the deadline. Thus, keeping ambiguous language is profitable for S, because by doing so, S gets the payoff of 1 with a high probability.

Moreover, the equilibrium is essentially unique. We say that the equilibrium is essentially unique if there exists a finite set $\{t_1, t_2, ..., t_k\}$ (possibly an empty set) such that, for each subgame perfect equilibrium σ and σ', each player i, and each history h_i with $t \notin \{t_1, t_2, ..., t_k\}$, we have $\sigma_i(h_i) = \sigma'_i(h_i)$. That is, each subgame perfect equilibrium coincides except for finitely many timings.

Proposition 3 *For each $p \neq \frac{1}{1+e}$, the equilibrium is essentially unique.*[23] *In this equilibrium, the*

[23] If $p = \frac{1}{1+e}$, then there is indeterminacy about S's equilibrium strategy at all $-t < -t^*$ since she is indifferent.

following are satisfied if the previous policy sets are both $\{0,1\}$*:*

1. *If* $p > \frac{1}{1+e}$*, there exists* $t^* := \frac{1}{\lambda}$ *such that the following hold:*
   (a) S *announces* $\{0,1\}$ *for all* $-t \in (-\infty, 0]$.
   (b) W *announces* $\{0,1\}$ *for all* $-t \in (-\infty, -t^*)$ *and* $\{1\}$ *for all* $-t \in (-t^*, 0]$.
2. *If* $p < \frac{1}{1+e}$*, then there exist* t_S *and* t_W *(that depend on* p*) such that the following hold:*
   (a) S *announces* $\{1\}$ *for all* $-t \in (-\infty, -t_S)$ *and* $\{0,1\}$ *for all* $-t \in (-t_S, 0]$.
   (b) W *announces* $\{0,1\}$ *for all* $-t \in (-\infty, -t_W)$ *and* $\{1\}$ *for all* $-t \in (-t_W, 0]$.
   (c) *Moreover,* $\frac{dt_S}{dp} < 0$ *and* $\frac{dt_W}{dp} > 0$.

Note that the cutoffs are independent of T. Hence, when T and p are large, we expect that candidates use ambiguous language for most of the campaign period. Note that stretching T and enlarging λ with the same ratio are equivalent. Hence, this also implies that for fixed length of campaign period T, if we consider the situation in which the opportunities arrive frequently, candidates spend most of the time in T on using ambiguous languages.

In Figure 3, we depict the times t^*, t_S, and t_W that appear in Proposition 3, for different values of p. For example, $p = .4$ ($> \frac{1}{1+e}$) corresponds to part 1 of the proposition. In this case, there is one point at which the graph in the figure intersects with the $p = .4$ line, so as a result, the time spectrum is divided into two regions: In the left region, no candidate enters. In the right region, S does not enter while W enters. When $p = .2$ ($< \frac{1}{1+e}$), there are two intersections, and as a result the time spectrum is divided into three regions: In the left-most region, S enters while W does not enter. In the middle region, both candidates enter. Finally, in the right-most region, S does not enter while W enters.

Notice that this particular model predicts that when the distribution of voters is ex ante very skewed (p is very small), S enters as soon as possible, so if T is large, then there would be almost no ambiguity in equilibrium. This hinges on our assumption that even if W enters after S, S does not incur any loss. In Section 5.4, we show that if there is a small loss, then S prefers to use ambiguous language until some point in time that does not depend on the horizon length T, and so the modified model is consistent with ambiguity even if the distribution of voters is ex ante very

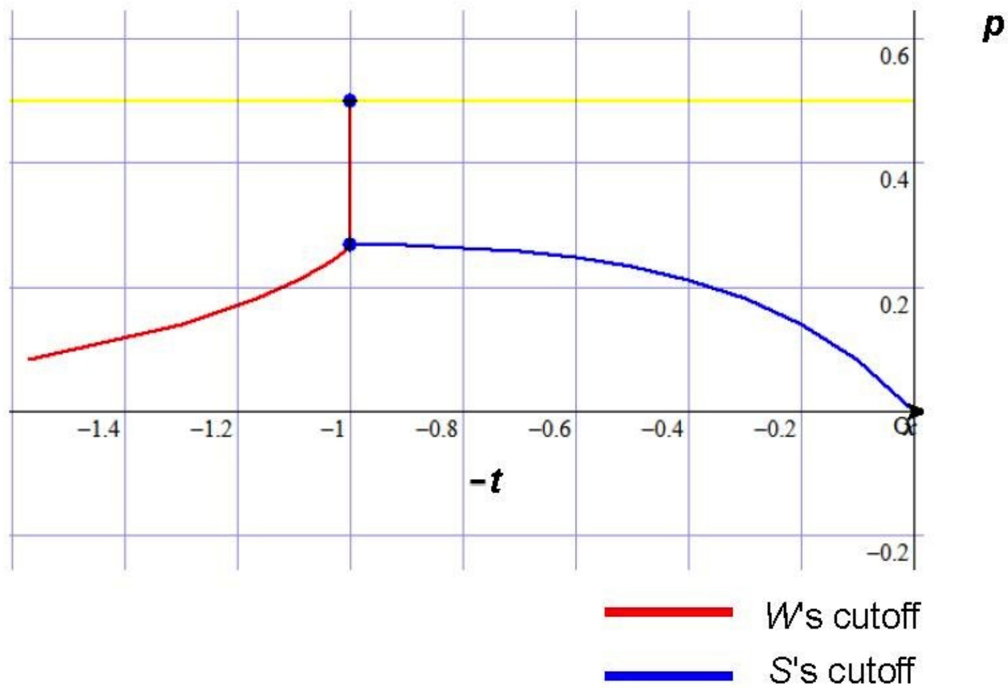

Figure 3: Cutoffs for the base model

skewed. Despite this feature, we believe that the simple base model provides a basic intuition about the dynamic incentives that candidates face. The basic take-away is that the nature of the election game with valence leads candidates to strategically "time" their announcements, since the benefit and cost of maintaining flexibility of choice varies over time. The benefit comes from the fact that the election game is constant-sum, so it is better to be a second-mover. On the other hand, the cost comes from the difference in valence. For example, the weak candidate does not want to end up in making the same choice (that is, taking $\{0, 1\}$) as the strong candidate. This is the general trade-off of timing strategies faced by electoral candidates, and our model succinctly captures such a trade-off.

5 Discussions

This section provides discussions of our model. In the first subsection we compare the outcome of our dynamic game with that of the one-shot game analyzed in Section 3, and discuss how the addition of the campaign phase changes the likelihood of eventual ambiguity and the welfare of candidates.

Next, in Section 5.2, we derive empirical implications of our model. Although we see these finding as only suggestive, these results are consistent with the empirical findings such as Campbell (1983).

The dynamic model we have analyzed so far was kept as simple as possible to highlight the complexity added by the fact that candidates face dynamic incentive problems. In the remaining three subsections (5.3-5.5), we extend and modify this model in various directions, to examine robustness of our prediction that candidates use ambiguous language at early stages of the campaign, and also to discuss the new implications that arise in the respective models.

5.1 The One-Shot Game versus the Dynamic Game

Let us now compare the ex ante probability distribution of the policy profile in our model with that of the one-shot simultaneous-move game with the same payoff structure. Given Proposition 3, the limit ex ante distribution of the policy profile at the election date and the expected payoffs as $T \to \infty$ are calculated as follows:

1. If $p > \frac{1}{1+e}$, W announces $\{1\}$ after $-t^* = -\frac{1}{\lambda}$ and S tries to copy W's policy after W enters. Hence, the following three cases are possible:

   (a) W cannot enter and $(\{0,1\}, \{0,1\})$ is realized. The probability of this event is $e^{-\lambda t^*} = e^{-1}$. The payoff profile in this case is $(1, 0)$.[24]

   (b) W enters and S cannot enter afterwards, and $(\{0,1\}, \{1\})$ is realized. The probability of such an event is $e^{-\lambda t^*} = e^{-1}$. The payoff profile in this case is $(p, 1-p)$.[25]

   (c) Both candidates enter and $(\{1\}, \{1\})$ is realized. The probability of this event is $1 - 2e^{-\lambda t^*} = 1 - 2e^{-1}$. The payoff profile in this case is $(1, 0)$.

[24]Throughout the paper, the first component of a payoff vector denotes S's payoff, and the second component denotes W's payoff.

[25]To see how the probability is calculated, note that nobody enters until $-t^*$ and that W is indifferent at $-t^*$. Hence, W's expected payoff from the entire game is the one given by W's entry at $-t^*$. This payoff is equal to $1-p$ times the probability with which S cannot enter after $-t^*$. Hence, W's equilibrium payoff is $(1-p) \cdot e^{-\lambda t^*}$. In equilibrium, W gets a positive payoff of $1-p$ only when he enters and S cannot enter afterwards. Hence, the equilibrium payoff can also be written as $1-p$ times the probability that W enters and S cannot enter afterwards. Therefore, the probability of the event is $\frac{(1-p)\cdot e^{-\lambda t^*}}{(1-p)} = e^{-\lambda t^*} = e^{-1}$.

Overall, the probability distribution over outcomes at the election date is

$$(\Pr(\{0,1\},\{0,1\}),\Pr(\{0,1\},\{1\}),\Pr(\{1\},\{1\})) = (e^{-1}, e^{-1}, 1-2e^{-1}).$$

2. If $p < \frac{1}{1+e}$, then the probability of S's entering first goes to 1 as $T \to \infty$. After S's entry, W is indifferent between entering and not entering. The expected payoff profile is $(1-p, p)$.

Thus, in the equilibrium under $p > \frac{1}{1+e}$, the probability of each candidate taking the ambiguous policy at the election date is at least e^{-1}. We note that this probability is higher than the probability assigned to $\{0,1\}$ in the one-shot case, which is $\frac{p(1-p)}{1-p+p^2}$ for each p. That is, we find that ambiguity is more likely in the dynamic game. The basic intuition for this result is that in the dynamic game with $p > \frac{1}{1+e}$, the only occasion on which S stops using ambiguous language is when W has already specified his policy, and W tries to minimize the probability that such an occasion will occur. This is why ambiguity is likely in the dynamic game than in the one-shot game.

Notice that the probability distribution over outcomes at the election date corresponds to a correlated strategy profile. This is because the sequential nature of the game serves as the correlation device. On the other hand, by definition, in the unique Nash equilibrium in the one-shot simultaneous-move game, the strategies are given by an independent mixture (cf. (1) of Section 3).

Now let us move on to the analysis of expected payoffs. In the one-shot game, the expected payoff profile is

$$\left(\frac{1-p}{1-p+p^2}, \frac{p^2}{1-p+p^2}\right).$$

In the dynamic game, by the above calculation, the expected payoff profile in the limit as $T \to \infty$ is

$$\begin{array}{ll} \left(1-(1-p)e^{-1}, (1-p)e^{-1}\right) & \text{if } p > \frac{1}{1+e}; \\ (1-p, p) & \text{if } p \leq \frac{1}{1+e}. \end{array}$$

Above we include the case of $p = \frac{1}{1+e}$ since the equilibrium payoff is unique although there is multiplicity of equilibrium (cf. footnote 23).[26] Figure 4 graphs S's expected payoffs for different values of p. Notice that the payoff in the one-shot game is decreasing in p, while the payoff in the dynamic game takes its minimum at $p = \frac{1}{1+e}$. The latter payoff exceeds the former when p is

[26]See also the proof of Proposition 3.

sufficiently large. This can be explained as follows: When p is small, S gives up on copying W's policy, and simply goes for the policy $\{1\}$, which guarantees her the payoff of $1-p$. In the one-shot game, however, the equilibrium is mixed, as we have seen in Section 3; S must thus be getting the payoff greater than $1-p$ which she can guarantee by taking policy $\{1\}$. This is why when $p < \frac{1}{1+e}$, S is worse off in the dynamic game. Thus, the fact that the moves are sequential helps W in the dynamic game.

However, when p becomes large, the value of committing to $\{1\}$ become small, and hence S tries to match W in the dynamic game (Case 1). In the dynamic game, the only case in which S loses is when she cannot enter after W's entry, and S's payoff when only W enters is p, which is increasing in p. Since the probabilities that S cannot enter after W are independent of p in this region (since W enters at $-t^* = -\frac{1}{\lambda}$, which is independent of p), S's payoff is increasing in p in this region. Eventually, her payoff exceeds that of the one-shot case. In other words, the relative cost for waiting vanishes as the difference between policies 0 and 1 becomes negligible, which is why the dynamic game is favorable to S when p is high.

The following proposition summarizes the findings so far:

Proposition 4 *The following are true in equilibria.*

1. *If $p > \frac{1}{1+e}$, the probability of ambiguity is greater in the dynamic game than in the one-shot game.*

2. *There exists $p^* \in (\frac{1}{1+e}, \frac{1}{2})$ such that S is strictly better off in the dynamic game than in the one-shot game if $p > p^*$, and she is strictly worse off if $p < p^*$.*[27]

5.2 Empirical Implications

In this section, we derive empirical implications of our base dynamic model. We see these results as only suggestive, but as will be seen, it is possible to enrich the model by incorporating various features (such as heterogenous arrival rates, general utilities from the outcomes, and so forth). Hence, if one wants to conduct empirical research, then it will be possible to extend the model to

[27] p^* is a solution to the equation $\frac{1-p}{1-p+p^2} = 1-(1-p)e^{-1}$ that is unique in the domain $(\frac{1}{1+e}, 1)$. Numerically, it is about 0.4069.

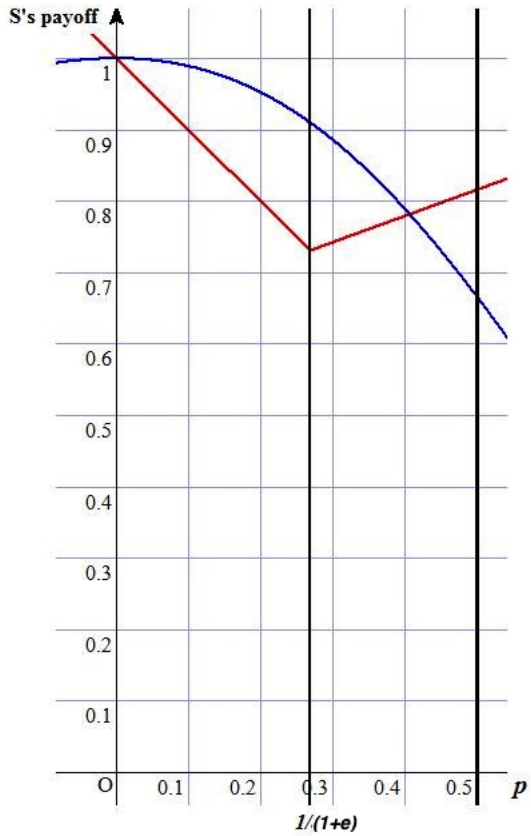

Figure 4: S's expected payoffs in the one-shot game and in the dynamic game (blue: S's expected payoff in the one-shot game; red: S's expected payoff in the dynamic game)

incorporate more characteristics and to derive the testable implication from such a general model, as we do here for the base model.

First, we show that ambiguity is likely when the probability distribution of the median voter's position is close to uniform, that is, when p is close to $\frac{1}{2}$. Specifically, fix a horizon length $T \in (\frac{1}{\lambda}, \infty)$. Let p^W be the p such that $t^W = T$.[28] By definition. $p^W < p^*$. Proposition 3 implies the following:

1. For $p \in (0, \frac{1}{2}) \setminus \{p^*\}$, the probabilities of W and S announcing the ambiguous policy are both nondecreasing in p.

2. For $p \in (0, p^W)$, the probability of W announcing the ambiguous policy is constant in p, and that of S announcing the ambiguous policy is strictly increasing in p.

[28]Such p^W exists and is unique due to Proposition 3 2(c) and $t^* = \frac{1}{\lambda}$.

3. For $p \in (p^W, p^*)$, the probabilities of W and S announcing the ambiguous policy are both strictly increasing in p.

4. For $p \in (p^*, \frac{1}{2})$, the probabilities of W and S announcing the ambiguous policy is constant in p.

Hence, roughly, as the position of the median voter becomes more unpredictable, the probability of ambiguous policy announcement at the election date increases. This is consistent with Campbell (1983) who suggests that opinion dispersion has a strong positive effect on the ambiguity in candidates' language.

Next, suppose that there are two candidates 1 and 2, and outside researchers know $p > \frac{1}{1+e}$ but do not know which candidate is strong and which candidate is weak. They have a prior that assigns positive probability to both candidate 1's being strong and candidates' 2's being strong. If the researchers can observe the campaign phase, the first entrant can be inferred to be weak (and if there is no entrance, then the posterior about valence is the same as the prior). If, on the other hand, they cannot observe the campaign phase but only the final policy choices by the candidates, then if only one candidate enters, then such a candidate can be inferred to be weak. Otherwise, the posterior about valence is the same as the prior.

5.3 Heterogeneous Arrival Rates

In this section we discuss the effect of heterogeneous arrival rates. Let the arrival rate for candidate i be $\lambda_i > 0$, and allow for the possibility that $\lambda_S \neq \lambda_W$. We define

$$r = \frac{\lambda_S}{\lambda_W}$$

as the relative frequency of the opportunities to enter between the candidates.

First, it is straightforward to show that the basic structure of the equilibrium does not change even if $\lambda_S \neq \lambda_W$: the equilibrium behaviors after some candidate has already entered are the same as before. When both candidates are announcing the ambiguous policy, there exist p^* and t^* such that if $p > p^*$, then W enters if $-t < -t^*$, he does not if $-t > -t^*$, and S never enters. If $p < p^*$, then W enters after some cutoff and S enters as soon as possible until another cutoff. The former cutoff for W to start entering precedes in time the latter for S to stop entering.

When $r \neq 1$, the cutoff p^* can be calculated as

$$p^* = \frac{r^{\frac{r}{1-r}}}{1 + r^{\frac{r}{1-r}}},$$

and the expected payoff profile for S and W when $p > p^*$ is

$$\left(1 - r^{\frac{r}{1-r}}, r^{\frac{r}{1-r}}\right).$$

Note that these values converge to the ones in the base model as $r \to 1$.

Since $r^{\frac{r}{1-r}}$ is decreasing in $r = \frac{\lambda_S}{\lambda_W}$, it follows that p^* is decreasing in r and S's payoff is increasing in r. Thus, having a relatively higher arrival rate makes the candidate better off. This is intuitive. With W's strategy being fixed, if S has a higher arrival rate, she has a greater chance to copy W's position. In contrast, with S's strategy being fixed, if W has a higher arrival rate, then he can wait longer at the policy profile $\{0, 1\}$ to reduce the probability of being copied afterwards. Of course W's strategy is not constant in the former case and S's is not in the latter, so determination of the equilibrium strategy profile is more complicated, but these are the main driving force of the comparative statics.

Note that Calcagno, Kamada, Lovo, and Sugaya (2014) show that having a higher arrival rate makes the player worse off since it decreases his/her commitment power. The difference is in the nature of the stage game being analyzed. Calcagno, Kamada, Lovo, and Sugaya (2014) analyze coordination games. Hence, player i's ability to commit to an action a_i can help induce his or her opponent to take a_j such that (a_i, a_j) constitutes a Nash equilibrium. On the other hand, in this paper, the game is a constant-sum game, so being unable to change an action over a longer time means that the player can react to the opponent less quickly and suffers a low payoff with a larger probability.

5.4 A Generalized Model

The simple model presented in Section 2 was intended to provide a basic intuition for the dynamic incentive problems faced by candidates. This section extends this base model to more general cases. In particular, we assume that the model specification is the same as in the base model, except that the arrival rates and the payoff functions are more general: S's arrival rate is $\lambda_S > 0$, W's arrival rate is $\lambda_W > 0$, and

$$(S\text{'s payoff}, W\text{'s payoff}) = \begin{cases} (\alpha, 1-\alpha) & \text{if only } W \text{ enters;} \\ (1-\beta, \beta) & \text{if only } S \text{ enters;} \\ (1-\gamma, \gamma) & \text{if } S \text{ enters and then } W \text{ enters;} \\ (1, 0) & \text{if } W \text{ enters and then } S \text{ enters, or if neither enters.} \end{cases}$$

We assume $\alpha \in [0, 1)$ and $\beta, \gamma \in [0, 1]$.[29]

Note that the crucial assumptions that we make here are (i) the payoff from the game is determined solely by the policy sets at the election, (ii) S wins for sure if S and W choose the same policy, and (iii) the position in the policy space that S enters does not depend on the timing of entry[30]; these are the only restrictions that we impose. These assumptions are satisfied in our base model, with $\lambda_S = \lambda_W = \lambda$ and $\alpha = \beta = \gamma = p$.

Moreover, the specification fits other cases as well. For example, this general model can be applied to the case of a continuous policy space, the model that the literature on elections often considers. Specifically, the set of possible policy announcements is $\{x\}_{x\in[0,1]} \cup [0, 1]$. That is, we allow the candidates to announce either a specific policy $x \in [0, 1]$ or an ambiguous policy $[0, 1]$. Analogous to the base model, the policy set at time $-T$ is $[0, 1]$. If candidate i wins the election and implements policy $x \in [0, 1]$, then the voter's utility with position $y \in [0, 1]$ is defined as $u(x, y) + \delta \cdot \mathbb{I}_{i=S}$, where the utility function u is strictly concave with respect to x (i.e., the voters are risk-averse). If a candidate with the ambiguous policy $[0, 1]$ wins, then the voter believes that the candidate will implement the policies in $[0, 1]$ according to the uniform distribution. Hence, the expected payoff is $\int_0^1 u(x, y)dx + \delta \cdot \mathbb{I}_{i=S}$.[31] The probability distribution of the median voter is uniform over the policy space $[0, 1]$. Again, we assume that the valence term is $\delta > 0$, but is sufficiently small so that W at $\frac{1}{2}$ beats S with the ambiguous policy.[32,33]

[29] We assume $\alpha \neq 1$ because otherwise W obtains the payoff of 0 in any equilibrium.

[30] (i) and (ii) imply that W's payoff is independent of the timing of his entry. This is because since W loses if S enters afterwords by (ii), when W chooses his policy to enter, he can condition on the event that S will not enter afterwards. Under such an event, by (i), W's payoff is determined solely by his policy announcement. Hence, the maximized payoff for W from his entry is independent of its timing.

[31] The integral is well-defined because u is concave and thus it is measurable.

[32] Specifically, $\int_0^1 u(x, y)dx + \delta < u(\frac{1}{2}, y)$ for all y. Note that such a $\delta > 0$ exists by the strict concavity of u.

[33] As we mentioned in the introduction, if we assume convexity, ambiguity does not need valence: if candidates are symmetric, it is optimal for a candidate to announce $[0, 1]$ when the opponent is announcing $\{\frac{1}{2}\}$.

In this model with the continuous policy space, if S enters before W does, she enters at policy $\frac{1}{2}$ regardless of the timing of her entry. This is because (i) this policy uniquely maximizes her payoff if W enters afterwards, and (ii) it guarantees the payoff of 1 if W does not enter. If W enters before S does, he enters at a policy around $\frac{1}{2}$ regardless of the timing of her entry. This is because (i) if S enters afterwards then S copies W's policy so W loses for sure, and (ii) if S does not enter afterwards, policies around $\frac{1}{2}$ guarantee the payoff of 1 since voters are risk-averse.

In this class of model, we obtain the following result. To state our result, we define three pieces of notation. First, write $Q_t = (E, N)$ if in all subgame perfect equilibria, (i) S enters if she receives an opportunity at t when W has not entered, and (ii) W does not enter if he receives an opportunity at t when S has not entered. That is, the first element Q_t denotes S's action at time $-t$ and the second element denotes W's action at the same time. The symbol E stands for "entering" and the symbol N stands for "not entering." Define $Q_t = (E, E)$, $Q_t = (N, E)$, and $Q_t = (N, N)$ analogously.

Second, we define functions

$$f_S(t) \quad : \quad = \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1-e^{-\lambda_W t}\right)\max\{\gamma-\beta, 0\}}{1-\alpha} & \text{if } r \neq 1; \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta + \left(1-e^{-\lambda_W t}\right)\max\{\gamma-\beta, 0\}}{1-\alpha} & \text{if } r = 1, \end{cases}$$

$$f_W(t) \quad : \quad = \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1; \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1, \end{cases}$$

where $r = \frac{\lambda_S}{\lambda_W}$ as before.[34]

Finally, let t_S be the smallest positive solution for $f_S(t) = 0$ (if there is no solution, then define $t_S = \infty$); and let t_W be the smallest positive solution for $f_W(t) = 0$ (since $f_W(t)$ is negative for sufficiently small $t > 0$, positive for sufficiently large t, and continuous, there always exists a positive solution).[35]

The equilibrium behavior is characterized as follows:

Proposition 5 *For any parameter profile $(r, \alpha, \beta, \gamma)$, S enters at the same position as W once W has entered but S has not. In addition, the following hold.*

[34] One can show that $f_S(t)$ and $f_W(t)$ are continuous in r at $r = 1$.

[35] The smallest positive solutions always exist because f_S and f_W are both continuous.

1. *If $\beta \geq \gamma$, then*

   (a) *If $-t_S < -t_W$, then $Q_t = (N, E)$ for all $-t \in (-t_W, 0]$; and $Q_t = (N, N)$ for all $-t \in (\infty, -t_W)$.*

   (b) *If $-t_S > -t_W$, then there exists $t^* \in (t_S, \infty)$ such that $Q_t = (N, E)$ for all $-t \in (-t_S, 0]$; $Q_t = (E, E)$ for all $-t \in (-t^*, -t_S)$; and $Q_t = (E, N)$ for all $-t \in (-\infty, -t^*)$.*

2. *If $\beta < \gamma$, then*

   (a) *If $-t_S < -t_W$, then $Q_t = (N, E)$ for all $-t \in (-t_W, 0]$; and $Q_t = (N, N)$ for all $-t \in (\infty, -t_W)$.*

   (b) *If $-t_S > -t_W$, there exists $t^{**} \in (t_S, \infty)$ such that $Q_t = (N, E)$ for all $-t \in (-t_S, 0]$; $Q_t = (E, E)$ for $(-t^{**}, -t_S)$. The equilibrium behavior for $-t < -t^{**}$ depends on the details of the parameters, but the following properties hold:*

      i. *There exists $t_W^{***} \in [t^{**}, \infty)$ such that W does not enter for all $-t \in (-\infty, -t_W^{***})$; and*

      ii. *There exists $\bar{r} \leq 1$ such that $r \geq \bar{r}$ if and only if there exists $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$.*

3. *All the time-cutoffs described above are independent of T.*

This means that, for a sufficiently long election campaign phase, W uses ambiguous language (and for many cases S uses such language as well) for a long time during the early stages of the election campaign, but the candidates' incentive to do so changes as the election date approaches. This basic pattern is common across a wide range of parameter specifications, although the exact way the incentives change varies across different specifications. Notice that in the base model, the parameters satisfy $\beta = \gamma$. In this case, if p is very small, then S enters as soon as possible. Thus, Proposition 5 claims that, if S expects even the slightest cost of W entering after her own entry (i.e., $\beta < \gamma$), then she will not wish to enter when the election date is far away.

Recall that the model includes the case of a continuous policy space with a concave payoff function. Thus, the proposition implies that the essence of our result is orthogonal to the convexity of payoff functions. This is in contrast to the models of Shepsle (1972) and Aragonès and Postlewaite (2002) in which the convexity of payoff functions is essential to the ambiguous policy announcement.

We now offer comparative statics of the cutoff timing with respect to the parameter values:

Proposition 6 *The following comparative statics hold:*

1. *For each (α, β, γ), there exists $r^* \in (0, \infty)$ such that $-t_S < -t_W$ if and only if $r^* < r$.*

2. *For each (r, β, γ), there exists $\alpha^* \in [0, 1)$ such that $-t_S < -t_W$ if and only if $\alpha^* < \alpha$.*

3. *For each (r, α, γ), there exists $\beta^* \in [0, 1)$ such that $-t_S < -t_W$ if and only if $\beta^* < \beta$.*

4. *For each (r, α, β), there exists $\gamma^* \in [0, 1]$ such that $-t_S < -t_W$ if and only if $\gamma^* < \gamma$.*

5. *For each (r, α, β), there exists $\bar{\gamma} \in [0, 1)$ such that, for each $\bar{\gamma} < \gamma$, there exists $-\bar{t}$ such that S does not enter at all $-t < -\bar{t}$.*

Part 1 of this proposition implies that, for sufficiently large r, Case 1(a) or 2(a) in Proposition 5 apply. Intuitively, since S can move quickly compared to W, W enters only if the deadline is very close ($-t_W$ is close to 0).

Parts 2 and 3 imply that for sufficiently large α or β, Case 1(a) or 2(a) in Proposition 5 apply. To see the intuition, notice that high α implies that S gets a high payoff when only W enters, and high β implies that S gets a low payoff when only S enters. Hence, in these situations, S has a small incentive to enter.

If $\beta \geq \gamma$, since W never enters after S enters, the value of γ does not affect the cutoff times. On the other hand, if $\beta < \gamma$, Part 4 implies that for sufficiently large γ, Case 1(a) or 2(a) in Proposition 5 apply. Intuitively, high γ implies that S gets a small payoff when W enters after S's entry. In such a situation, S has a small incentive to enter.

Part 5 implies that, if γ is sufficiently large, then S does not enter if the election is sufficiently far away. To see this, consider the extreme case with $\gamma = 1$. In this case, S's payoff is zero if S enters first and then W enters afterwards. Hence, if S enters when the election is far away, then with a high probability W will enter and S's payoff is close to zero. Therefore, in equilibrium S does not enter when the election is far away.

Remark 7 *The numbers t_S and t_W that appear in Proposition 5 are only implicitly defined as the smallest solutions of $f_S(t) = 0$ and $f_W(t) = 0$, respectively. There is a sufficient condition to ensure that $-t_S < -t_W$. The sufficient condition is that $\phi < 0$, where*[36]

$$\phi := \begin{cases} -\frac{\gamma}{1-\alpha} & \text{if } \gamma > \beta \text{ and } r < 1 - \frac{1-\alpha}{\gamma-\beta}; \\ e^{\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}-1} - \frac{\max\{\beta,\gamma\}}{1-\alpha} & \text{if } r = 1; \\ \left(\frac{1}{r} - \frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}\right)^{\frac{r}{r-1}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} & \text{otherwise.} \end{cases}$$

In fact, we use the condition in Remark 7 to show that r^* is finite and α^* and β^* are strictly less than 1 in Proposition 6. Moreover, Part 5 of Proposition 6 ensures the existence of $\bar{\gamma}$ such that $\gamma > \bar{\gamma}$ implies S does not enter if the deadline is far. In total, if at least one of these parameters is sufficiently high then there is a long period of no entry by any candidate.

Recall that in the base model, $r = 1$ and $\alpha = \beta = \gamma = p$. Proposition 3 implies that for sufficiently large p, there is a cutoff time $-t^*$ such that no candidate enters for all $-t < -t^*$. The specification of the base model implies that the three parameters α, β, and γ move simultaneously as p varies, so it is not possible in the base model to examine the effects of individual parameters. Proposition 6 ensures that if at least one of these parameters is sufficiently high, then no candidate enters when the deadline is far, as in the case of high p's. In addition, in Section 5.3, we define p^* to be a cutoff of p such that $p > p^*$ implies the existence of t^* with which (i) no candidate enters for all $-t < -t^*$ and (ii) W enters and S does not for all $-t > -t^*$. Part 1 of Proposition 6 generalizes the claim that p^* is decreasing in r (and it approximates 0 as $r \to \infty$). Overall, the insight from the base model carries over to the general setting.

5.5 Synchronous Policy Announcements

So far, we have assumed that candidates' policy announcements are asynchronous. In practice, not all the announcements are asynchronous; for example, televised political debates would be better modeled as synchronous policy announcements. To understand the role of the move structure on our ambiguity result, in this section we consider the case in which all the opportunities are synchronous. That is, time flows from $-T$ to 0 and, according to the Poisson process with arrival rate λ, both

[36] One can show that ϕ is continuous in r at $r = 1$.

of the candidates receive opportunities to announce their policy platforms simultaneously. We will show that the ambiguous policy announcements are robust to this setting. The very basic intuition— S wants to wait for W who does not want to be copied, which makes both candidates announce ambiguous policies when the election date is still far away— is the same as in the base model, but the detailed equilibrium structure is different. In particular, candidates use mixed strategies at any time point close to the election date.

We assume the same voter's utility and the same distribution of the median voter as in the original model explained in Section 2. For sufficiently small valence, the payoffs at the deadline 0 are given by the payoff matrix in Figure 2.

In this model, it is straightforward to see that parts 1 and 2 of Lemma 2 continue to hold. Therefore, the only relevant state is the state in which no player has entered so far. Assume for now that a Markov perfect equilibrium exists, and fix one of them.[37] Let V_t^i be the value of candidate i when no one has yet entered at $-t$ and an opportunity to enter arrives at $-t$ but actions have not been taken. Note that this value is independent of the other histories since we condier a Markov equilibrium. Suppose neither candidate enters at $-t$. Then, if they have an opportunity at $-\tau > -t$, they will then get (V_τ^S, V_τ^W). Otherwise, $\{0,1\},\{0,1\}$ will be realized at time 0 and they will get $(1,0)$. Hence, the value profile of choosing $\{0,1\},\{0,1\}$ at time $-t$ is[38]

$$\left(\int_0^t \lambda e^{-\lambda\tau} V_{t-\tau}^S d\tau + e^{-\lambda t}, \int_0^t \lambda e^{-\lambda\tau} V_{t-\tau}^W d\tau\right).$$

For other action profiles, parts 1 and 2 of Lemma 2 determine the value profile. As in the base model, the game has a constant sum since the winning probabilities must sum up to 1, so it suffices to keep track of S's payoffs. Specifically, when the candidates have an opportunity at $-t$, S's payoffs for the choices of policy platforms are given by the payoff matrix in Figure 5. and V_t^S

[37]We will show in Proposition 8 that a subgame perfect equilibrium exists, all subgame perfect equilibria are essentially Markov, and they have a unique continuation payoff at each time.

[38]The integration is well-defined because V_t^i is continuous in t for each $i \in \{S,W\}$ for the following reason: Let W_t^S be S's continuation payoff at time $-t$ when no opportunity arrives. Since expected payoffs are continuous in probability, W_t^S is continuous in t.

In Markov equilibria, the continuation play after taking $(\{0,1\},\{0,1\})$ at $-t$ and that after not receiving an opportunity are the same. Hence, we can replace the right-bottom entry of the payoff matrix with W_t^S in Figure 5. Since the minimax value of a constant-sum normal-form game is continuous in its payoff function, this means that the expected payoff from the Nash equilibrium of the game in Figure 5 is also continuous in t. Since by definition V_t^S is the expected payoff from the Nash equilibrium of the game, V_t^S is continuous in t. Since $V_t^W = 1 - V_t^S$, both integrations in these payoffs are well-defined.

$S \setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	1	p	p
$\{1\}$	$1-p$	1	$1-p$
$\{0,1\}$	$1-pe^{-\lambda t}$	$1-(1-p)\,e^{-\lambda t}$	$\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + e^{-\lambda t}$

Figure 5: The payoff matrix at time $-t$

is the unique minimax value of this constant-sum game.

Unfortunately, a complete characterization of the equilibria for all parameter values is hard to obtain. However, we can show that a Markov perfect equilibrium exists (and so does a subgame perfect equilibrium), and the Markov perfect equilibrium value V_t^S is unique. Moreover, all the subgame perfect equilibria are essentially Markov, meaning that for each subgame perfect equilibrium σ, there exists σ' such that the following two conditions are satisfied:

1. For each $i \in \{S, W\}$ and h_t, candidate i's continuation payoff at h_t given strategy profile σ coincides with the one given σ'.

2. For each h_t, if the minimax strategy profile is unique in the payoff matrix represented by Figure 5, then $(\sigma_S(h_t), \sigma_W(h_t)) = (\sigma'_S(h_t), \sigma'_W(h_t))$.

Moreover, we provide two analytical results on the basic dynamics of the equilibrium behaviors.

Proposition 8 *A Markov perfect equilibrium exists and the Markov perfect equilibrium value V_t^S is unique. Moreover, all the subgame perfect equilibria are essentially Markov. In addition, in each subgame perfect equilibrium, the following are true:*

1. *There exists $t^* > 0$ such that for all time $-t \in (-t^*, 0]$, both candidates use completely mixed strategies conditional on the event that the opponent has not entered.*

2. *There exists $t^{**} < \infty$ such that for all $-t < -t^{**}$, the probability with which a candidate enters at $\{0\}$ or $\{1\}$, conditional on the event that the opponent has not entered, is zero.*

Part 1 of the proposition states that if the election date is close, both candidates have to mix. This is in stark contrast to the asynchronous case, but is a natural consequence of the game representation above. The continuation payoff matrix approaches the original payoff matrix in the one-shot game whose unique equilibrium is completely mixed, and by the upper hemi-continuity of the set of Nash equilibria, the result holds.

Part 2 of the proposition shows the robustness of our ambiguity result with respect to the move structure. The intuition is the same as before. If W enters at $-t$ sufficiently far from the election date with positive probability, then it is optimal for S to wait and try to copy W's policy later. Given this, W does not enter. S gains a lot by copying W's policy, so she has an option value of waiting. Thus S does not enter either, when the election date is sufficiently far away.

As part 1 shows, the equilibrium involves mixing when the election date is close if opportunities arrive simultaneously. The mixing probabilities have to change over time, since the Nash equilibrium of the game matrix above changes as t changes. The transition of mixing probabilities is complicated and the incentive problems faced by the two candidates are subtle. We illustrate its complexity with an example with spacific p and λ in Section 7.7 in the appendix.

6 Conclusion

We proposed a model of a "policy announcement game" in which candidates stochastically obtain opportunities to announce their policies. We showed that, if two candidates are perfectly symmetric, they specify their policy positions as soon as possible. On the other hand, if one candidate is slightly stronger than the other, both candidates may have incentives to defer a clear announcement of their policies, depending on the opponent's latest announcement and the time left until the election.

We have introduced the first model of dynamic campaigns into the literature on elections by analyzing one particular simple setting, and have demonstrated that candidates face nontrivial dynamic incentive problems. Our work raises a wide range of new questions. Here we mention a few of them. First, we restricted ourselves to the case in which policies are either perfectly ambiguous or perfectly precise. One could allow for "intermediate language" (e.g., by letting the candidates choose any subintervals of $[0, 1]$ for the initial opportunity and then any subintervals included in their most recent announcements from the second opportunity on) and analyze how gradually candidates shift from ambiguous to clear policy language over the course of the campaign. Second, it would be more realistic to assume that policy announcements are sometimes synchronous and sometimes asynchronous. This problem seems nontrivial, as Ishii and Kamada (2011) show in their analysis of revision games with synchronous and asynchronous revisions. However, there should remain the incentive to announce ambiguous policy when the deadline is far. Third, we restricted ourselves to the case in which, once a candidate commits to a particular policy, he or she cannot overturn it later. Although we believe that this is a reasonable starting point for analysis, one could also assume that candidates can change their policies if they are willing to incur a "reputational cost" for announcing "inconsistent" policies. The idea is that if a candidate overturns his or her policy announcement, voters would infer that it is likely that the candidate would change it even after the election. Fourth, it would be interesting to enrich the model by assuming that the median voter's position gets gradually revealed over the course of the campaign (for example, because of polls), so that candidates have an additional reason to wait.

Finally, our work raises empirical questions as well. First, our model predicts different patterns for the timing of policy clarification for different parameter values such as p, which measures how much uncertainty candidates face with respect to the position of the median voters. One may want to test whether this prediction is supported by the data.[39] Second, in our analysis we have essentially assumed that λT is large so that candidates have sufficiently many chances to announce their policies and successfully communicate with the voters about such policies. It would be desirable to examine whether this assumption is correct in real election campaigns.

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7 Appendix

7.1 Backward Induction Argument for Proposition 1

Lemma 9 *Suppose that for any t, there exists $\epsilon > 0$ such that statement $A_{t'}$ is true for all $t' \in (t-\epsilon, t]$ if statement $A_{t''}$ is true for any $t'' > t$. Then, for any t, statement A_t is true.*

Proof. Suppose that the premise of the lemma holds. Let $-t^*$ be the supremum of $-t$ such that A_t is false. If $t^* = \infty$, we are done. So suppose that $t^* < \infty$. Then it must be the case that for any $\varepsilon > 0$, there exists $-\tau \in (-t^* - \varepsilon, -t^*]$ such that A_τ is false. But by the definition of t^*, there exists $\tilde{\varepsilon} > 0$ such that statement A_τ is true for all $-\tau \in (-t^* - \tilde{\varepsilon}, -t^*]$ because the premise of the lemma is true. This is a contradiction. ■

7.2 Proof of Proposition 3

Let $h_t = \left((t_S^k, x_S^k)_{k=0}^{k_S}, (t_W^k, x_W^k)_{k=0}^{k_W}, t \right)$ be the public history at $-t$, where $-t_j^k$ is the time at which candidate j receives his or her k'th revision opportunity; x_j^k is the policy set that j has chosen at time $-t_j^k$; and t denotes the current remaining time. For notational convenience, let $\theta(h_t) = (x_S^{k_S}, x_W^{k_W})$ be the profile of policy sets that are chosen most recently.

From Lemma 2, the following statements are true at h_t:

- If $\theta(h_t) = (\{x\}, \{0,1\})$ with $x \in \{0,1\}$ and if W can move, then W is indifferent between entering at $x' \in \{0,1\}$ with $x' \neq x$ and announcing $\{0,1\}$. S wins if and only if the median is located at x.
- If $\theta(h_t) = (\{0,1\}, \{x\})$ with $x \in \{0,1\}$ and if S can move, then S enters at x and wins.

When $-t$ is sufficiently close to the deadline 0, then at h_t with $\theta(h_t) = (\{0,1\}, \{0,1\})$, the following are true:

- If W can move, then W enters at 1. Note that, since $-t$ is sufficiently close to zero, with a high probability there is no more opportunity to announce a policy. Hence, $\{1\}$ gives W the payoff close to p, $\{0\}$ gives W the payoff close to $1-p$, and $\{0,1\}$ gives W the payoff close to zero. S wins if and only if the median voter is located at 0.
- If S can move, then S does not enter. Note that, since $-t$ is sufficiently close to zero, with a high probability there is no more opportunity to announce a policy. Hence, $\{1\}$ gives S the payoff close to p, $\{0\}$ gives S the payoff close to $1-p$, and $\{0,1\}$ gives S the payoff close to 1.

Let us define $V^S(h_t)$ as S's continuation payoff at $-t$ when h_t is the public history and no candidates receive an opportunity at $-t$. Note that it is always true that $V^W(h_t) = 1 - V^S(h_t)$.

From the above argument, for $-t$ sufficiently close to zero, for each h_t, we have

$$\begin{aligned} V^S(h_t) &= 1 - (1-p)\,\lambda t \exp(-\lambda t) \text{ if } \theta(h_t) = (\{0,1\}, \{0,1\})\,; \text{ and} \\ V^S(h_t) &= 1 - (1-p) \exp(-\lambda t) \text{ if } \theta(h_t) = (\{0,1\}, \{1\})\,. \end{aligned}$$

Subtracting these from 1, we obtain

$$
\begin{aligned}
V^W(h_t) &= (1-p)\,\lambda t \exp(-\lambda t) \text{ if } \theta(h_t) = (\{0,1\},\{0,1\})\,; \text{ and} \\
V^W(h_t) &= (1-p)\exp(-\lambda t) \text{ if } \theta(h_t) = (\{0,1\},\{1\})\,.
\end{aligned}
$$

Given the above value functions, at h_t with $\theta(h_t) = (\{0,1\},\{0,1\})$, the following conclusions hold:

- If W can move, then W enters at 1 as long as the following holds:

$$
(1-p)\,\lambda t \exp(-\lambda t) < (1-p)\exp(-\lambda t) \Leftrightarrow t < \frac{1}{\lambda}.
$$

- If S can move, then S does not enter as long as the following holds:

$$
\begin{aligned}
1-(1-p)\,\lambda t \exp(-\lambda t) &> \text{the value of entering at } \{1\} \\
&\Leftrightarrow 1-(1-p)\,\lambda t \exp(-\lambda t) > 1-p \\
&\Leftrightarrow 1 > \frac{p}{1-p} > \lambda t \exp(-\lambda t).
\end{aligned}
$$

To fully characterize the candidates' strategies, we examine the following three possible cases.

Case (1): $\frac{p}{1-p} > \exp(-1)$.

Let t^* be $\frac{1}{\lambda}$. Note that W becomes indifferent between entering at 1 and announcing $\{0,1\}$ at $-t^*$. We will now show that the following claims hold for each $-t < -t^*$:

- If S can move, then S has a strict incentive to announce $\{0,1\}$.
- If W can move, then W has a strict incentive to announce $\{0,1\}$.

First, by continuity, there exists $-\bar{t} < -t^*$ such that for all $-t \in (-\bar{t},-t^*)$ and all h_t with $\theta(h_t) = (\{0,1\},\{0,1\})$, if S can move at h_t, then S has a strict incentive to announce $\{0,1\}$.

Given S's strategy above, W can ensure that $\theta(h_{t^*}) = (\{0,1\},\{0,1\})$ by not entering for all $-t \in (-\bar{t},-t^*)$. On the other hand, W's payoff for entering at 1 monotonically decreases as $-t$

decreases. Since W is indifferent between entering at 1 and having $\theta(h_t) = (\{0,1\},\{0,1\})$ at $-t^*$, W has a strict incentive to announce $\{0,1\}$.

Similarly, we can show that if both S and W have a strict incentive to announce $\{0,1\}$ for all time in $(-t,-t^*)$, then there exists $\varepsilon > 0$ such that for each $-t' \in (-t-\varepsilon,-t]$, both S and W has the strict incentive to announce $\{0,1\}$. By Lemma 9, we have shown the claims.

Case (2): $\frac{p}{1-p} < \exp(-1)$**.**

In this case, there exists t_S such that $\frac{p}{1-p} = \lambda t_S \exp(-\lambda t_S)$ and $t_S < \frac{1}{\lambda}$. Moreover, by the implicit function theorem, we have

$$\begin{aligned} \frac{dt_S}{dp} &= -\frac{\frac{d\lambda t_S \exp(-\lambda t_S)}{dt_S}}{\frac{d\left(\frac{p}{1-p}\right)}{dp}} \\ &= -(1-p)^2 \lambda e^{-\lambda t_S} (1-\lambda t_S) < 0. \end{aligned} \tag{2}$$

At $-t_S$, S becomes indifferent between entering at 1 and announcing $\{0,1\}$. By continuity, at h_t with $-t < -t_S$ sufficiently close to $-t_S$ and $\theta(h_t) = (\{0,1\},\{0,1\})$, if W can move, then W strictly prefers entering at 1.

Let us now consider the candidates' incentive for entering when $-t < -t_S$. The payoff of S entering at $\{1\}$ and ensuring $1-p$ is always strictly greater than her payoff of announcing $\{0,1\}$. To see why, suppose that S announces $\{0,1\}$ at $-t$. To calculate S's payoff for announcing $\{0,1\}$, we have the following three subcases to consider.

(a) If W can move next by $-t_S$, then one strategy that W can take is to always announce $\{0,1\}$. The following two cases are possible: If S enters at $\{1\}$ by $-t_S$, W gets p. If S does not enter by $-t_S$, by the definition of $-t_S$ (that is, S is indifferent between $\{1\}$ and $\{0,1\}$ at $-t_S$), S gets $1-p$ and W gets p. In both cases, W gets at least p. Furthermore, if W can get a revision opportunity close to $-t_S$, W gets more than p since W strictly prefers entering at $\{1\}$ to announcing $\{0,1\}$. Overall, W gets more than p, which means S gets less than $1-p$.

(b) If S can move next by $-t_S$, S enters and gets $1-p$. Here, we assume the "inductive hypothesis" that S will enter for $-\tau \in (-t,-t_S)$.[40]

[40]Formally, we can use Lemma 9 for this step.

(c) If no candidate can move by $-t_S$, then by definition, S gets $1-p$.

Therefore, the value of announcing $\{0,1\}$ is strictly less than $1-p$.

We now consider W's incentive at $-t < -t_S$. Suppose that W enters at each $-\tau > -t$. Given this continuation strategy profile, if W enters, then W's payoff is equal to

$$(1-p)\, e^{-\lambda t}$$

since W receives $1-p$ if and only if S cannot enter afterwords. On the other hand, if W does not enter, W's payoff is equal to

$$\int_0^{t-t_S} e^{-2\lambda\tau}\lambda\,(1-p)\, e^{-\lambda(t-\tau)}d\tau + p\left(1-\int_0^{t-t_S}\lambda e^{-2\lambda\tau}d\tau\right).$$

Here we used the fact shown above that S enters for each $-(t-\tau) < -t_S$ when W has not entered. Note that $e^{-2\lambda\tau}$ is the probability that no candidates enters until $-(t-\tau)$, that $\lambda d\tau$ is the probability with which W can enter at $-(t-\tau)$, and that $(1-p)\, e^{-\lambda(t-\tau)}$ is W's payoff when W enters at $-(t-\tau)$. Hence, $\int_0^{t-t_S} e^{-2\lambda\tau}\lambda\,(1-p)\, e^{-\lambda(t-\tau)}d\tau$ is W's payoff when W enters first by $-t_S$.

At time $-t$, the probability that W does not enter first by $-t_S$ is $1-\int_0^{t-t_S}\lambda e^{-2\lambda\tau}d\tau$. There are two possible cases: (i) S enters by $-t_S$. In this case, S's payoff is $1-p$ and W's payoff is p. (ii) S does not enter by $-t_S$. Since S is indifferent between entering and not entering at $-t_S$, S's payoff is $1-p$ and W's payoff is p. In both cases, therefore, W's payoff is p. Hence, $p\left(1-\int_0^{t-t_S}\lambda e^{-2\lambda\tau}d\tau\right)$ is W's payoff when W does not enter first by $-t_S$.

Hence, W enters if and only if

$$(1-p)\, e^{-\lambda t} \geq \int_0^{t-t_S} e^{-2\lambda\tau}\lambda\,(1-p)\, e^{-\lambda(t-\tau)}d\tau + p\left(1-\int_0^{t-t_S}\lambda e^{-2\lambda\tau}d\tau\right)$$

$\Leftrightarrow$

$$(1-p)\, e^{-\lambda t} \geq (1-p)\left(e^{-\lambda t} - e^{-\lambda(2t-t_S)}\right) + p\left(1-\frac{1}{2}\left(1-e^{-2\lambda(t-t_S)}\right)\right)$$

$\Leftrightarrow$

$$e^{-\lambda(2t-t_S)} \geq \frac{p}{1-p}\frac{1}{2}\left(1+e^{-2\lambda(t-t_S)}\right).$$

Since $\frac{p}{1-p} = \lambda t_S e^{-\lambda t_S}$ by definition, this inequality is equivalent to

$$e^{-\lambda(2t-t_S)} \geq \lambda t_S e^{-\lambda t_S} \frac{1}{2} \left(1 + e^{-2\lambda(t-t_S)}\right)$$

$\Leftrightarrow$

$$e^{-2\lambda t} \geq \frac{\frac{1}{2}\lambda t_S}{1 - \frac{1}{2}\lambda t_S} e^{-2\lambda t_S}.$$

Taking the log of both sides and rearranging, we obtain

$$t \leq t_S - \frac{1}{2\lambda} \log \left(\frac{\frac{1}{2}\lambda t_S}{1 - \frac{1}{2}\lambda t_S} \right).$$

Hence, there exists t_W such that

$$t_W = t_S - \frac{1}{2\lambda} \log \left(\frac{\frac{1}{2}\lambda t_S}{1 - \frac{1}{2}\lambda t_S} \right)$$

and, at $-t_W$, W is indifferent between entering at $\{1\}$ and announcing $\{0,1\}$.

Note that, for $-t < -t_W$, W always prefers $\{0,1\}$. To see this, note that the payoff of W entering at 1 monotonically decreases if $-t$ becomes smaller; this is because (i) W gets the payoff of 0 if S has at least one revision opportunity after $-t$ and otherwise his payoff is $1-p>0$, and (ii) the probability of S having at least one opportunity monotonically increases as $-t$ becomes smaller. In addition, assuming that W does not enter until $-t_W$, W's payoff is the same. Hence, it is a strict best response for W not to enter.

Moreover, we have

$$\frac{dt_W}{dp} = \frac{dt_W}{dt_S}\frac{dt_S}{dp} = \left(1 - \frac{1}{\lambda t_S (2-\lambda t_S)}\right) \frac{dt_S}{dp}.$$

Recall that $\lambda t_S \in (0,1)$. Hence, we have

$$\sqrt{\lambda t_S (2-\lambda t_S)} < \frac{1}{2} (\lambda t_S + (2-\lambda t_S)) = 1,$$

and so

$$\frac{1}{\lambda t_S (2-\lambda t_S)} > 1.$$

Therefore, together with (2), we have

$$\text{sign}\frac{dt_W}{dp} = \text{sign}\left(1 - \frac{1}{\lambda t_S\,(2-\lambda t_S)}\right)\text{sign}\frac{dt_S}{dp} = 1. \tag{3}$$

The inequalities (2) and (3) prove part 2(c) of Proposition 3.

Case (3): $\frac{p}{1-p} = \exp(-1)$**.**

At time $-t^* = \frac{1}{\lambda}$ with $\theta(h_{t^*}) = (\{0,1\},\{0,1\})$, S is indifferent between "entering at $\{1\}$ and thereby ensuring $1-p$," and "announcing $\{0,1\}$." At the same time, W is indifferent between entering at $\{1\}$ and $\{0,1\}$.

For $-t < -t^*$, when W can move, his value of not entering is at least p since he gets p if S enters at $\{1\}$ by $-t^*$. If S does not enter by $-t^*$, by the definition of $-t^*$, S gets $1-p$ and W gets p. On the other hand, entering at $\{1\}$ gives W the payoff of $1-p$ times the probability of S not having any future revision opportunity, which is equal to $(1-p)\exp(-\lambda t) < (1-p)\exp(-\lambda t^*) = p$. Therefore, W strictly prefers not entering.

Given this, S is always indifferent between "entering at $\{1\}$ and thereby ensuring $1-p$," and "announcing $\{0,1\}$."

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7.3 Proof of Proposition 5

7.3.1 Uniqueness of the Value Function

We will first show uniqueness of the value functions. The uniqueness implies that, if we show that an action is a unique best response in one equilibrium, then such an action is the unique best response in all the equilibria.

As in the proof of Proposition 3, let

$$h_t = \left(\left(t_S^k, x_S^k \right)_{k=0}^{k_S}, \left(t_W^k, x_W^k \right)_{k=0}^{k_W}, t \right) \tag{4}$$

be the public history at $-t$. Let $\theta(h_t) = (x_S^{k_S}, x_W^{k_W}) \in \{E, N\} \times \{E, N\}$ be the action profile that is chosen most recently. Here, E means "entering" and N means "not entering." Let $\theta_i(h_t) = x_i^{k_i}$ be candidate i's element of $\theta(h_t)$. Define $V^S(\sigma, h_t)$ to be S's continuation payoff at h_t when σ is a strategy profile, h_t is the public history at $-t$, and no candidates receive an opportunity. As in the proof of Proposition 3, $V^W(h_t) = 1 - V^S(h_t)$. On the other hand, let $W^S(\sigma, h_t, i, \theta_i)$ be S's continuation payoff at $-t$ when h_t is the public history at $-t$ and candidate $i \in \{S, W\}$ has an opportunity and takes $\theta_i \in \{E, N\}$ at $-t$.

We first show that there exists a function $v_t^S(\cdot)$ such that for each equilibrium strategy σ and each history h_t, we have $V(\sigma, h_t) = v_t^S(\theta(h_t))$.[41] Moreover, for each σ, h_t, and $i \in \{S, W\}$, we also have $W^S(\sigma, h_t, i, \theta_i) = v_t^S(\theta_i, \theta_j(h_t))$.

If $\theta(h_t) = (N, E)$, then S enters as soon as possible. Hence, for each h_t with $\theta(h_t) = (N, E)$, we have

$$V^S(h_t) = \alpha + \left(1 - e^{-\lambda_S t}\right)(1 - \alpha).$$

Let $v_t^S(\theta) = \alpha + \left(1 - e^{-\lambda_S t}\right)(1 - \alpha)$ for $\theta = (N, E)$.

On the other hand, if $\theta(h_t) = (E, N)$, then W enters (or does not enter) if $\gamma > \beta$ (or $\gamma < \beta$). Hence, for each h_t with $\theta(h_t) = (E, N)$, we have $V^S(h_t) = 1 - \beta_t$ with

$$\beta_t = \beta + \left(1 - e^{-\lambda_W t}\right) \max\{\gamma - \beta, 0\}.$$

[41] The proof technique follows the one developed in Kamada and Muto (2014).

Let $v_t^S(\theta) = 1 - \beta_t$ for $\theta = (E, N)$.

Hence, we have $V(\sigma, h_t) = v_t^S(\theta(h_t))$ if $\theta(h_t) \neq (N, N)$. The above argument also implies that $W^S(\sigma, h_t, i, \theta_i) = v_t^S(\theta_i, \theta_j(h_t))$ with $j \neq i$ if $\theta(h_t) \neq (N, N)$. Therefore, we concentrate on the case with $\theta(h_t) = (N, N)$ or the case in which candidate i who has an opportunity at $-t$ takes N.

For each h_t with $\theta(h_t) = (N, N)$, let $W_t^S(\sigma, h_t, i)$ be S's continuation payoff when candidate $i \in \{S, W\}$ receives an opportunity and takes N at $-t$, and the candidates take the strategy profile σ such that $\sigma|_{h_t, z_i = yes, N}$ is subgame perfect in the subgame at which h_t is the public history and candidate i has an opportunity and takes N. Here, $\sigma|_{h_t, z_i = yes, N}$ is the continuation strategy in such a subgame. Moreover, let $\bar{W}_t^S$ be the supremum of S's continuation payoff: $\bar{W}_t^S \equiv \sup_{\sigma, h_t, i} W_t^S(\sigma, h_t, i)$, where the supremum is taken over all the possible histories and strategies such that h_t is the public history at $-t$, candidate i receives an opportunity and takes N, and $\sigma|_{h_t, z_i = yes, N}$ is subgame perfect in the subgame after $(h_t, z_i) = (yes, N)$ (henceforth, analogous restrictions on the domains over which supremums and infimums are taken apply whenever we take supremums and infimums). Similarly, let $\underline{W}_t^S$ be the infimum of S's continuation payoff: $\underline{W}_t^S \equiv \inf_{\sigma, h_t, i} W_t^S(\sigma, h_t, i)$. Let w_t^S be the difference between the supremum and infimum: $w_t^S \equiv \bar{W}_t^S - \underline{W}_t^S$. Note that $w_0^S = 0$.

We first show that w_t^S is continuous in t. To this end, let $V_t^S(\sigma, h_t)$ be S's payoff when there is no opportunity at $-t$ and the candidates take the strategy profile σ such that $\sigma|_{h_t, no}$ is subgame perfect in the subgame at which h_t is the public history and $z_S = z_W = no$ at $-t$. Here, $\sigma|_{h_t, no}$ is the continuation strategy in such a subgame. Note that $V_t^S(\sigma, h_t)$ is continuous in t given σ since expected payoffs are continuous in probability. Hence, $\bar{W}_t^{S,\text{no}} \equiv \sup_{\sigma, h_t} V_t^S(\sigma, h_t)$ and $\underline{W}_t^{S,\text{no}} \equiv \inf_{\sigma, h_t} V_t^S(\sigma, h_t)$ are continuous, where the supremum and infimum are taken over all the possible histories and strategies such that h_t is the public history at $-t$, $z_S = z_W = no$, and $\sigma|_{h_t, no}$ is subgame perfect in the subgame after $(h_t, z_S = z_W = no)$.

To show w_t^S is continuous in t, it suffices to show that $\bar{W}_t^S = \bar{W}_t^{S,\text{no}}$ and $\underline{W}_t^S = \underline{W}_t^{S,\text{no}}$. To see this, let $\sigma|_{h_t, z_i = yes, N}$ be the continuation equilibrium strategy in the subgame after $(h_t, z_i = yes, N)$. Let us define $\tilde{\sigma}$ as the strategy profile such that, in the subgame after $(h_t, z_S = z_W = no)$, the candidates follow the strategy profile $\sigma|_{h_t, z_i = yes, N}$. That is, they take actions as if candidate i had an opportunity and took N at $-t$. Since the strategic environment in the subgame after $(h_t, z_S = z_W = no)$ is the same as in the one after $(h_t, z_i = yes, N)$, this continuation strategy profile is subgame perfect in the subgame after $(h_t, z_S = z_W = no)$ if $\sigma|_{h_t, z_i = yes, N}$ is subgame perfect in the subgame after $(h_t, z_i = yes, N)$. Therefore, for each h_t and $W_t^S(\sigma, h_t, i)$ such that $\sigma|_{h_t, z_i = yes, N}$ is subgame perfect, $W_t^S(\sigma, h_t, i) = V_t^S(\tilde{\sigma}, h_t)$ holds and $\tilde{\sigma}|_{h_t, no}$ is subgame perfect. Therefore, we have $\bar{W}_t^S = \bar{W}_t^{S,\text{no}}$ and $\underline{W}_t^S = \underline{W}_t^{S,\text{no}}$. The equality $W_t^S(\sigma, h_t, i) = V_t^S(\tilde{\sigma}, h_t)$ also implies that $W_t^S(\sigma, h_t, i)$ does not depend on i. Hence, we will write $W_t^S(\sigma, h_t)$ to denote $W_t^S(\sigma, h_t, i)$ from now on.

We will now show that $w_t^S = 0$ for each $t \geq 0$. To this end, fix $-\tau \in (-t, 0]$ arbitrarily. In addition, we introduce the following notation: Let $z_{i,\tau'} \in \{yes, no\}$ represent whether candidate i receives an opportunity at $-\tau'$. For example, $z_{i,\tau'} = no$ for each $i \in \{S, W\}$ and $-\tau' \in (-t, -\tau)$ and $z_{S,\tau} = yes$ represents the event that S receives the first opportunity after $-t$ at $-\tau$. Given $z_{i,t} = yes$, since the Poisson process is asynchronous, $z_{j,\tau} = no$ with probability one for $j \neq i$. Moreover, let $W_t^S(\sigma, h_t \mid \tau, i)$ be candidate S's continuation payoff at $-t$ given σ and h_t, conditional on $z_{i',\tau'} = no$ for each $i' \in \{S, W\}$ and $-\tau' \in (-t, -\tau)$ and $z_{i,\tau} = yes$.

Suppose that S receives an opportunity at $-\tau$ for the first time after $-t$: $z_{i,\tau'} = no$ for each $i \in \{S, W\}$ and $-\tau' \in (-t, -\tau)$ and $z_{S,\tau} = yes$. Then, S's continuation payoff $W_t^S(\sigma, h_t \mid \tau, S)$ varies at most by w_τ^S. To see why, note that if $z_{S,\tau} = yes$, then candidate S's continuation payoff of taking E at $-\tau$ is equal to $v_\tau(E, \theta_W(h_\tau))$ irrespective of σ, and this is equal to $v_\tau(E, N)$ because $\theta_W(h_t) = N$ and we assume $z_{W,\tau'} = no$ for each $-\tau' \in (-t, -\tau)$ and $z_{W,\tau} = no$ with probability one. On the other hand, S's continuation payoff of taking N at $-\tau$ is $W_\tau^S(\sigma, h_\tau)$. Hence, S's continuation payoff after S takes an optimal action varies at most by

$$\sup_{\sigma, h_t, \hat{\sigma}, \hat{h}_t} \left| W_t^S(\sigma, h_t \mid \tau, S) - W_t^S(\hat{\sigma}, \hat{h}_t \mid \tau, S) \right| \leq w_\tau^S.$$

Symmetrically, suppose that W receives an opportunity at $-\tau$ for the first time after $-t$: $z_{i,\tau'} = no$ for each $i \in \{S, W\}$ and $-\tau' \in (-t, -\tau)$ and $z_{W,\tau} = yes$. Then, S's continuation payoff $W_t^S(\sigma, h_t \mid \tau, W)$ varies at most by w_τ^S. To see why, note that if $z_{W,\tau} = yes$, then candidate W's continuation payoff of taking E at $-\tau$ is equal to $1 - v_\tau(\theta_S(h_\tau), E)$ irrespective of σ, and this is equal to $v_\tau(N, E)$ because $\theta_S(h_t) = N$ and we assume $z_{S,\tau'} = no$ for each $-\tau' \in (-t, -\tau)$ and $z_{S,\tau} = no$ with probability one. On the other hand, W's continuation payoff of taking N at $-\tau$ is

$1 - W^S_\tau(\sigma, h_\tau)$. Hence, S's continuation payoff after W takes an optimal action varies at most by

$$\sup_{\sigma, h_t, \hat{\sigma}, \hat{h}_t} \left| 1 - W^S_t(\sigma, h_t \mid \tau, W) - \left(1 - W^S_t(\hat{\sigma}, \hat{h}_t \mid \tau, W)\right) \right| \leq w^S_\tau.$$

On the other hand, let $W^S_t(\sigma, h_t \mid no)$ be candidate S's continuation payoff at 0 given σ and h_t, conditional on the event that no opportunity comes after $-t$ (that is, $z_{i',\tau'} = no$ for each $i' \in \{S, W\}$ and $-\tau' \in (-t, 0]$). Since candidate S wins for sure, we have

$$\sup_{\sigma, h_t, \hat{\sigma}, \hat{h}_t} \left| W^S_t(\sigma, h_t \mid no) - W^S_t(\hat{\sigma}, \hat{h}_t \mid no) \right| = 0.$$

Since the probability that some candidate i receives the first opportunity after $-t$ at some $-\tau \in (-t, 0]$ (that is, $z_{i,\tau'} = no$ for each $i \in \{S, W\}$ and $-\tau' \in (-t, -\tau)$ and $z_{i,\tau} = yes$ for some $-\tau \in (-\tau, 0]$ and $i \in \{S, W\}$) is $1 - \exp(-(\lambda_S + \lambda_W)\, t)$, we have

$$\begin{aligned} w^S_t &\leq (1 - \exp(-(\lambda_S + \lambda_W)\, t)) \times \underbrace{\max_{\tau \leq t} w^S_\tau}_{\substack{\text{Supremum difference} \\ \text{in } W^S_t(\sigma, h_t \mid \tau, i)}} + \exp(-(\lambda_S + \lambda_W)\, t) \times 0 \\ &= (1 - \exp(-(\lambda_S + \lambda_W)\, t)) \max_{\tau \leq t} w^S_\tau. \end{aligned}$$

Fix $\bar{w} > 0$ arbitrarily, and let $\bar{t}$ be the minimum of t such that $w^S_t \geq \bar{w}$: $\bar{t} \equiv \min_{t \geq 0, w^S_t \geq \bar{w}} t$. The minimum exists since w^S_t is continuous with respect to t. In other words, we have

$$\max_{\tau \leq \bar{t}} w^S_\tau \leq w. \tag{5}$$

Since $w^S_0 = 0$ and w^S_t is continuous in t, we have $\bar{t} > 0$. Hence, there exists $\varepsilon > 0$ such that $\frac{1}{1 - \exp(-(\lambda_S + \lambda_W)\bar{t})} \geq 1 + \varepsilon$. Since $w^S_{\bar{t}} \leq (1 - \exp(-(\lambda_S + \lambda_W)\, \bar{t})) \max_{\tau \leq \bar{t}} w^S_\tau$, we have

$$\max_{\tau \leq \bar{t}} w^S_\tau \geq (1 + \varepsilon)\, \bar{w}, \tag{6}$$

which is a contradiction to (5).

Therefore, there exists $v^S_t(N, N)$ such that $W^S_t(\sigma, h_t, i) = V^S_t(\sigma, h_t) = v^S_t(N, N)$ for each equilibrium σ, history h_t with $\theta(h_t) = (N, N)$, and t.

In total, we have proven that there exists $v_t^S(\theta)$ such that for each $\theta \in \{E, N\} \times \{E, N\}$, each equilibrium strategy profile σ, each history h_t, and each $i \in \{S, W\}$, we have $V^S(\sigma, h_t) = v_t^S(\theta(h_t))$ and $W^S(\sigma, h_t, i, \theta_i) = v_t^S(\theta_i, \theta_j(h_t))$.

We will now prove the statements in Proposition 5. Note that S enters and receives a payoff of 1 if S can move after W enters. In addition, by the same proof as the one for Proposition 3, if $-t$ is close to zero, then $Q_t = (N, E)$. Below, we consider the transition of Q_t in the following two cases: $\beta \geq \gamma$ and $\beta < \gamma$.

7.3.2 Case 1: $\beta \geq \gamma$

In this case, for all $-t$, W does not enter after S enters. (If $\beta = \gamma$, then W is indifferent. The following analysis goes through when $\beta = \gamma$ regardless of W's strategy after S enters.)

First, let us consider S's incentive. At time $-t$, if W has not entered, S's payoff is $1 - \beta$ if S enters; if S does not enter, then her payoff is $1 - (1 - \alpha) \int_0^t e^{-\lambda_S \tau} \lambda_W e^{-\lambda_W (t-\tau)} d\tau$, given $Q_\tau = (N, E)$ for all $-\tau \in (-t, 0)$. Hence, S enters if and only if

$$1 - \beta \leq 1 - (1 - \alpha) \int_0^t e^{-\lambda_S \tau} \lambda_W e^{-\lambda_W (t-\tau)} d\tau.$$

This is equivalent to $f_S(t) \leq 0$ where

$$f_S(t) = \begin{cases} \frac{1}{1-r} \left(e^{-\lambda_S t} - e^{-\lambda_W t} \right) - \frac{\beta}{1-\alpha} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta}{1-\alpha} & \text{if } r = 1 \end{cases}. \tag{7}$$

Recall that we define t_S as the smallest positive solution for $f_S(t) = 0$ in Section 5.4. If there is no solution, then we define $t_S = \infty$. Note that the function f_S is continuous, so the smallest positive solution always exists or there is no solution.

Second, let us consider W's incentive. At time $-t$, if S has not entered, W's payoff is $(1 - \alpha) e^{-\lambda_S t}$ if W enters; if W does not enter, then his payoff is $(1 - \alpha) \int_0^t e^{-\lambda_S \tau} \lambda e^{-\lambda_W (t-\tau)} d\tau$, given $Q_\tau = (N, E)$ for $-\tau \in (-t, 0)$. Hence, W enters if and only if

$$(1 - \alpha) e^{-\lambda_S t} \geq (1 - \alpha) \int_0^t e^{-\lambda_S \tau} \lambda_W e^{-\lambda_W (t-\tau)} d\tau$$

$\Leftrightarrow$

$$f_W(t) \leq 0, \text{ where } f_W(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1 \end{cases}.$$

Recall that we define t_W as the smallest solution for $f_W(t) = 0$ in Section 5.4:

$$\begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t_W} - e^{-\lambda_W t_W}\right) - e^{-\lambda_S t_W} = 0 & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t_W} - e^{-\lambda_W t_W} = 0 & \text{if } r = 1. \end{cases} \tag{8}$$

Since $f_W(t)$ is continuous, negative for sufficiently small $t > 0$, and positive for sufficiently large t, there exists the smallest positive t such that $f_W(t) = 0$.

The transition of Q_t depends on the relationship between t_S and t_W. Recall that (i) at $-t < -t_W$, W has a strict incentive not to enter assuming that S will not enter after $-t$, and that (ii) at $-t_W < -t$, he has a strict incentive to enter under the same assumption. Also, at $-t < -t_S$, S has a strict incentive to enter assuming that W will enter after $-t$, and (ii) at $-t_S < -t$, she has a strict incentive not to enter under the same assumption.

Case 1(a): $-t_S < -t_W$**.**

This inequality means that W's cutoff $-t_W$ is closer to the deadline than S's cutoff (if any) $-t_S$. Between $(-\infty, -t_W)$, S never enters since her payoff for not entering is constant over this time interval and that for entering is constant too as W never enters after S's entry. Hence, in total,

- $Q_t = (N, E)$ for $-t \in (-t_W, 0]$.
- $Q_t = (N, N)$ for $-t \in (-\infty, -t_W)$.

Hence, part 1(a) of Proposition 5 holds.

Case (1)(b): $-t_S > -t_W$**.**

This inequality means that S's cutoff $-t_S$ is closer to the deadline than W's cutoff $-t_W$. At $-t < -t_S$, W enters if and only if

$$(1-\alpha)\, e^{-\lambda_S t} \geq \int_0^{t-t_S} e^{-(\lambda_S+\lambda_W)\tau} \left(\lambda_S \beta + \lambda_W (1-\alpha)\, e^{-\lambda_S (t-\tau)}\right) d\tau + e^{-(\lambda_S+\lambda_W)(t-t_S)} \beta$$

$\Longleftrightarrow$

$$g_W(t) \equiv e^{-(\lambda_S+\lambda_W)(t-t_S)} \left(\frac{1}{1+r}\beta - (1-\alpha) e^{-\lambda_S t_S} \right) + \frac{r}{1+r}\beta \leq 0.$$

Here, we use the fact that, since S is indifferent between entering and not entering at $-t_S$, her payoff at $-t_S$ is $1-\beta$ if no candidates enter by $-t_S$. Hence, W's payoff is β if no candidates enter by $-t_S$.

At $t = t_S$, we have

$$\begin{aligned} g_W(t_S) &= \beta - (1-\alpha) e^{-\lambda_S t_S} \\ &= (1-\alpha) \left(\frac{\beta}{1-\alpha} - e^{-\lambda_S t_S} \right). \end{aligned}$$

Since t_S is the solution for $f_S(t) = 0$, we have

$$\frac{\beta}{1-\alpha} = \begin{cases} \frac{1}{1-r} \left(e^{-\lambda_S t} - e^{-\lambda_W t} \right) & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t} & \text{if } r = 1. \end{cases}$$

Hence, we have

$$\begin{aligned} g_W(t_S) &= \begin{cases} (1-\alpha) \left(\frac{1}{1-r} \left(e^{-\lambda_S t_S} - e^{-\lambda_W t_S} \right) - e^{-\lambda_S t_S} \right) & \text{if } r \neq 1 \\ (1-\alpha) \left(\lambda_W t e^{-\lambda_W t_S} - e^{-\lambda_S t_S} \right) & \text{if } r = 1 \end{cases} \\ &= (1-\alpha) f_W(t_S). \end{aligned}$$

Since we assume $t_S < t_W$, we have $f_W(t_S) < 0$. Hence, we have

$$g_W(t_S) = \beta - (1-\alpha) e^{-\lambda_S t_S} = (1-\alpha) f_W(t_S) < 0.$$

This implies the following two claims. First, since

$$\frac{1}{1+r}\beta - (1-\alpha) e^{-\lambda_S t_S} = \beta - (1-\alpha) e^{-\lambda_S t_S} - \frac{r}{1+r}\beta < 0,$$

$g_W(t)$ is increasing in t. Second, W strictly prefers to enter at $-t_S$: $g_W(t_S) < 0$. Moreover, $g_W(\infty) = \frac{r}{1+r}\beta > 0$. Therefore, there exists a unique solution for $g_W(t) = 0$ such that $t \in (t_S, \infty)$. Let t^* be the unique solution.

On the other hand, S always prefers entering at $-t < -t_S$ as long as W prefers entering at the same time $-t$ for the following reason. Since S (weakly) prefers entering at $-t_S$, if W has not entered by $-t_S$, S's payoff at $-t_S$ is no more than $1-\beta$. (Even if S enters by $-t_S$, S gets at most $1-\beta$.) That is, W can guarantee β if W does not enter until $-t_S$. The fact that W prefers entering implies that W's payoff when W can enter before S is no less than β. Therefore, S's payoff when W can enter before S is no more than $1-\beta$. On the other hand, by entering, S can guarantee a payoff of $1-\beta$. Hence, entering is S's strict betst response at $-t$.

Given the above characterization of t^*, the transition of Q_t is as follows:

- $Q_t = (N, E)$ for $-t \in (-t_S, 0]$.
- $Q_t = (E, E)$ for $-t \in (-t^*, -t_S)$.
- $Q_t = (E, N)$ for $-t \in (-\infty, -t^*)$. In this region, no candidate changes their actions. To see this, first observe that S's incentive at $-t \in (-\infty, -t^*)$ is the same as $-t = -t^*$ since (i) W does not enter before S enters for $-t \in (-\infty, -t^*)$ and (ii) W does not enter after S enters.

  Second, we show that W's incentive does not change. Consider the following two possibilities of the realization of S's opportunities: (i) If S can enter by $-t^*$, then entering at $-t$ gives W the payoff of 0 while not entering until $-t^*$ gives him at least the payoff of β. (ii) If S cannot enter by $-t^*$, then W is indifferent between entering and not entering at $-t^*$. Hence, conditional on the event that S cannot enter by $-t^*$, not entering is one of W's optimal actions at $-t$. Since S can enter by $-t^*$ with positive probability, not entering is W's strict best response.

Hence, part 1(b) of Proposition 5 holds.

7.3.3 Case 2: $\gamma > \beta$

In this case, for all $-t$, W enters after S enters. The value profile when only S enters at $-t$ is given by $(1-\beta_t, \beta_t)$ with

$$\beta_t = \beta + \left(1 - e^{-\lambda_W t}\right)(\gamma - \beta). \tag{9}$$

When we replace β with β_t in (7), the analysis for the case with $\beta \geq \gamma$ implies the following: Given $Q_\tau = (N, E)$ for $-\tau \in (-t, 0)$, at time $-t$, S does not enter if and only if $f_S(t) \leq 0$, where

$$f_S(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta_t}{1-\alpha} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta_t}{1-\alpha} & \text{if } r = 1 \end{cases}. \tag{10}$$

Recall that we define t_S as the smaller positive solution for $f_S(t) = 0$ in Section 5.4. If there is no solution, then we define $t_S = \infty$.

On the other hand, given $Q_\tau = (N, E)$ for $-\tau \in (-t, 0)$, at time $-t$, W enters if and only if $f_W(t) \leq 0$, where

$$f_W(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1 \end{cases}.$$

Recall that we define t_W as the smallest solution for $f_W(t) = 0$ in Section 5.4:

$$\begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t_W} - e^{-\lambda_W t_W}\right) - e^{-\lambda_S t_W} = 0 & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t_W} - e^{-\lambda_W t_W} = 0 & \text{if } r = 1. \end{cases}$$

Since $f_W(t)$ is continuous, negative for sufficiently small t, and positive for sufficiently large t, there exists the smallest t such that $f_W(t) = 0$.

The transition of Q_t depends on the relationship between t_S and t_W.

Case 2(a): $-t_S < -t_W$

This inequality means that W's cutoff $-t_W$ is closer to the deadline than S's cutoff (if any) $-t_S$. In this case, S does not enter for any $-t$ since (i) S's payoff for not entering does not change between $(-\infty, -t_W)$ and (ii) S's payoff for entering β_t decreases as $-t$ decreases from (9) (recall $\gamma > \beta$). Hence,

- $Q_t = (N, E)$ for $-t \in (-t_W, 0]$.
- $Q_t = (N, N)$ for $-t \in (-\infty, -t_W)$.

Hence, part 2(a) of Proposition 5 holds.

Case 2(b): $-t_S > -t_W$

This inequality means that S's cutoff $-t_S$ is closer to the deadline than W's cutoff $-t_W$. In this case, there exists $\varepsilon > 0$ such that $Q_t = (E, E)$ for all $-t \in (-t_S - \varepsilon, -t_S)$. Hence, as part 2(b) of Proposition 5 states, there exists t^{**} with $0 < t_S < t^{**}$ such that (i) $Q_t = (N, E)$ for all $-t \in (-t_S, 0]$, and (ii) $Q_t = (E, E)$ for all $-t \in (-t^{**}, -t_S)$.

The equilibrium behavior for $-t < -t^{**}$ depends on the details of the parameters, but we can derive the differential equation that characterizes the transition. Let x_t be W's continuation payoff at time $-t$ when W has entered and S has not entered at $-t$; let y_t be W's continuation payoff at time $-t$ when W has not entered and S has entered at $-t$; and let z_t be W's continuation payoff at time $-t$ when no candidate has entered at $-t$.

Suppose x_t, y_t, and z_t satisfy the following differential equations:

$$\frac{dx_t}{dt} = \lambda_S (0 - x_t), \tag{11}$$
$$\frac{dy_t}{dt} = \lambda_W \max\{\gamma - y_t, 0\}, \tag{12}$$
$$\frac{dz_t}{dt} = \lambda_W \max\{x_t - z_t, 0\} + \lambda_S \min\{y_t - z_t, 0\}, \tag{13}$$

with this initial condition:

$$x_0 = 1 - \alpha, y_0 = \beta, z_0 = 0.$$

Since this system of ordinary differential equations satisfies Lipschitz continuity, there exists a solution. Such a solution is an equilibrium payoff for the following reasons : Equation (11) means that whenever S can enter after W enters, W's payoff is 0. Equation (12) means that when W can enter after S enters, W enters if and only if his payoff for entering γ is bigger than the payoff for not entering y_t. In addition, the first term of (13) means that when W can enter, W enters if and only if his payoff for entering x_t is bigger than the payoff for not entering z_t. The second term of (13) means that when S can enter, S enters if and only if her payoff for entering $1 - y_t$ is bigger than her payoff for not entering $1 - z_t$ (that is, y_t is smaller than z_t). Since we have shown the uniqueness of the value function, the solution for the system of (11), (12), and (13) is the unique equilibrium payoff.

Consider the transition of z_t. At $t = 0$, $z_0 < \min\{x_0, y_0\}$. Hence, by continuity, there exist

$t_+ > 0$ and $\varepsilon > 0$ such that $z_{t_+} \geq \varepsilon$. Note that z_t is nondecreasing if $z_t \leq y_t$. Hence, $z_t \geq \min\{\varepsilon, y_t\} \geq \min\{\varepsilon, \beta\}$ since y_t is nondecreasing in t and $y_0 = \beta$. On the other hand, x_t converges to zero as $t \to \infty$. Hence, for sufficiently small $-t$, z_t (W's continuation payoff for not entering) is larger than x_t (W's continuation payoff for entering). Therefore, there exists $t_W^{***} \in [t^{**}, \infty)$ such that W does not enter for all $-t \in (-\infty, -t_W^{***})$. Hence, part 2(b)i of Proposition 5 holds.

To show that there exists $\bar{r} \leq 1$ such that $r \geq \bar{r}$ if and only if there exists $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$, we prove the following three claims:

1. [$\bar{r} \leq 1$] For $r \geq 1$, there exists $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$.
2. [cutoff from below] If there does not exist $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$ for (λ_S, λ_W), then such t_S^{***} does not exist for (λ'_S, λ_W) with $\lambda'_S < \lambda_S$.
3. [cutoff from above] If there exists $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$ for (λ_S, λ_W), then such t_S^{***} exists for (λ'_S, λ_W) with $\lambda'_S > \lambda_S$.

[Proof of "$\bar{r} \leq 1$"] To analyze the conditions under which there exists $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$, let us consider a sufficiently large t. Since x_t is sufficiently small compared to z_t for sufficiently large t, the first term of (13) is zero. Hence, as long as $y_t \leq z_t$, we have

$$\frac{dz_t}{dt} = \lambda_S (y_t - z_t).$$

Since

$$\frac{dz_t}{dt} + \lambda_S z_t = \lambda_S y_t,$$

we have

$$e^{\lambda_S t} z_t = C + \int_a^t e^{\lambda_S \tau} \lambda_s y_\tau d\tau, \tag{14}$$

where a is the supremum of τ with $x_\tau \geq z_\tau$, and C is determined by the condition $x_a = z_a$. As we have shown above, a is finite, and so is C.

To show that we have $z_t < y_t$ for sufficiently large t for each (λ_S, λ_W) with $\lambda_S \geq \lambda_W$, we consider the following two cases: $r > 1$ and $r = 1$. Suppose first that $r > 1$. The second term of

(14) can be explicitly written as follows:

$$\begin{aligned}\int_a^t e^{\lambda_S \tau}\lambda_s y_\tau d\tau &= \int e^{\lambda_S \tau}\lambda_s\left(\beta+\left(1-e^{-\lambda_W \tau}\right)(\gamma-\beta)\right)d\tau \\ &= \frac{r}{1-r}(\gamma-\beta)e^{(\lambda_S-\lambda_W)t}+\gamma e^{\lambda_S t} \\ &\quad -\frac{r}{1-r}(\gamma-\beta)e^{(\lambda_S-\lambda_W)a}-\gamma e^{\lambda_S a}.\end{aligned}$$

Hence, the payoff z_t is characterized as follows:

$$z_t = C_a e^{-\lambda_S t}+\gamma+\frac{r}{1-r}(\gamma-\beta)e^{-\lambda_W t}, \tag{15}$$

with

$$C_a = C-\frac{r}{1-r}(\gamma-\beta)e^{(\lambda_S-\lambda_W)a}-\gamma e^{\lambda_S a}.$$

On the other hand, the payoff y_t is characterized as follows:

$$y_t = \beta+\left(1-e^{-\lambda_W t}\right)(\gamma-\beta).$$

Therefore, the difference between z_t and y_t (as long as $y_t \leq z_t$) is:

$$z_t - y_t = C_a e^{-\lambda_S t}+\frac{1}{1-r}(\gamma-\beta)e^{-\lambda_W t}.$$

As a result, whether there exists $t_S^{***} \in [t^{**},\infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$ or not depends on

$$\lim_{t\to\infty}\left(C_a e^{-\lambda_S t}+\frac{1}{1-r}(\gamma-\beta)e^{-\lambda_W t}\right). \tag{16}$$

If $r > 1$, for sufficiently large t, the second term of (16) dominates. Since $r > 1$ for this case, (16) is negative for sufficiently large t. That is, there exists $-t_S^{***}$ such that S does not enter for $-t \in (-\infty, -t_S^{***})$, as stated in part 2(b)ii of Proposition 5.

We now consider the case with $r = 1$. In this case, we can write $\lambda_S = \lambda_W = \lambda$. The second term of (14) can be explicitly written as follows:

$$\int_a^t e^{\lambda_S \tau}\lambda_s y_\tau d\tau = -\lambda(t-a)(\gamma-\beta)+\gamma\left(e^{\lambda t}-e^{\lambda a}\right).$$

Hence, the payoff z_t is characterized as

$$z_t = \gamma + e^{-\lambda t}\left(C - \lambda\left(t-a\right)\left(\gamma-\beta\right) - \gamma e^{-\lambda a}\right).$$

On the other hand, again, the payoff y_t is characterized as

$$y_t = \gamma + e^{-\lambda t}\left(\beta - \gamma\right).$$

Therefore, the difference between z_t and y_t (as long as $y_t \leq z_t$) is:

$$z_t - y_t = e^{-\lambda t}\left(C - \lambda\left(t-a\right)\left(\gamma-\beta\right) - \gamma e^{-\lambda a} - \left(\beta-\gamma\right)\right).$$

For sufficiently large t, the term $-\lambda\left(t-a\right)\left(\gamma-\beta\right)$ dominates the other terms in the parentheses, and so $z_t - y_t < 0$. That is, there exists $-t_S^{***}$ such that S does not enter for $-t \in (-\infty, -t_S^{***})$, as stated in part 2(b)ii of Proposition 5.

[Proof of "cutoff from below"] We show that, if there does not exist $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$ for (λ_S, λ_W), then such t_S^{***} does not exist for (λ_S', λ_W) with $\lambda_S' < \lambda_S$.

To show this monotonicity, we first arbitrarily fix λ_W. Note that x_t and y_t are independent of λ_S. Let $z_t(\lambda_S)$ be the value of z_t, given λ_S for the fixed λ_W. For sufficiently small t, $z_t(\lambda_S) < z_t(\lambda_S')$. Define $t^* \equiv \inf_t \{z_t(\lambda_S) \geq z_t(\lambda_S')\} \in \mathbb{R}_+ \cup \{+\infty\}$.

If $t^* = +\infty$, then we have $z_t(\lambda_S) \leq z_t(\lambda_S')$ for all $\lambda_S' < \lambda_S$. Since y_t is independent of λ_S, we are done. Hence, we concentrate on the case with $t^* < \infty$.

At $-t^*$, it must be the case that $z_{t^*}(\lambda_S) = z_{t^*}(\lambda_S')$ and $\dot{z}_{t^*}(\lambda_S) \geq \dot{z}_{t^*}(\lambda_S')$.[42] From $z_{t^*}(\lambda_S) =$

[42]The first equality follows from the continuity of z_t with respect to t. The second inequality follows from the first equality and the definition of the derivative: for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\dot{z}_{t^*}(\lambda_S) &\approx \frac{z_{t^*}(\lambda_S) - z_{t^*-\varepsilon}(\lambda_S)}{\varepsilon}, \\
\dot{z}_{t^*}(\lambda_S') &\approx \frac{z_{t^*}(\lambda_S') - z_{t^*-\varepsilon}(\lambda_S')}{\varepsilon} \\
&= \frac{z_{t^*}(\lambda_S) - z_{t^*-\varepsilon}(\lambda_S')}{\varepsilon}.
\end{aligned}$$

Since $z_{t^*-\varepsilon}(\lambda_S') > z_{t^*-\varepsilon}(\lambda_S)$, it follows that $\dot{z}_{t^*}(\lambda_S) \geq \dot{z}_{t^*}(\lambda_S')$.

$z_{t^*}(\lambda'_S)$, we have

$$\begin{aligned}\dot{z}_{t^*}(\lambda_S) &= \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} + \lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} \\ \dot{z}_{t^*}(\lambda'_S) &= \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda'_S), 0\right\} + \lambda'_S \min\left\{y_{t^*} - z_{t^*}(\lambda'_S), 0\right\} \\ &= \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} + \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.\end{aligned}$$

Note that, by definition, we have $x_{t^*}(\lambda'_S) > x_{t^*}(\lambda_S)$. Given this inequality, the following two cases are possible:

1. If $x_{t^*}(\lambda_S) \geq z_{t^*}(\lambda_S)$, then we have $\lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} < \lambda_W \max\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\}$. In addition, we have $\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} \leq \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}$. Hence, we have $\dot{z}_{t^*}(\lambda'_S) > \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

2. If $x_{t^*}(\lambda_S) < z_{t^*}(\lambda_S)$, then we consider the following subcases:

   (a) If $y_{t^*} > z_{t^*}(\lambda_S)$, then we have

   $$\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} < \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.$$

   Since we have

   $$\lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} \leq \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\},$$

   we have $\dot{z}_{t^*}(\lambda'_S) > \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

   (b) If $y_{t^*} \leq z_{t^*}(\lambda_S)$, then we have

   $$\lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} = 0$$

   and

   $$\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} = 0.$$

   Therefore, $\dot{z}_{t^*}(\lambda_S) = 0$. For $t > t^*$, since $x_t(\lambda_S)$ is decreasing in t and y_t is increasing in t, we have $\dot{z}_t(\lambda_S) = 0$. Hence, S does not enter for $-t < -t^*$. This contradicts the assumption that there does not exist $t_S^{***} \in [t^{**}, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{***})$ for λ_S.

[Proof of "cutoff from above"] We prove that if S does not enter for sufficiently large t for a pair (λ_S, λ_W), then S does not enter for sufficiently large t for all the pairs (λ_S', λ_W) with $\lambda_S' > \lambda_S$.

This proof is symmetric to the one for "cutoff from below." We first arbitrarily fix λ_W. Again, x_t and y_t are independent of λ_S. Let $z_t(\lambda_S)$ be the value of z_t, given λ_S for the fixed λ_W. For sufficiently small t, $z_t(\lambda_S) > z_t(\lambda_S')$. Define $t^* \equiv \inf_t \{z_t(\lambda_S) \leq z_t(\lambda_S')\} \in \mathbb{R}_+ \cup \{+\infty\}$.

If $t^* = +\infty$, then we have $z_t(\lambda_S) \geq z_t(\lambda_S')$ for all $\lambda_S' < \lambda_S$. Since y_t is independent of λ_S, we are done. Hence, we concentrate on the case with $t^* < \infty$.

At $-t^*$, it must be the case that $z_{t^*}(\lambda_S) = z_{t^*}(\lambda_S')$ and $\dot{z}_{t^*}(\lambda_S) \leq \dot{z}_{t^*}(\lambda_S')$ by an argument analogous to footnote 42. From $z_{t^*}(\lambda_S) = z_{t^*}(\lambda_S')$, we have

$$\begin{aligned}
\dot{z}_{t^*}(\lambda_S) &= \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} + \lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} \\
\dot{z}_{t^*}(\lambda_S') &= \lambda_W \max\left\{x_{t^*}(\lambda_S') - z_{t^*}(\lambda_S'), 0\right\} + \lambda_S' \min\left\{y_{t^*} - z_{t^*}(\lambda_S'), 0\right\} \\
&= \lambda_W \max\left\{x_{t^*}(\lambda_S') - z_{t^*}(\lambda_S), 0\right\} + \lambda_S' \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.
\end{aligned}$$

Note that, by definition, we have $x_{t^*}(\lambda_S') < x_{t^*}(\lambda_S)$. Given this inequality, the following two cases are possible:

1. If $x_{t^*}(\lambda_S) > z_{t^*}(\lambda_S)$, then we have $\lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} > \lambda_W \max\{x_{t^*}(\lambda_S') - z_{t^*}(\lambda_S), 0\}$. In addition, we have $\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} \geq \lambda_S' \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}$. Hence, we have $\dot{z}_{t^*}(\lambda_S') < \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

2. If $x_{t^*}(\lambda_S) \leq z_{t^*}(\lambda_S)$, then we have

   $$\lambda_W \max\left\{x_{t^*}(\lambda_S') - z_{t^*}(\lambda_S), 0\right\} \leq \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} = 0$$

   and so

   $$\lambda_W \max\left\{x_{t^*}(\lambda_S') - z_{t^*}(\lambda_S), 0\right\} = \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} = 0.$$

   We consider the following subcases:

(a) If $y_{t^*} > z_{t^*}(\lambda_S)$, then we have

$$\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} > \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.$$

Hence, we have $\dot{z}_{t^*}(\lambda'_S) < \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

(b) If $y_{t^*} \leq z_{t^*}(\lambda_S)$, then we have

$$\lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} = 0$$

and

$$\lambda'_S \min\left\{y_{t^*} - z_{t^*}(\lambda'_S), 0\right\} = \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} = 0.$$

Therefore, $\dot{z}_{t^*}(\lambda'_S) = 0$. For $t > t^*$, since $x_t(\lambda'_S)$ is decreasing in t and y_t is increasing in t, we have $\dot{z}_t(\lambda'_S) = 0$. Hence, S does not enter for $-t < -t^*$ with λ'_S, as desired.

In the proof above, all the time-cutoffs described above are finite and independent of T, as stated in part 3 of Proposition 5.

7.4 Proof of Remark 7

Before proving Proposition 6, we prove Remark 7. It suffices to show that $\phi < 0$ implies $t_S = \infty$.

By definition, we can write

$$f_S(t) = \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} & \text{if } r = 1. \end{cases}$$

If $\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha} \leq 0$, then

$$\begin{aligned} f_S(t) &= \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} \\ &= \frac{1}{1-r}e^{-\lambda_S t} + \left(\frac{\max\{(\gamma-\beta),0\}}{1-\alpha} - \frac{1}{1-r}\right)e^{-\lambda_W t} - \frac{\beta + \{\max(\gamma-\beta),0\}}{1-\alpha} \end{aligned}$$

is always decreasing in t. Since $f_S(0) = -\frac{\beta}{1-\alpha}$, we have $f_S(t) < 0$ for all t. Hence, we have $t_S = \infty$, as desired.

Hence, for the rest of the proof, we focus on the case in which $\frac{1}{r}-\frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}>0$. Then, the first- and second- order conditions for $f_S(t)$ imply that $f_S(t)$ is single-peaked at

$$t^{\text{peak}}=\begin{cases}\frac{\log\left(\frac{1}{r}-\frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)}{\lambda_W-\lambda_S} & \text{if } r\neq 1,\\ \frac{1}{\lambda_W}\left(1-\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right) & \text{if } r=1.\end{cases}$$

For $r\neq 1$, since

$$\begin{aligned} f_S(t) &= \frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}\\ &= \left(\frac{1}{1-r}\left(re^{-\lambda_S t}-e^{-\lambda_W t}\right)+\frac{e^{-\lambda_W t}\max\{(\gamma-\beta),0\}}{1-\alpha}\right)\\ &\quad -\frac{1}{1-r}(r-1)e^{-\lambda_S t}-\frac{\beta+\max\{(\gamma-\beta),0\}}{1-\alpha}\\ &= -\frac{1}{\lambda_W}f_S'(t)+e^{-\lambda_S t}-\frac{\max\{\beta,\gamma\}}{1-\alpha},\end{aligned}$$

substituting $f_S'(t^{\text{peak}})=0$ and $t^{\text{peak}}=\frac{\log\left(\frac{1}{r}-\frac{1-r}{r}\frac{\gamma-\beta}{1-\alpha}\right)}{\lambda_W-\lambda_S}$ into $f_S(t)$ yields

$$\begin{aligned} f_S(t^{\text{peak}}) &= e^{-\lambda_S\frac{\log\left(\frac{1}{r}-\frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)}{\lambda_W-\lambda_S}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\\ &= e^{-r\frac{\log\left(\frac{1}{r}-\frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)}{1-r}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\\ &= e^{\log\left(\frac{1}{r}-\frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)^{\frac{r}{r-1}}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\\ &= \left(\frac{1}{r}-\frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)^{\frac{r}{r-1}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\\ &= \phi.\end{aligned}$$

Therefore, if $\phi<0$, then there is no solution for $f_S(t)=0$ and so $t_S=\infty$, as desired.

On the other hand, for $r=1$, since

$$\begin{aligned} f_S(t) &= \lambda_W te^{-\lambda_W t}-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}\\ &= -\frac{1}{\lambda_W}f_S'(t)+e^{-\lambda_W t}-\frac{\max\{\beta,\gamma\}}{1-\alpha},\end{aligned}$$

substituting $f_S'(t^{\text{peak}}) = 0$ and $t^{\text{peak}} = \frac{1}{\lambda}\left(1 - \frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)$ into $f_S(t)$ yields

$$
\begin{aligned}
f_S(t^{\text{peak}}) &= e^{-\lambda_W \frac{1}{\lambda_W}\left(1-\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= e^{\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}-1} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= \phi.
\end{aligned}
$$

Therefore, if $\phi < 0$, then there is no solution for $f_S(t) = 0$ and so $t_S = \infty$ holds, as desired.

7.5 Proof of Proposition 6

Recall that t_S is the smallest positive solution for $f_S(t) = 0$ where

$$
f_S(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta}{1-\alpha} & \text{if } r = 1, \end{cases} \tag{17}
$$

while t_W is the smallest solution for $f_W(t) = 0$ where

$$
f_W(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1. \end{cases} \tag{18}
$$

We prove each part of the proposition in what follows.

7.5.1 Proof of Part 1 of Proposition 6

When we change r, without loss, we keep λ_W fixed and make λ_S larger. First, note that, for sufficiently large r, ϕ is negative:

$$
\begin{aligned}
&\lim_{r\to\infty}\left(\frac{1}{r} - \frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}\right)^{\frac{r}{r-1}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
= \quad & -\max\left(\frac{\gamma-\beta}{\alpha-1},0\right) - \frac{\max\{\beta,\gamma\}}{1-\alpha} < 0.
\end{aligned}
$$

Hence, for sufficiently large r, we have $-t_W > -t_S$.

On the other hand, since

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\left\{(\gamma-\beta),0\right\}}{1-\alpha}\right)\\
=\;&\lim_{r\to 0}\frac{1}{1-r}\left(e^{-r\lambda_W t}-e^{-\lambda_W t}\right)-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\left\{(\gamma-\beta),0\right\}}{1-\alpha}\\
=\;&1-e^{-\lambda_W t}-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\left\{(\gamma-\beta),0\right\}}{1-\alpha}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)-e^{-\lambda_S t}\right)\\
=\;&\lim_{r\to 0}\left(\frac{1}{1-r}\left(e^{-r\lambda_W t}-e^{-\lambda_W t}\right)-e^{-r\lambda_W t}\right)\\
=\;&-e^{-t\lambda_W}
\end{aligned}$$

hold for each t, $\lim_{r\to 0} t_S<\infty$ and $\lim_{r\to 0} t_W=\infty$. Hence, for sufficiently small r, we have $-t_W<-t_S$.

Therefore, we are left to show that

$$\frac{\partial\left(-t_W\right)}{\partial r}-\frac{\partial\left(-t_S\right)}{\partial r}>0.$$

To this end, in (17) and (18), when λ_S goes up with λ_W fixed, the first terms in $f_S(t)$ and $f_W(t)$ move in the same way while the second terms ($-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ in $f_S(t)$ and $-e^{-\lambda_S t}$ in $f_W(t)$) become larger only in $f_W(t)$. Hence, we have $\frac{\partial(-t_W)}{\partial r}-\frac{\partial(-t_S)}{\partial r}>0$, as desired.

7.5.2 Proof of Part 2 of Proposition 6

First, note that, for sufficiently large α, ϕ is negative for the following reason: If $r\neq 1$, since

$$\lim_{\alpha\to 1}\left(\frac{1}{r}-\frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}\right)<0,$$

$\phi<0$ in the limit as $\alpha\to 1$. If $r=1$, since

$$\lim_{\alpha\to 1}\left(e^{-\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\right)<0,$$

$\phi < 0$ in the limit as $\alpha \to 1$. Hence, for sufficiently large α, we have $-t_W > -t_S$.

Therefore, we are left to show that

$$\frac{\partial(-t_W)}{\partial \alpha} - \frac{\partial(-t_S)}{\partial \alpha} > 0.$$

In (17) and (18), when α goes up, only the second terms $-\frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta), 0\}}{1-\alpha}$ and $\frac{\beta}{1-\alpha}$ in $f_S(t)$ become smaller, while $f_W(t)$ is unchanged. Hence, we have $\frac{\partial(-t_W)}{\partial \alpha} - \frac{\partial(-t_S)}{\partial \alpha} > 0$, as desired. To see why $-t_S$ becomes smaller (that is, t_S becomes larger), we examine $f_S(t)$ in (17). Notice that $\frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right)$ is single-peaked at $\frac{\log \lambda_W - \log \lambda_S}{\lambda_W - \lambda_S}$. The term $\lambda_W t e^{-\lambda_W t}$ is single-peaked at $t = \frac{1}{\lambda_W}$. Since $\frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta), 0\}}{1-\alpha}$ and $\frac{\beta}{1-\alpha}$ become larger, the smallest positive number such that $\frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) = \frac{\beta}{1-\alpha}$ increases.

7.5.3 Proof of Part 3 of Proposition 6

First, note that, for sufficiently large β, ϕ is negative: If $r \neq 1$, since

$$\begin{aligned}
&\lim_{\beta \to 1}\left(\left(\frac{1}{r} - \frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha}, 0\right\}\right)^{\frac{r}{r-1}} - \frac{\max\{\beta, \gamma\}}{1-\alpha}\right) \\
=\ & \left(\frac{1}{r}\right)^{\frac{r}{r-1}} - \frac{1}{1-\alpha} \\
\leq\ & \max_r \left(\frac{1}{r}\right)^{\frac{r}{r-1}} - \frac{1}{1-\alpha} \\
=\ & 1 - \frac{1}{1-\alpha} < 0,
\end{aligned}$$

$\phi < 0$ in the limit as $\beta \to 1$. If $r = 1$, since

$$\begin{aligned}
&\lim_{\beta \to 1}\left(e^{-\max\left\{\frac{\gamma-\beta}{1-\alpha}, 0\right\}} - \frac{\max\{\beta, \gamma\}}{1-\alpha}\right) \\
=\ & 1 - \frac{1}{1-\alpha} < 0,
\end{aligned}$$

$\phi < 0$ in the limit as $\beta \to 1$. Hence, for sufficiently large β, we have $-t_W > -t_S$.

Therefore, we are left to show that

$$\frac{\partial(-t_W)}{\partial \beta} - \frac{\partial(-t_S)}{\partial \beta} < 0.$$

In (17) and (18), when β goes up, the second terms $-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ and $-\frac{\beta}{1-\alpha}$ in $f_S(t)$ become smaller while $f_W(t)$ is unchanged. Hence, by the same proof as in the case where α increases, $-t_W-(-t_S)$ increases. Hence, we have $\frac{\partial(-t_W)}{\partial\beta}-\frac{\partial(-t_S)}{\partial\beta}>0$, as desired.

7.5.4 Proof of Part 4 of Proposition 6

It suffices to show that

$$\begin{aligned}\frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma} &= 0 \text{ if } \gamma\leq\beta\\ \frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma} &> 0 \text{ if } \gamma>\beta.\end{aligned}$$

In (17) and (18), if $\beta\geq\gamma$, then neither $f_S(t)$ nor $f_W(t)$ depends on γ. Hence, we have $\frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma}=0$. Hence, let us focus on the case $\gamma>\beta$. If γ goes up in (17) and (18), then the second term $-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ in $f_S(t)$ becomes smaller while $f_W(t)$ is unchanged. Hence, by the same proof as in part 3 of Proposition 6, $-t_W-(-t_S)$ increases, which means $\frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma}>0$, as desired.

7.5.5 Proof of Part 5 of Proposition 6

Fix (r,α,β). If $\beta=1$, then part 3 of Proposition 6 ensures that, for each γ, we have $-t_S<-t_W$, and so S does not enter at $-t<-t_W$. Hence, let us focus on the case with $\beta<1$. Take $\bar{\gamma}$ sufficiently large so that, for each $\gamma>\bar{\gamma}$, we have $\gamma>\beta$ and

$$1-\gamma<\min\left\{1-e^{-\lambda_S},e^{-\lambda_W}\right\}. \tag{19}$$

For each $\gamma>\bar{\gamma}$, once S enters before W, W will enter afterwards if W has an opportunity. Hence, S's payoff of entering at $-t$ is equal to

$$\left(1-e^{-\lambda_W t}\right)(1-\gamma)+e^{-\lambda_W t}. \tag{20}$$

We want to show that S's payoff of not entering at $-t$ is higher than $\left(1-e^{-\lambda_W t}\right)(1-\gamma)+e^{-\lambda_W t}$.

A possible strategy that S can take if she does not enter at $-t$ is not to enter until -1. With this strategy, if W has entered by $-t=-1$, then S's payoff is no less than $1-e^{-\lambda_S}$ since S gets one if S can enter between $-t$ and 0. If W has not entered, then S's payoff is no less than $e^{-\lambda_W}$ since S can ensure, by not entering until the deadline, that she gets one if W cannot enter. In total, S's payoff is no less than

$$\min\left\{1-e^{-\lambda_S}, e^{-\lambda_W}\right\}.$$

By (19), this payoff is higher than (20) for sufficiently large t. Hence, S is better off by not entering at $-t$ for sufficiently large t, as desired.

7.6 Proof of Proposition 8

We first show the result that will be useful for the following proof. Fix t arbitrarily. Suppose that the candidates play the one-shot constant-sum game, where S's payoff is given by

$S \setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	1	p	p
$\{1\}$	$1-p$	1	$1-p$
$\{0,1\}$	$1-pe^{-\lambda\tau}$	$1-(1-p)\,e^{-\lambda\tau}$	w

Let $V(w)$ be the unique minimax value given w. We will show that

$$\left|V(w)-V(w')\right| \leq \left|w-w'\right| \tag{21}$$

for each w and w'. Without loss, we can assume $w \geq w'$.

We first derive an upper bound for $V(w)-V(w')$. By Minimax Theorem, we can assume that W moves first to minimize S's payoff and then S moves to maximize S's payoff. Let $\sigma^W(w)$ be an optimal strategy for W given w. When W takes the same strategy $\sigma^W(w)$ given w', then S can improve her payoff compared to $V(w)$ at most by $w-w'$. Hence, $V(w)-V(w') \leq w-w'$.

We second derive a lower bound for $V(w)-V(w')$. By Minimax Theorem, we can assume that S moves first to maximize S's payoff and then W moves to minimize S's payoff. Let $\sigma^S(w)$ be an optimal strategy for S given w. When S takes the same strategy $\sigma^S(w)$ given w', then W can improve his payoff at most by $w-w'$. Hence, $V(w)-V(w') \geq -(w-w')$. In total, we have shown (21).

We now show that Markov equilibria exist for each T. Consider the following functional equation f: Given $v^S : [0,T] \to [0,1]$ such that v^S is continuous in t, $f(v^S)(t)$ is equal to the unique minimax value of the following payoff matrix

$S \setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	1	p	p
$\{1\}$	$1-p$	1	$1-p$
$\{0,1\}$	$1-pe^{-\lambda t}$	$1-(1-p)\,e^{-\lambda t}$	$\int_0^t e^{-\lambda\tau}\lambda v^S(t-\tau)d\tau + e^{-\lambda t}$

If v^S is continuous, then each element of the payoff matrix is continuous in t, and $f(v^S)(t)$ is also continuous in t. Hence, f is the mapping from the set of continuous functions such that $v^S : [0,T] \to [0,1]$ to itself.

Consider the sup norm: $\left\|v^S - \hat{v}^S\right\| \equiv \sup_{t\in[0,T]} \left|v^S(t) - \hat{v}^S(t)\right|$. Given this norm, the mapping f is contraction. To see why, note that, for each $t \in [0,T]$, we have

$$\begin{aligned} \left|f(v^S)(t) - f(\hat{v}^S)(t)\right| &\leq \left|\int_0^t e^{-\lambda\tau}\lambda\left(v^S(t-\tau) - \hat{v}^S(t-\tau)\right)d\tau\right| \\ &\leq \sup_{t\in[0,1]} \left|v^S(t) - \hat{v}^S(t)\right| \int_0^t e^{-\lambda\tau}\lambda d\tau \\ &= \left(1 - e^{-\lambda T}\right)\left\|v^S - \hat{v}^S\right\|. \end{aligned}$$

The first inequality follows from (21). Hence, there exists a unique fixed point $\bar{v}^S$ for the mapping f. When we define $V_t^S = \bar{v}^S(t)$ for each t, such V_t^S is the equilibrium value. Moreover, taking the minimax strategy of the game

$S \setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	1	p	p
$\{1\}$	$1-p$	1	$1-p$
$\{0,1\}$	$1-pe^{-\lambda t}$	$1-(1-p)\,e^{-\lambda t}$	$\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + e^{-\lambda t}$

(22)

in each period t is an equilibrium. Therefore, the existence is proven.

Next, we will prove that the equilibrium value in the subgame perfect equilibrium is unique.

Let $h_{<-t}$ be the history before time $-t$:

$$h_{<-t} = \left(\left(t^k, x_S^k, x_W^k \right)_{k=0}^{K} \right),$$

where $-T < -t^1 < ... < -t^k < -t$ and $x_i^k \in 2^X \setminus \{\emptyset\}$ for all k and $i = S, W$. The interpretation is that $-t^k$ is the time at which the candidates receive their k'th revision opportunity, and x_i^k is the policy set that i has chosen at time $-t_i^k$.

Intuitively, the same proof as in the proof of Proposition 5 establishes the uniqueness, with h_t in (4) replaced with $h_{<-t}$. In addition, since the opportunity arrives synchronously, we consider the event such that the candidates receive an opportunity at $-t$ and both of them take $\{0,1\}$, instead of $z_i = yes$ (that is, candidate i receives an opportunity) and i taking N.

The formal proof proceeds as follows. Let $\tilde{W}_t^S(\sigma, h_{<-t})$ be S's payoff when both candidates take $\{0,1\}$ at $-t$ and take a strategy σ such that $\sigma|_{h_{<-t}, yes, \{0,1\}, \{0,1\}}$ is subgame perfect in the game after $(h_{<-t}, yes, \{0,1\}, \{0,1\})$, where $\sigma|_{h_{<-t}, yes, \{0,1\}, \{0,1\}}$ denotes a continuation strategy defined for such a subgame given by restriction of σ on such a subgame. That is, $h_{<-t}$ is the record of what has been observed before $-t$, *yes* means that the candidates receive an opportunity at $-t$, and both of them take $\{0,1\}$ at $-t$. Moreover, let $\bar{W}_t^S$ be the supremum of S's continuation payoff: $\bar{W}_t^S \equiv \sup_{\sigma, h_{<-t}} \tilde{W}_t^S(\sigma, h_{<-t})$, where the supremum is taken over all the possible histories and strategies such that $h_{<-t}$ is the history at $-t$, the candidates receive an opportunity at $-t$, and both of them take $\{0,1\}$, and $\sigma|_{h_{<-t}, yes, \{0,1\}, \{0,1\}}$ is subgame perfect after $(h_{<-t}, yes, \{0,1\}, \{0,1\})$. Similarly, let $\underline{W}_t^S$ be the infimum of S's continuation payoff: $\underline{W}_t^S \equiv \inf_{\sigma, h_{<-t}} \tilde{W}_t^S(\sigma, h_{<-t})$. Let w_t^S be the difference between the supremum and infimum: $w_t^S \equiv \bar{W}_t^S - \underline{W}_t^S$. Note that $w_0^S = 0$ since the game that the candidates play at time 0 has a unique equilibrium payoff because it is a constant-sum game.

We first show that w_t^S is continuous in t. To this end, as we do in footnote 38, let $W_t^S(\sigma, h_{<-t})$ be S's payoff when there is no opportunity at $-t$ and the candidate takes a strategy σ such that $\sigma|_{h_{<-t}, no}$ is subgame perfect in the game after $(h_{<-t}, no)$ (that is, $h_{<-t}$ is what has been observed before $-t$ and *no* means that the candidates do not receive an opportunity at $-t$). As seen in footnote 38, $W_t^S(\sigma, h_{<-t})$ is continuous in t given σ. Hence, $\bar{W}_t^{S,\text{no}} \equiv \sup_{\sigma, h_{<-t}} W_t^S(\sigma, h_{<-t})$ and $\underline{W}_t^{S,\text{no}} \equiv \inf_{\sigma, h_{<-t}} W_t^S(\sigma, h_{<-t})$ are continuous, where supremum and infimum are taken over all the possible histories and strategies such that there is no opportunity at $-t$ and $\sigma|_{h_{<-t},no}$ is subgame perfect.

To show w_t^S is continuous in t, it suffices to show that $\bar{W}_t^S = \bar{W}_t^{S,\text{no}}$ and $\underline{W}_t^S = \underline{W}_t^{S,\text{no}}$. Let us define $\tilde{\sigma}$ as the strategy such that, after $(h_{<-t}, no)$, the candidates at time $-\tau$ follows the strategy $\sigma|_{h_{<-t},yes,\{0,1\},\{0,1\}}$. That is, they take actions as if there were an opportunity at $-t$ and both of them took $\{0,1\}$ at $-t$. Since the strategic environment is the same between $(h_{<-t}, no)$ and $(h_{<-t}, yes, \{0,1\}, \{0,1\})$, this continuation strategy is subgame perfect after $(h_{<-t}, no)$ if $\sigma|_{h_{<-t},yes,\{0,1\},\{0,1\},h_{t,\tau}}$ is subgame perfect after $(h_{<-t}, yes, \{0,1\}, \{0,1\}, h_{t,\tau})$. Therefore, for each $\tilde{W}_t^S(\sigma, h_{<-t})$ such that $\sigma|_{h_{<-t},yes,\{0,1\},\{0,1\}}$ is subgame perfect, there exists $\tilde{\sigma}$ such that $\tilde{W}_t^S(\sigma, h_{<-t}) = W_t^S(\tilde{\sigma}, h_{<-t})$ and $\tilde{\sigma}|_{h_{<-t},no}$ is subgame perfect. Therefore, we have $\bar{W}_t^S = \bar{W}_t^{S,\text{no}}$ and $\underline{W}_t^S = \underline{W}_t^{S,\text{no}}$.

We will now show that $w_t^S = 0$ for each $t \geq 0$. To this end, fix $-\tau \in (-t, 0]$ arbitrarily. Suppose that the candidates receive an opportunity at $-\tau$ for the first time after $-t$, that is, $z_{\tau'} = no$ for each $-\tau' \in (-t, -\tau)$ and $z_\tau = yes$. Here, $z_t \in \{yes, no\}$ represents whether the candidates receive an opportunity at $-t$. Let $W_t^S(\sigma, h_t \mid \tau)$ be S's continuation payoff from $-\tau$, conditional on that $z_{\tau'} = no$ for each $-\tau' \in (-t, -\tau)$ and $z_\tau = yes$. This S's continuation payoff from $-\tau$, denoted by $W_t^S(\sigma, h_t \mid \tau)$, varies at most by w_τ^S by (21).

On the other hand, let $W_t^S(\sigma, h_t \mid no)$ be candidate S's continuation payoff at 0 given σ and h_t, conditional on that no opportunity comes after $-t$ (that is, $z_\tau = no$ for each $-\tau \in (-t, 0]$). By definition, this difference is equal to $w_0^S = 0$.

Since the probability that the candidates receive the first opportunity after $-t$ at $-\tau \in (-t, 0]$ is $1 - \exp(-\lambda t)$ (that is, $z_{\tau'} = no$ for each $-\tau' \in (-t, -\tau)$ and $z_\tau = yes$ for some $-\tau \in (-\tau, 0]$), we have

$$
\begin{aligned}
w_t^S &\leq (1 - \exp(-\lambda t)) \times \underbrace{\max_{\tau \leq t} w_\tau^S}_{\substack{\text{Supremum difference} \\ \text{in } W_\tau^S(\sigma, h_{<\tau} \mid \tau)}} + \exp(-\lambda t) \times 0 \\
&= (1 - \exp(-\lambda t)) \max_{\tau \leq t} w_\tau^S.
\end{aligned}
$$

The same proof as in (6) with $\lambda_S + \lambda_W$ replaced with λ establishes the uniqueness. Let V_t^S be the unique value. Given V_t^S, the candidates at $-t$ play the constant-sum game with payoff matrix (22). Hence, as long as the minimax strategy for (22) is unique, the strategies for the candidates are unique. Hence, the equilibrium is essentially Markov.

Now we prove parts 1 and 2. Part 1 holds since (i) each candidate takes a completely mixed strategy at $-t = 0$ and (ii) the payoff function is continuous in t. Hence, we focus on proving part 2.

In equilibrium, there are following three possibilities:

1. S takes a pure strategy $\{x\}$ at $-t$. W then takes $\{x'\}$ or $\{0,1\}$, with $x' = \{0,1\} \setminus \{x\}$. For x to be optimal, it must be the case that $x = 1$. Consider the following two possible subcases:

   (a) If W takes a pure strategy $\{x'\}$, then S takes $\{x'\}$. This is a contradiction.

   (b) If W takes $\{0,1\}$ with positive probability, then the payoff of S's taking $\{0,1\}$ is $1-p$ if W enters at $x' = 0$, and strictly greater than $1-p$ if W takes $\{0,1\}$. To see this, we calculate S's payoff for taking each action when W takes $\{0,1\}$. Conditional on W taking $\{0,1\}$, S's payoffs are given by the following table:

$S \setminus W$	$\{0,1\}$
$\{0\}$	p
$\{1\}$	$1-p$
$\{0,1\}$	$\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau}d\tau + e^{-\lambda t}$

   Since S can always enter at $\{1\}$ and thereby guarantee payoff $1-p$, it follows that $V^S_{t-\tau} \geq 1-p$ for all τ. Therefore, $\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau}d\tau + e^{-\lambda t} \geq \left(1-e^{-\lambda t}\right)(1-p) + e^{-\lambda t} > 1-p$. This means it is a strict best response for S to announce $\{0,1\}$. This is a contradiction.

2. S takes a mixed strategy only over $\{0\}$ and $\{1\}$ at $-t$. It is then a strict best response for W to take $\{0,1\}$ since the probability of S and W entering at the same platform would then be zero. This means it is a strict best response for S to announce $\{0,1\}$ by the same argument as above. This is a contradiction.

3. S takes $\{0,1\}$ with positive probability. In order to show that it is a strict best response for W to take $\{0,1\}$, we compare W's payoff for entering at $\{x\}$ at $-t$ and that of taking $\{0,1\}$ in the following three possible subcases:

(a) Conditional on the event that S enters at $\{x\}$ at $-t$, W gets zero if W enters at $\{x\}$. Compared to this, announcing $\{0,1\}$ is strictly better for W since that gives him at least $1-p$.

(b) Conditional on the event that S enters at $\{x'\}$ at $-t$, W gets p by entering at $\{x\}$ if $x=0$, and gets $1-p$ if $x=1$. Announcing $\{0,1\}$ also gives W the same payoff.

(c) Conditional on the event that S does not enter, W gets at most $p\Pr(S$ will not have an opportunity$)=p\exp(-\lambda t)$ by entering at $\{x\}$. On the other hand, consider the strategy in which W announces $\{0,1\}$ until $-\bar{t}=-\frac{1}{\lambda}$. If player S has entered by $-\bar{t}$, W will get at least $1-p$. Otherwise, when the candidates have an opportunity to enter at $-s\geq-\bar{t}$, then the value for S should be less than the minimax value of the following constant-sum game.

$S\setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	1	p	p
$\{1\}$	$1-p$	1	$1-p$
$\{0,1\}$	$1-pe^{-1}$	$1-(1-p)\,e^{-1}$	1

This is because this payoff matrix is the same as the original payoff matrix except that we replace the payoffs when S takes $\{0,1\}$ with higher payoffs. The value is bounded away from 1, which means the payoff for W is bounded away from 0. Let $\underline{v}$ be this lower bound. When we take into account the probability of the candidates having an opportunity between $-\bar{t}$ and 0, the expected payoff is no less than $\left(1-e^{-1}\right)\underline{v}$. For sufficiently large t, $p\exp(-\lambda t)<\min\left\{1-p,\left(1-e^{-1}\right)\underline{v}\right\}$, which means taking $\{0,1\}$ is strictly better.

To summarize Case 3, since we assume that S takes $\{0,1\}$ with a positive probability, it follows that $\{0,1\}$ is a strict best response for W for sufficiently large t.

Let us consider S's incentive, given that W takes $\{0,1\}$. Recall that S's payoffs given that W

takes $\{0,1\}$ for sure are given by the following table:

$S \setminus W$	$\{0,1\}$
$\{0\}$	p
$\{1\}$	$1-p$
$\{0,1\}$	$\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau}d\tau + e^{-\lambda t}$

For the same reason as in Case 1(b) above, S should take $\{0,1\}$ with probability 1.

7.7 An Example of Equilibrium Dynamics with Simultaneous Arrivals

As part 1 of Proposition 8 shows, equilibria in the synchronous announcement model involve mixing when the election date is close. The mixing probabilities have to change over time, since the Nash equilibrium of the game matrix in Figure 5 changes as t changes. The transition of mixing probabilities is complicated. We illustrate its complexity in the numerical results for $p = 0.45$ and $\lambda = 1$. This example illuminates the subtle incentive problems faced by the two candidates.

The values of S when she takes $\{0,1\}$ against various announcements of W are as depicted in Figure 6 as a function of $-t$. Note that S's payoffs at policy profiles $(\{0,1\},\{0\})$ and $(\{0,1\},\{1\})$ at $-t$ increase as $-t$ decreases since the probability with which S can enter afterwards and copy W's policy increases. On the other hand, S's payoff at $(\{0,1\},\{0,1\})$ at $-t$ decreases since the weight for the highest payoff 1 decreases.

Figure 7 depicts S's and W's strategies as functions of $-t$. When $-t$ is sufficiently close to zero, each candidate mixes over all the announcements, as we stated in part 1 of Proposition 8. Now we consider the strategies of the candidates one by one for time $-t$ close to the deadline.

(α) Consider the transition of S's strategy. Since S's mixing probability is determined in order to make W indifferent between his actions, we hypothetically fix S's mixing probability over time and examine how W's incentive changes over time; we then use this transition of W's incentive to determine how S's mixing probability should change over time.

To this end, suppose that S enters at $x \in \{0,1\}$ with the same probability at time 0 and time $-t < 0$. Then, it must be the case that W's incentive to enter at x is weaker at time $-t$ than at time 0. To show this, we compare W's payoff for taking each action at time 0 with

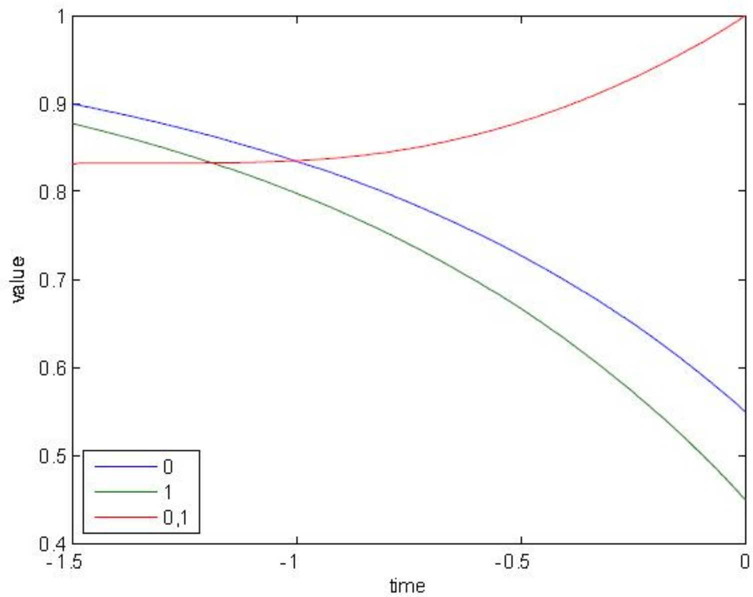

Figure 6: S's value V_t^S in the synchronous model when the candidates do not receive an opportunity at $-t$ given W's most recent announcement, given that S has been taking $\{0,1\}$. For example, the blue line corresponds to S's value given that W has taken $\{0\}$ and S has been taking $\{0,1\}$.

his payoff at time $-t < 0$. At time 0, entering at x gives W a positive payoff if and only if S either enters at the other point or takes $\{0,1\}$; but taking $\{0,1\}$ gives W a positive payoff if and only if S does not take $\{0,1\}$. On the other hand, at time $-t$, entering at x gives W a positive payoff if and only if S either enters at the other point at $-t$ or "takes $\{0,1\}$ and cannot enter until the deadline"; but taking $\{0,1\}$ gives W a positive payoff if S does not take $\{0,1\}$. Furthermore, if both take $\{0,1\}$ at $-t$, then the payoff depends on the continuation play after $-t$ but is weakly higher than the payoff for both candidates taking $\{0,1\}$ at time 0.

To summarize, W's payoff for entering at x is smaller at time $-t < 0$ than at time 0 while W's payoff for taking $\{0,1\}$ is no less at time $-t < 0$ than at time 0, if S entered at x with the same probability over time. Hence, to incentivize W to enter at x, S should reduce the probability of her taking $x \in \{0,1\}$ over time.

(β) Consider the transition of W's strategy. In an approach similar to Argument (α) with the

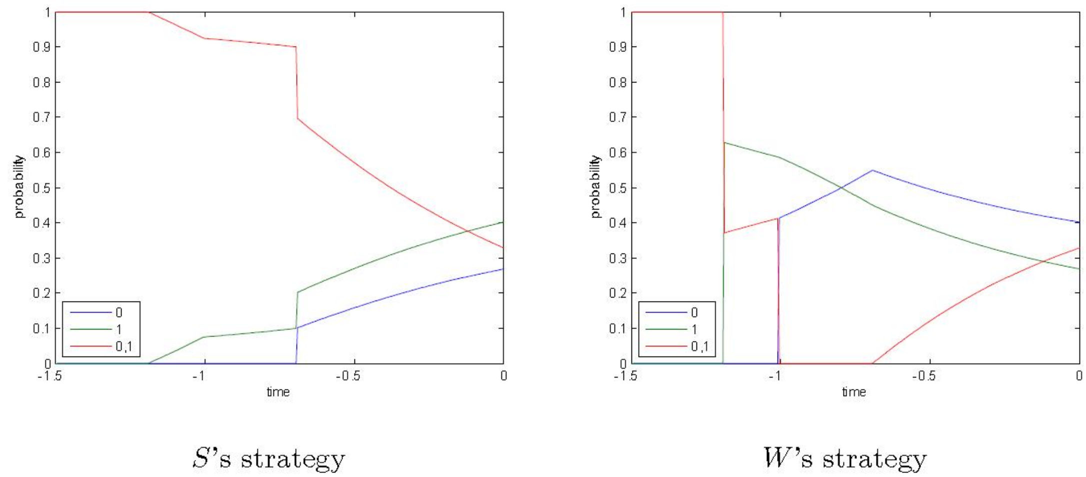

Figure 7: Strategies in the synchronous model

roles of S and W reversed, suppose that W enters at $x \in \{0, 1\}$ with the same probability over time $[-t, 0]$. Given this assumption about W's strategy, we will show that S's incentive to enter at x is stronger at time 0 than at time $-t$.

To compare S's payoff for each action at time 0 with her payoff at time $-t$, we first show that S's payoff for entering at $x \in \{0, 1\}$ is the same between time $-t$ and time 0. At time 0, entering at x gives S a positive payoff if and only if either W takes x or the median voter is at x. At time $-t$, entering at x gives S a positive payoff if and only if either W takes x or the median voter is at x. Since we assume that W enters at x with the same probability at both times 0 and $-t$, the two payoffs are the same.

Next, we show that S's payoff for taking $\{0, 1\}$ is lower at time 0 than at time $-t$. Playing $\{0, 1\}$ at time 0 gives S the mixed-strategy equilibrium payoff in the one-shot game. If there is no opportunity after time $-t$, then since we assume that W enters at $x \in \{0, 1\}$ with the same probability between time 0 and time $-t$, S's expected payoff for taking $\{0, 1\}$ is the same as this mixed-strategy equilibrium payoff. If there is an opportunity to enter, S's payoff for taking $\{0, 1\}$ depends on W's realized action at time $-t$. If W takes $\{0, 1\}$ at time $-t$, then S's payoff again corresponds to the mixed-strategy equilibrium payoff at time 0.[43] On

[43]Here, we assume that this another opportunity to enter is the last opportunity until the deadline because the probability to have one more opportunity is small for $-t$ close to 0.

the other hand, if W specifies his policy, then S's payoff is 1. Since W's strategy assigns a strictly positive probability to specifying his policy, S's expected payoff for taking $\{0,1\}$ is lower at time 0 than at time $-t$.

The above comparison implies that S's payoff for entering at x would be constant but S's payoff for taking $\{0,1\}$ would increase as $-t$ becomes smaller, if W took each action with the same probability between time 0 and time $-t$. Hence, in order to incentivize S to enter at x, W should increase the probability of his taking x as $-t$ becomes smaller, so that both S and W enter at x with a higher probability. Therefore, W puts higher probabilities on $\{0\}$ and $\{1\}$ as $-t$ becomes smaller.

Now we consider the candidates' strategies for times further away from the deadline.

Around $-t = -0.7$, the constraint that the probability of $\{0,1\}$ is nonnegative binds for W. As $-t$ becomes further away from the deadline than such a cutoff time, W cannot increase the probability of entering both at $\{0\}$ and $\{1\}$. Then, as seen in the comparison of S's payoff above (Argument (β)), entering becomes less attractive for S. Since the median voter is located with a lower probability at $\{0\}$, S stops entering at $\{0\}$.

Now let us consider the transition of the mixing probabilities in the time interval $(-1,-0.7)$. Again, as seen in the comparison of S's payoff above, W increases the probability of entering at $\{1\}$ as $-t$ becomes smaller in order to incentivize S to enter at $\{1\}$. On the other hand, as seen in the comparison of W's payoff above (Argument (α)), S reduces the probability of taking $\{1\}$ as $-t$ becomes smaller in order to incentivize W to enter at $\{1\}$.

Consider the incentive at $-t < -1$. For each time $-t \in (-1,0]$, W is indifferent between $\{0\}$ and $\{0,1\}$. As in the comparison of W's payoff above, if S took each action with the same probability between times -1 and $-t < -1$, W's incentive to enter at $x \in \{0,1\}$ would decrease as $-t$ becomes smaller. As $-t$ gets smaller, this incentive gets even weaker since if S has not yet specified her policy, then S can enter with a higher probability later and W's risk of being copied by S later goes up. In general, entering at $\{0\}$ is less attractive for W than entering at $\{1\}$ since the median voter is less likely to be at policy 0. Hence, there is a time $-\bar{t}$ such that for each $-t < -\bar{t}$, W strictly prefers $\{0,1\}$ to $\{0\}$.

Again, as seen in Argument (α) (that is, the comparison of W's payoff), as $-t$ becomes smaller, S reduces the probability of taking $\{1\}$ in order to incentivize W to enter at $\{1\}$. Finally, the probability of S taking $\{0, 1\}$ hits 1. If $-t$ gets further away from the deadline, then no player enters.

To wrap up the discussion, although the exact transition of incentives is complicated, the basic reason for the ambiguous policy announcements with synchronous arrivals is the same as in the case with asynchronous arrivals— S wants to wait for W who does not want to be copied, which makes both candidates announce ambiguous policies when the election date is still far away.