# Strongly Consistent Self-Confirming Equilibrium

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**Abstract**

Fudenberg and Levine (1993a) introduce the notion of self-confirming equilibrium, which is generally less restrictive than Nash equilibrium. Fudenberg and Levine also define a concept of consistency and claim in their Theorem 4 that with consistency and other conditions on beliefs, a self-confirming equilibrium has a Nash equilibrium outcome. We provide a counterexample that disproves Theorem 4 and prove an alternative by replacing consistency with a more restrictive concept, which we call *strong consistency*. In games with observed deviators, self-confirming equilibria are strongly consistent self-confirming equilibria. Hence, our alternative theorem ensures that despite the counterexample, the corollary of Theorem 4 is still valid.

Keywords : Self-confirming equilibrium, consistency, strong consistency, Nash equilibrium

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## 1. INTRODUCTION

Fudenberg and Levine (1993a; henceforth "FL") introduce the notion of self-confirming equilibrium.[2] In general, it is less restrictive than the notion of Nash equilibrium. This is mainly because beliefs can be incorrect at off-path information sets in a self-confirming equilibrium, which results in the possibility that two players have different beliefs about the strategy used by a third player. This is illustrated in the "horse" example of Fudenberg and Kreps (1988). FL define a concept of consistency in an attempt to preclude this possibility, and claim in their Theorem 4 that with consistency and other conditions on beliefs, a self-confirming equilibrium has a Nash equilibrium outcome. We provide a counterexample that disproves Theorem 4 and prove an alternative by replacing consistency with a more restrictive notion, which we call *strong consistency.*

Briefly, consistency requires that each player's belief be correct at the information sets that are reachable if he sticks to his equilibrium strategy and the opponents deviate.[3] Strong consistency further requires that each player's belief be correct at certain other information sets —those that are reachable if he sticks to actions that he plays on the equilibrium path, the opponents deviate, and he himself deviates at off-path information sets.

As a consequence of the alternative theorem proved here, we have that in games with observed deviators, in particular in two-person games, strong consistency is sufficient to ensure a Nash equilibrium outcome, so the corollary of FL's Theorem 4 is valid.

[2] For related issues, see also Battigalli (1987), Fudenberg and Kreps (1995), Fudenberg and Levine (1993b), and Kalai and Lehrer (1993).

[3] For a justification of consistent self-confirming equilibrium, see FL and Fudenberg, Kreps, and Levine (1988).

## 2. NOTATION, DEFINITIONS, AND THEOREM 4 OF FL

We follow the same notation as in pp. 525–527 of FL. Here we review and expand it.

Fix an $I$-player game in extensive form with perfect recall. $X$ is the set of nodes; 0 are the moves of Nature; $H_i$, $H$, $H_{-i}$ are the sets of information sets; $A(h_i)$ is the set of actions at $h_i$; $S_i$, $S$, $S_{-i}$ are the sets of pure strategies and profiles; $\Sigma_i$, $\Sigma$, $\Sigma_{-i}$ are the sets of mixed strategies and profiles; $\Pi_i$, $\Pi$, $\Pi_{-i}$ are the sets of behavior strategies and profiles; $\hat{\pi}_i(\cdot|\sigma_i)$ is the behavior strategy induced by (giving the same outcome as) $\sigma_i$; $H(\cdot)$ is the set of the information sets reachable under the argument (strategy or strategy profile)[4]; $\mu_i$ is $i$'s belief (a probability measure over $\Pi_{-i}$); $u_i(\cdot)$ is $i$'s expected utility given the argument (a strategy profile or a strategy-belief pair). We assume that each player knows (at least) his own payoff function, the extensive form of the game, and the probability distribution over Nature's moves.

Some new notation follows: $N$ is the set of players. We let $\Pi_{-i,j} = \times_{k \neq i,j} \Pi_k$. Let $p(z|b)$ denote the probability of reaching the terminal node $z$ given a strategy profile or strategy-belief pair $b$.

FL define the following concepts: An information set $h_j$, $j \neq i$, is *relevant to player $i$ given a belief* $\mu_i$ if there exists $s_i \in S_i$ such that $p(h_j|s_i, \mu_i) > 0$. The set of information sets that are relevant to $i$ given $\mu_i$ is denoted $R_i(\mu_i)$. A game has *observed deviators* if for all players $i$, all strategy profiles $s \in S$ and all deviations $s'_i \neq s_i$, $h \in H(s'_i, s_{-i}) \backslash H(s)$ implies that there is no $s'_{-i}$ with $h \in H(s_i, s'_{-i})$. In FL, it is proved that every two-player game of perfect recall has observed deviators.[5]

[4]Notice that, in contrast to FL, we do not distinguish between what they denote $H(\cdot)$ and $\bar{H}(\cdot)$.

[5]See Lemma 2 of FL.

We say that profile $\sigma \in \Sigma$ is *equivalent* to another profile $\sigma' \in \Sigma$ if they lead to the same distribution over terminal nodes, that is, $p(z|\sigma) = p(z|\sigma')$ for all $z \in Z$.

**Definitions 1 and 2 of FL** Profile $\sigma \in \Sigma$ is a *self-confirming equilibrium* if $\forall i \in N$, $\forall s_i \in \text{support}(\sigma_i)$, $\exists \mu_i$ *s.t.*

(i) $s_i$ maximizes $u_i(\cdot, \mu_i)$, and

(ii) $\mu_i[\{\pi_{-i} | \pi_j(h_j) = \hat{\pi}_j(h_j|\sigma_j)\}] = 1 \quad \forall j \neq i, \forall h_j \in H(s_i, \sigma_{-i})$.

It is a *consistent self-confirming equilibrium* if "$\forall h_j \in H(s_i, \sigma_{-i})$" in (ii) above is replaced by a stronger requirement, "$\forall h_j \in H(s_i)$".

A self-confirming equilibrium $\sigma$ is said to have *unitary beliefs* if for each player $i$ a single belief $\mu_i$ can be used to rationalize every $s_i \in \text{support}(\sigma_i)$. That is, in Definition 1 of FL, we would replace "$\forall s_i \in \text{support}(\sigma_i)$, $\exists \mu_i$ *s.t.*" with "$\exists \mu_i$ *s.t.* $\forall s_i \in \text{support}(\sigma_i)$,". A self-confirming equilibrium $\sigma$ is said to have *independent beliefs* if for each player $i$ and each $s_i \in \text{support}(\sigma_i)$, the associated belief $\mu_i$ satisfies $\mu_i(\times_{j \neq i} \bar{\Pi}_j) = \times_{j \neq i} \mu_i(\bar{\Pi}_j \times \Pi_{-i,j})$ for all $(\times_{j \neq i} \bar{\Pi}_j) \subseteq \Pi_{-i}$ where $\bar{\Pi}_j \subseteq \Pi_j$ for all $j \in N$.

The set of consistent self-confirming equilibria is strictly smaller than that of self-confirming equilibria, while it is strictly larger than that of Nash equilibrium.[6] It is defined in FL in an attempt to rule out the possibility of a non-Nash outcome, as is claimed in Theorem 4 of FL.

**Theorem 4 of FL** *Every consistent self-confirming equilibrium with independent, unitary beliefs is equivalent to a Nash equilibrium.*

The next section provides a counterexample to this theorem. It also establishes

[6]The examples in which consistent self-confirming equilibrium distinguishes itself from self-confirming equilibrium and Nash equilibrium are given in FL. See Example 1 of FL for self-confirming equilibrium, and see Examples 2, 3, and 4 of FL (and the game in Figure 1 of the present paper which we will explain in the next section) for Nash equilibrium.

that Theorems 1 and 3 of FL are incorrect as well, and the proof of their Theorem 2 needs a correction.

## 3. A COUNTEREXAMPLE

Consider the game depicted in Figure 1. In this game, player 1 moves first at his information set $h_1$, choosing between $L_1$ and $R_1$. Knowing 1's choice, player 2 moves next. After $L_1$ (resp. $R_1$), 2 chooses between $L_2$ and $R_2$ at his information set $h_2$ (resp. $l_2$ and $r_2$ at $h_2'$). If $L_1$ and $L_2$ are chosen, each player receives the payoff 2, and if $R_1$ and $r_2$ are chosen, each player receives the payoff 1. Otherwise, 3 gets the move at his information set $h_3$, not knowing 1 and 2's choices. Regardless of 1 and 2's choices, payoffs are $(3, 0, 0)$ if 3 plays $L_3$ and $(0, 3, 0)$ if 3 plays $R_3$.

We will show that $(R_1, r_2)$ is played in a consistent self-confirming equilibrium with independent, unitary beliefs while it is not a Nash equilibrium outcome.

To see that $(R_1, r_2)$ is played in a consistent self-confirming equilibrium with independent, unitary beliefs, consider the strategy profile $s^* = (R_1, (L_2, r_2), R_3)$. We first verify that this is a consistent self-confirming equilibrium by considering the beliefs of players 1, 2, and 3 to be $((R_2, r_2), R_3)$, $(R_1, L_3)$, and $(R_1, (L_2, r_2))$, respectively. Table 1 presents these strategies and beliefs. It is easy to see that no player has an incentive to deviate from $s^*$ under these beliefs: By playing $L_1$ player 1 expects the payoff 0; by playing $l_2$ player 2 expects the payoff 0; and $h_2$ and $h_3$ lie off the equilibrium path so that there is no incentive to deviate at these information sets. Thus, it suffices to show that each player has the correct belief at the information sets reachable under his equilibrium strategy.[7] The beliefs

[7]Player $i$'s belief is defined as a measure on the space $\Pi_{-i}$, so the term "correct belief at information set $h_j$" is not appropriate; throughout this paper, we use it to mean "belief that is correct at $h_j$" in order to simplify exposition.

specified above (or in Table 1) are incorrect only in that player 1 believes 2 will play $R_2$ at $h_2$ and player 2 believes 3 will play $L_3$ at $h_3$. These incorrect beliefs hold in a consistent self-confirming equilibrium because $h_2$ is not included in $H(R_1)$, and $h_3$ is not included in $H((L_2, r_2))$. Thus $s^*$ is a consistent self-confirming equilibrium. Moreover, $s^*$ has independent, unitary beliefs, because correlations are not allowed in each player's belief, and beliefs are concentrated on singletons.

Next, we show that $(R_1, r_2)$ cannot be played in a Nash equilibrium. To see this, suppose the contrary, i.e., that $(R_1, r_2)$ is played in a Nash equilibrium. If player 3 played $R_3$ with a probability greater than 1/3, then player 2 would take $l_2$ with probability 1. So player 3 must be playing $L_3$ with a probability at least 2/3. This means that if player 1 plays $L_1$, he obtains at least 2 as his payoff. Thus no matter how player 3 plays, at least one of players 1 and 2 has an incentive to deviate. This means that $(R_1, r_2)$ cannot be played in a Nash equilibium.[8]

In FL's proof of their Theorem 4, they construct a strategy profile $\pi'$ that is supposed to be a Nash equilibrium. In our example, $\pi'$ is $(R_1, (R_2, r_2), L_3)$, but this is not a Nash equilibrium since player 1 would have an incentive to deviate.[9]

The example also establishes that Theorems 1 and 3 of FL are incorrect as well, and the proof of their Theorem 2 needs a correction. Theorem 1 claims that, even if we relax condition (ii) for consistent self-confirming equilibrium by allowing beliefs to also be incorrect at information sets that cannot be reached when opponents' equilibrium strategies are fixed, the set of possible strategy profiles does not change. In our example, strategy profile $(R_1, (L_2, r_2), L_3)$ is not a consistent self-confirming

[8]It would be easy to see that this example holds for an open set of payoffs around the payoffs we give. Thus the example is not a trivial one.

[9]There is an illogical jump when FL claim "$u_i(s_i, \pi'_{-i}) = u_i(s_i, \pi^i_{-i})$" holds for all $s_i \in S_i$. In fact, it is satisfied only for all $s_i \in \text{support}(\sigma_i)$. Moreover, $\pi'$ is not well defined for information sets which are reached only by deviations made by more than one player.

equilibrium since player 1, being restricted to believe that player 3 will play $L_3$, has an incentive to deviate. On the other hand, this strategy profile is allowed in the relaxed condition: Since player 2's strategy makes $h_3$ unreachable, 1's belief about 3's strategy can be arbitrary.[10]

Theorem 3 claims that, for each consistent self-confirming equilibrium of a game whose information sets are ordered by precedence, there is an equivalent extensive-form correlated equilibrium of Forges (1986). It is straightforward to check that $s^*$ is not an extensive-form correlated equilibrium.[11]

Finally, Theorem 2 claims that, in games with observed deviators, self-confirming equilibria are consistent self-confirming. This claim itself is correct, but FL's proof is incorrect because it uses the result of Theorem 1. The claim is a consequence of our Proposition 2.

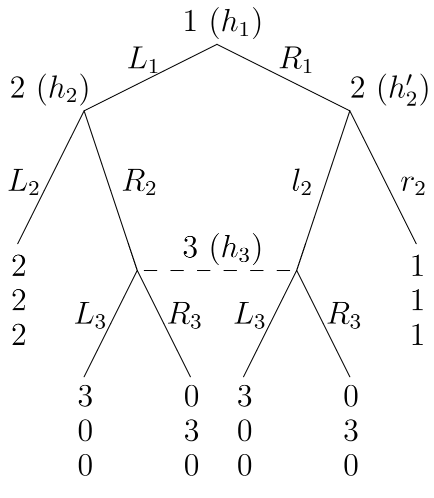


Figure 1

[10]Theorem 1 can be modified without loss of the intuition by slightly modifying one of its conditions. Specifically, replace condition (iii) of the Definition in Section 4 with condition (iii') as follows:

Condition (iii'): $\mu_i[\{\pi_{-i}|\pi_j(h_j) = \hat{\pi}_j(h_j|\sigma_j)\}] = 1 \quad \forall j \neq i,\ \forall h_j \in H_i(s_i, \sigma_{-i}) \cap H(\sigma_j)$.

[11]Theorem 3 and its corollary can be shown to be true by replacing consistency with strong consistency, which we will define in Section 4.

| | player 1 | player 2 | player 3 |
|---|---|---|---|
| equilibrium strategy profile $s^*$ | $R_1$ | $L_2, r_2$ | $R_3$ |
| player 1's belief | $\cdot$ | $R_2, r_2$ | $R_3$ |
| player 2's belief | $R_1$ | $\cdot$ | $L_3$ |
| player 3's belief | $R_1$ | $L_2, r_2$ | $\cdot$ |

Table 1

## 4. STRONGLY CONSISTENT SELF-CONFIRMING EQUILIBRIUM AND NASH EQUILIBRIUM

To define the equilibrium concept that enables us to rule out non-Nash outcomes, we need one more piece of notation. Let

$$H_i(s_i^*, \sigma_{-i}) = \left\{ h \;\middle|\; h \in H(s_i) \text{ for some } s_i \in S_i \; s.t. \; s_i(h') = s_i^*(h') \; \forall h' \in H(s_i^*, \sigma_{-i}) \right\}$$

be the set of *action-possible information sets* for player $i$ under $(s_i^*, \sigma_{-i})$, that is, the set of information sets that can be reached when player $i$ conforms to $s_i^*$ at nodes that are reached under $(s_i^*, \sigma_{-i})$. Note that the action-possible information sets for a player are determined not only by his own strategy but also by his opponents' strategies.

**Definition** Profile $\sigma \in \Sigma$ is a *strongly consistent self-confirming equilibrium* if $\forall i \in N$, $\forall s_i \in \text{support}(\sigma_i)$, $\exists \mu_i$ *s.t.*

(i) $s_i$ maximizes $u_i(\cdot, \mu_i)$, and

(iii) $\mu_i[\{\pi_{-i} | \pi_j(h_j) = \hat{\pi}_j(h_j|\sigma_j)\}] = 1 \quad \forall j \neq i$, $\forall h_j \in H_i(s_i, \sigma_{-i})$.

**Proposition 1** *Every strongly consistent self-confirming equilibrium with independent, unitary beliefs is equivalent to a Nash equilibrium.*

The difference between consistency and strong consistency can be best seen in the example in the previous section: $(R_1, r_2)$ is not played in any strongly consistent self-confirming equilibrium: If it were, then player 2 must have the correct belief at $h_3$. This is because $h_3$ can be reached by the action combination $(L_1, R_2)$, which does not contradict 2's on-path play, namely $r_2$. Because player 1 must also have the correct belief at $h_3$, players 1 and 2's beliefs about the strategy of player 3 must coincide. This implies $(R_1, r_2)$ cannot satisfy the best-response condition (condition (i) of the Definition), as we have seen already.

A strongly consistent self-confirming equilibrium can have a non-Nash outcome.[12] This is immediate because Proposition 2 below ensures that strongly consistent self-confirming equilibrium reduces to self-confirming equilibrium in games with observed deviators.

**Proposition 2** *In games with observed deviators, hence a fortiori in two-player games, self-confirming equilibria are strongly consistent self-confirming.*

We omit the proof for this proposition. It is just a matter of showing that in games with observed deviators, action-possible information sets for player $i$ that lie off the equilibrium path are not relevant to him.[13]

This establishes that the following result from FL holds without any modification as a corollary of Proposition 1.

**Corollary** *In games with observed deviators, and hence a fortiori in two-player games, every self-confirming equilibrium with independent, unitary beliefs is equivalent to a Nash equilibrium.*

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[12] Examples where outcomes (and hence strategy profiles) arise which are not Nash but strongly consistent self-confirming are given in FL. (See Examples 2, 3, and 4 of FL.)

[13] Notice that the game in Figure 1 does not have observed deviators. This is because player 3 cannot tell which of players 1 and 2 has deviated from $(R_1, (R_2, r_2))$ when he gets his turn to move at $h_3$. Thus the assumption of the proposition fails, allowing for the difference between self-confirming equilibrium and strong consistent self-confirming equilibrium, which we have seen already.

Now, we are going to prove Proposition 1. Intuitively, the proof of the main theorem is as follows: Fix a strongly consistent self-confirming equilibrium, $\sigma$, and construct a new strategy profile, $\sigma'$, as follows: $\hat{\pi}_k(\sigma'_k)(h_k)$ is how player $i$ believes $k$ will play at the information set $h_k$ if $h_k$ is relevant to player $i$. For an information set which is irrelevant to all the players, the strategy is specified arbitrarily by how player $j$ actually plans to play. This construction is well defined because strong consistency ensures, as we will see, that if an information set $h_k$ is relevant to both players $i$ and $j$, they have the correct beliefs at $h_k$ so that they have the same beliefs at $h_k$. Thus $\sigma'$ is a Nash equilibrium because $\sigma'_{-i}$ is constructed according to player $i$'s belief $\mu_i$ whenever an information set in question is relevant to $i$, and $i$ takes a best response against $\mu_i$ by the condition (i) of the Definition. A formal proof is given below.

**Proof of Proposition 1**

Let $\sigma$ be a strongly consistent self-confirming equilibrium with independent, unitary beliefs. The condition of unitary beliefs ensures that a single belief rationalizes all $s_i \in \text{support}(\sigma_i)$. For each player $i$, take one such belief, denoted $\mu_i$.

Before proceeding, we need one more piece of notation: $\pi_{-i}^{\mu_i} \in \Pi_{-i}$ is defined by $\pi_{-i}^{\mu_i} = \times_{j \neq i} \pi_j^{\mu_i}$ where for each $h_j \in H_j$ and each $a_j \in A(h_j)$:

$$\pi_j^{\mu_i}(h_j)(a_j) = \int_{\Pi_j} \pi_j(h_j)(a_j)\mu_i(d\pi_j \times \Pi_{-i,j}).$$

We now construct each player $k$'s behavior strategy $\pi'_k$ by the rule[14]:

$$\begin{cases} \pi'_k(h_k) = \pi_k^{\mu_i}(h_k) & \text{if} \quad \exists i \in N, i \neq k \quad \text{s.t.} \quad h_k \in R_i(\mu_i) \\ \pi'_k(h_k) = \hat{\pi}_k(\sigma_k)(h_k) & \text{if} \quad h_k \in H \setminus \bigcup_{i \in N \setminus \{k\}} R_i(\mu_i). \end{cases} \tag{1}$$

[14]This construction is different from the original in FL.

The construction of $\pi'_k$ in (1) is well defined. To see this, first observe that $R_i(\mu_i) \subseteq \bigcup_{s_i \in \text{support}(\sigma_i)} H_j(s_j, \sigma_{-j})$ holds. This is because player $i$'s belief about $j$'s strategy is correct at on-path information sets, by condition (iii) of the Definition. By the condition of unitary beliefs, this implies that player $j$ has the correct belief at all $h_k \in R_i(\mu_i)$. Similarly, player $i$ has the correct belief at all $h_k \in R_j(\mu_j)$. Thus if $h_k \in R_i(\mu_i)$ and $h_k \in R_j(\mu_j)$, $i \neq k \neq j$, the beliefs of players $i$ and $j$ about $k$'s strategy at the information set $h_k$ are correct, so in particular $\pi_k^{\mu_i}(h_k) = \pi_k^{\mu_j}(h_k)$ holds.

Now, construct each player $i$'s strategy $\sigma'_i$ and belief $\mu'_i$ by

$$\sigma'_i = \hat{\sigma}_i(\pi'_i), \qquad \mu'_i[\{\pi'_{-i}\}] = 1, \tag{2}$$

where $\hat{\sigma}_i(\pi_i)$ is a mixed strategy induced by (giving the same outcome as) $\pi_i$. We will show that this $\sigma'$ is a Nash equilibrium.

Because of the condition of independent beliefs, from the Lemma in the Appendix, for all $s'_i \in S_i$,

$$p(\cdot|s'_i, \pi_{-i}^{\mu_i}) = p(\cdot|s'_i, \mu_i), \tag{3}$$

$$u_i(s'_i, \pi_{-i}^{\mu_i}) = u_i(s'_i, \mu_i), \tag{4}$$

$$H(\pi_{-i}^{\mu_i}) \setminus H_i = R_i(\mu_i). \tag{5}$$

Now define $\mu''_i$ such that

$$\mu''_i(\{\pi_{-i}^{\mu_i}\}) = 1. \tag{6}$$

Then from (5) and (6),

$$R_i(\mu_i) = R_i(\mu''_i). \tag{7}$$

For $Q \subseteq H_{-i}$, define $\Pi_{-i}^Q = \times_{h \in Q} \Delta(A(h))$ and $\Pi_{-i}^{\sim Q} = \times_{h \in H_{-i} \setminus Q} \Delta(A(h))$, and then define $\mu_i^Q$ to satisfy $\mu_i^Q(B_{-i}^Q) = \mu_i(B_{-i}^Q \times \Pi_{-i}^{\sim Q})$ for every $B_{-i}^Q \subseteq \Pi_{-i}^Q$.

By (1), (2), and (6),

$$\mu_i'^{R_i(\mu_i)} = \mu_i''^{R_i(\mu_i)}.$$

From this and (7), we can apply Lemma 1 of FL[15] to have for all $s_i' \in S_i$,

$$p(\cdot|s_i', \mu_i') = p(\cdot|s_i', \mu_i'') \text{ and } u_i(s_i', \mu_i') = u_i(s_i', \mu_i''),$$

which mean, according to (2) and (6), for all $s_i' \in S_i$,

$$p(\cdot|s_i', \pi_{-i}') = p(\cdot|s_i', \pi_{-i}^{\mu_i}) \text{ and } u_i(s_i', \pi_{-i}') = u_i(s_i', \pi_{-i}^{\mu_i}). \tag{8}$$

From (4) and (8), we have for all $s_i' \in S_i$,

$$u_i(s_i', \pi_{-i}') = u_i(s_i', \mu_i).$$

Because of condition (i) of the Definition, for all $s_i \in \text{support}(\sigma_i)$, for all $s_i' \in S_i$,

$$u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i).$$

From the above two expressions and (2), we obtain: $\forall i \in N$, $\forall s_i \in \text{support}(\sigma_i)$, for all $s_i' \in S_i$,

$$u_i(s_i, \sigma_{-i}') \geq u_i(s_i', \sigma_{-i}').$$

This inequality implies that $\sigma'$ is a Nash equilibrium if we establish that $\sigma$ and $\sigma'$ are equivalent, because then we can replace "support$(\sigma_i)$" in the above inequality by "support$(\sigma_i')$".[16] Thus to conclude the proof, it now suffices to show

[15] Lemma 1 of FL states that: "If $\mu_i$ and $\hat{\mu}_i$ are two distributions on $\Pi_{-i}$ such that $\mu_i^{R(\mu_i)} = \hat{\mu}_i^{R(\mu_i)}$, then (a) $R(\mu_i) = R(\hat{\mu}_i)$, and (b) $u_i(s_i, \mu_i) = u_i(s_i, \hat{\mu}_i)$ for all $s_i$." Also $p(\cdot|s_i, \mu_i) = p(\cdot|s_i, \hat{\mu}_i)$ for all $s_i \in S_i$ is implicit in this result.

[16] To see this, first note that fixing $\sigma_{-i}'$, "$\forall s_i \in \text{support}(\sigma_i)$" in the last inequality can be replaced by "$\forall s_i$ s.t. $\left[\exists s_i^* \in \text{support}(\sigma_i) \text{ s.t. } [\forall h_i \in H(s_i^*, \sigma_{-i}'),\, s_i(h_i) = s_i^*(h_i)]\right]$". Now, we have that if a) $\exists s_i^* \in \text{support}(\sigma_i')$ s.t. $[\forall h_i \in H(s_i^*, \sigma_{-i}'),\, s_i(h_i) = s_i^*(h_i)]$," then b) $\exists s_i^* \in \text{support}(\sigma_i)$ s.t. $[\forall h_i \in H(s_i^*, \sigma_{-i}'),\, s_i(h_i) = s_i^*(h_i)]$" because if b) doesn't hold, we have $Z(s_i) \setminus Z(\sigma_i, \sigma_{-i}') \neq \emptyset$. But this implies $Z(s_i) \setminus Z(\sigma_i', \sigma_{-i}') \neq \emptyset$ by the assumption that $\sigma$ is equivalent to $\sigma'$. But this contradicts a). So we can replace "$\forall s_i \in \text{support}(\sigma_i)$" by "$\forall s_i$ s.t. $\left[\exists s_i^* \in \text{support}(\sigma_i') \text{ s.t. } [\forall h_i \in H(s_i^*, \sigma_{-i}'),\, s_i(h_i) = s_i^*(h_i)]\right]$". A special case of this is "$\forall s_i \in \text{support}(\sigma_i')$".

that $\sigma$ and $\sigma'$ are equivalent. To see this, first observe that, for each player $i$, condition (iii) and $H(s_i, \sigma_{-i}) \subseteq H_i(s_i, \sigma_{-i})$ imply that, for all $s_i' \in \text{support}(\sigma_i)$,

$$p(\cdot|s_i', \sigma_{-i}) = p(\cdot|s_i', \mu_i).$$

From this and (2), (3) and (8), we obtain, for all $s_i' \in \text{support}(\sigma_i)$,

$$p(\cdot|s_i', \sigma_{-i}) = p(\cdot|s_i', \sigma_{-i}')$$

This equality means that $\sigma$ and $\sigma'$ are equivalent. This concludes the proof. (Q.E.D.)

# Appendix

**Lemma** *If profile $\sigma \in \Sigma$ has independent beliefs, for all $i \in N$, $s_i \in$ support$(\sigma_i)$, and the associated belief $\mu_i$, then $p(\cdot|s_i', \pi_{-i}^{\mu_i}) = p(\cdot|s_i', \mu_i)$, $u_i(s_i', \pi_{-i}^{\mu_i}) = u_i(s_i', \mu_i)$, and $H(\pi_{-i}^{\mu_i}) \setminus H_i = R_i(\mu_i)$ hold for all $s_i' \in S_i$.*

**Proof of Lemma**

Define $h(a_j) = A^{-1}(a_j)$ to be the information set where the action $a_j$ is possible. The path of actions to $z \in Z$, $\tilde{a}(z)$, is the set of actions which are necessarily taken to get to the terminal node $z$. Let $Z(s_i)$ be the set of the terminal nodes reachable under $s_i$; For all $z \in Z$,

$$\begin{aligned}
p(z|s_i', \pi_{-i}^{\mu_i}) =& \big|\{z\} \cap Z(s_i')\big| \cdot \prod_{a_j \in \tilde{a}(z), j \neq i} \left( \int_{\Pi_j} \pi_j(h(a_j))(a_j) \mu_i(d\pi_j \times \Pi_{-i,j}) \right) \\
=& \big|\{z\} \cap Z(s_i')\big| \cdot \int_{\Pi_I} \cdots \int_{\Pi_{i+1}} \int_{\Pi_{i-1}} \cdots \int_{\Pi_1} \left( \prod_{a_j \in \tilde{a}(z), j \neq i} \pi_j(h(a_j))(a_j) \right) \mu_i(\underset{j \neq i}{\times} d\pi_j) \\
=& \big|\{z\} \cap Z(s_i')\big| \cdot \int_{\Pi_{-i}} \left( \prod_{a_j \in \tilde{a}(z), j \neq i} \pi_j(h(a_j))(a_j) \right) \mu_i(d\pi_{-i}) \\
=& \int_{\Pi_{-i}} p(z|s_i', \pi_{-i}) \mu_i(d\pi_{-i}) \\
=& \, p(z|s_i', \mu_i)
\end{aligned}$$

In particular, the second equality follows from the condition of independent beliefs. Thus under this condition, given $s_i'$ and $\mu_i$, the distribution over the terminal nodes when players $j \neq i$ follow $\pi_{-i}^{\mu_i}$ is identical to what $i$ believes in his belief $\mu_i$. The other two equations follow, that is, $u_i(s_i', \pi_{-i}^{\mu_i}) = u_i(s_i', \mu_i)$ and $H(\hat{\sigma}_{-i}(\pi_{-i}^{\mu_i})) \setminus H_i = R_i(\mu_i)$ hold. (Q.E.D.)

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