

For Online Publication

Online Appendix

Anything Goes in Squid Game

Sequential Voting with Informed and Uninformed Voters

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This Online Appendix consists of four sections. Appendix B complements the discussion in Section 5.2 by considering a version of an incentive to end the game and showing that long voting occurs in the unique equilibrium. In Appendix C, we consider the environment with asymmetry analyzed in Section 6.1 and solve for the welfare bounds of the inconsistent strategy equilibria in the general asymmetric model. Appendix D provides a complete proof of Proposition 3, which is an adaptation of other proofs as discussed in Appendix A.7. In Appendix E, we discuss two examples for the model with abstention in Section 6.3.

B Further Discussion for Section 5.2

In Section 5.2, we considered incentives not to end the game. Here, we provide a model with incentives to end the game, and show a long voting result. We note that the objective of this section is not to declare that long voting is a ubiquitous result. Rather, we would like to point out that the long voting result is not an artifact that only results from the incentive not to end.

We consider the following model. Suppose that, when voter k does not receive a signal, independently with probability $\eta > 0$, she is a *decisive type* that receives additional payoff $D > 1$ from voting for X when the history has n times of X .

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Suppose D does not depend on the action (thus, the last voter $2n + 1$ would simply ignore this additional D payoff). Call this game the *decisive-type game*.

Theorem 8. *In any decisive-type game, there is no pure consistent strategy equilibrium such that the game ends before reaching voter $2n$ when no voter receives a signal.*

Proof. The proof closely follows that of Theorem 5. Fix a pure consistent strategy equilibrium. Suppose that if no voter gets a signal and no voter is a decisive type, on the path of play, k is the voter who ends the game and $k < 2n$. Suppose without loss of generality that k plays A on the default sequence.

Voter k has seen n times of action A and $k - n - 1$ times of action B . Let l be the number of voters before k who has seen n times of action A . The posterior on α is thus

$$P(\alpha) = \frac{(1 - \varepsilon)^{k-n}}{(1 - \varepsilon)^{k-n} + (1 - \varepsilon)^{n+1}} = \frac{1}{1 + (1 - \varepsilon)^{2n+1-k}}.$$

If voter k plays A , then the game ends and her expected payoff becomes $P(\alpha)$.

If voter k votes for B instead, then by Bayes rule each subsequent voter who receives her turn to vote and receives no signal assigns probability 1 to state β . This implies that she votes for B because voting for A results in the expected payoff of 0 while voting for B ensures a strictly positive payoff due to the event in which all the subsequent voters receive signal b .

Let q be the probability that voter $2n + 1$ assigns to A when he does not receive a signal. Then, k 's payoff if she plays B is

$$P(\alpha) [1 - (1 - \eta)^{2n-k} (1 - \varepsilon)^{2n+1-k} (1 - q)] + (1 - P(\alpha)) [(1 - (1 - \varepsilon)\eta)^{2n-k} (\varepsilon + (1 - \varepsilon)(1 - q))].$$

Let this be $f(q)$. Note that

$$\begin{aligned} f(0) &= P(\alpha) [1 - (1 - \eta)^{2n-k} (1 - \varepsilon)^{2n+1-k}] + (1 - P(\alpha)) (1 - (1 - \varepsilon)\eta)^{2n-k} \\ &> P(\alpha) [1 - (1 - \eta)^{2n-k} (1 - \varepsilon)^{2n+1-k}] + (1 - P(\alpha)) (1 - \eta)^{2n-k} \\ &= \frac{1 - (1 - \eta)^{2n-k} (1 - \varepsilon)^{2n+1-k} + (1 - \varepsilon)^{2n+1-k} (1 - \eta)^{2n-k}}{1 + (1 - \varepsilon)^{2n+1-k}} \\ &= \frac{1}{1 + (1 - \varepsilon)^{2n+1-k}} = P(\alpha). \end{aligned}$$

Also,

$$f(1) = P(\alpha) + (1 - P(\alpha))(1 - (1 - \varepsilon)\eta)^{2n-k}\varepsilon > P(\alpha).$$

So $f(q) > P(\alpha)$ for both $q = 0, 1$. This means that k would be better off playing B than A , contradicting the assumption that k plays A in the fixed equilibrium. The proof is complete. \square

C Inconsistent Strategies under General Asymmetric Voting

Consider the general voting model with $(p, \varepsilon_a, \varepsilon_b, K, L, \gamma_\alpha, \gamma_\beta)$. Let

$$\bar{R} := \gamma_\alpha p + \gamma_\beta(1 - p) - \min\{\gamma_\alpha p(1 - \varepsilon_a)^N, \gamma_\beta(1 - p)(1 - \varepsilon_b)^N\}.$$

Note that in the symmetric case (i.e., $p = \frac{1}{2}$, $\varepsilon_a = \varepsilon_b = \varepsilon$, $K = L = n + 1$, and $\gamma_\alpha = \gamma_\beta = 1$), \bar{R} reduces to \bar{R}_n that we defined in Section 4.1.

Consider a social choice function $f : \{a, b, \emptyset\}^{2n+1} \rightarrow \Delta(\{A, B\})$ that returns a probability distribution over outcomes for each realization of signals of all the N voters. Note that for any strategy profile σ in our original game, there is a social choice function f that achieves the same distribution over outcomes conditional on any realization of the state and the signals. Let F be the space of all social choice functions. Notice that a social choice function $f \in F$ determines the ex ante expected payoff to each voter. Let this payoff be $R(f)$.

Theorem 9. *Consider the general voting model with $(p, \varepsilon_a, \varepsilon_b, K, L, \gamma_\alpha, \gamma_\beta)$.*

1. *For any $(p, \varepsilon_a, \varepsilon_b, K, L, \gamma_\alpha, \gamma_\beta)$, there is an equilibrium σ that achieves the payoff \bar{R} .*
2. *For all $f \in F$, $R(f) \leq \bar{R}$.*
3. (a) *If $K \geq 3$, then there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon_a \in (0, \bar{\varepsilon})$, there is an equilibrium that achieves the payoff of $p\varepsilon_a + 1 - p$.*
 (b) *If $L \geq 3$, then there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon_b \in (0, \bar{\varepsilon})$, there is an equilibrium that achieves the payoff of $p + (1 - p)\varepsilon_b$.*

4. The expected payoff from any pure strategy equilibrium is at least $\min\{p + (1 - p)\varepsilon_b, p\varepsilon_a + 1 - p\}$, that is, it is

$$\begin{cases} p + (1 - p)\varepsilon_b & \text{if } p(1 - \varepsilon_a) < (1 - p)(1 - \varepsilon_b) \\ p\varepsilon_a + 1 - p & \text{if } p(1 - \varepsilon_a) > (1 - p)(1 - \varepsilon_b) \end{cases}.$$

Recall that the welfare for the consistent strategy equilibria depend on K and L (Corollary 2). Parts 2 and 4 of the above theorem show that the welfare bounds for inconsistent strategy equilibria do not depend on K or L .

*Proof. **Part 1:***

Fix $(p, \varepsilon_a, \varepsilon_b, \gamma_\alpha, \gamma_\beta)$. Without loss of generality, assume $\gamma_\alpha p(1 - \varepsilon_a)^N \leq \gamma_\beta(1 - p)(1 - \varepsilon_b)^N$. For any K and L , the following strategy profile $\bar{\sigma}$ is an equilibrium and achieves the ex ante payoff \bar{R} .

- For any $k = 1, \dots, K - 1$, voter k chooses B if she receives a ; otherwise she plays A .
- For any $k = K, \dots, N$, voter k uses a consistent strategy. If he does not receive a signal, then (i) k plays B if he observes that voters $1, \dots, K - 1$ play A and voters $K, \dots, k - 1$ play B , and (ii) k plays A otherwise.

Note that this is a pure strategy profile and is not consistent. Its default sequence is $(A, \dots, A, B, \dots, B)$, where A continues $K - 1$ times and then B continues L times. To see that each voter chooses a best response, observe first that it is immediate that any voter who receives a signal is taking a best response. Also, observe that once there is a deviation from the default sequence, then voters take a best response given a belief that the first deviator from the default sequence has received a signal and followed the equilibrium strategy (such a belief can easily be shown to be consistent in the sense of Kreps and Wilson (1982)). So suppose that the voters 1 through $k - 1$ have followed the default sequence, and suppose that voter k receives no signal. The posterior on α is $P(\alpha) = \frac{p(1 - \varepsilon_a)^k}{p(1 - \varepsilon_a)^k + (1 - p)(1 - \varepsilon_b)}$. If k follows the specified strategy, then her payoff is

$$\gamma_\alpha P(\alpha)(1 - (1 - \varepsilon_a)^{N - k}) + \gamma_\beta(1 - P(\alpha)) \cdot 1. \quad (9)$$

If k deviates, her payoff is $\gamma_\alpha P(\alpha)$ if $k \geq K$. In this case, k is taking a best response

if and only if

$$\begin{aligned}
& \gamma_\alpha P(\alpha)(1 - (1 - \varepsilon_a)^{N-k}) + \gamma_\beta(1 - P(\alpha)) \geq \gamma_\alpha P(\alpha) \\
& \iff \gamma_\beta(1 - P(\alpha)) \geq \gamma_\alpha P(\alpha)(1 - \varepsilon_a)^{N-k} \\
& \iff \gamma_\beta(1 - p)(1 - \varepsilon_b) \geq \gamma_\alpha p(1 - \varepsilon_a)^k(1 - \varepsilon_a)^{N-k} \\
& \iff \gamma_\beta(1 - p)(1 - \varepsilon_b) \geq \gamma_\alpha p(1 - \varepsilon_a)^N.
\end{aligned} \tag{10}$$

Since we assumed $\gamma_\alpha p(1 - \varepsilon_a)^N \leq \gamma_\beta(1 - p)(1 - \varepsilon_b)^N$, (10) holds if $\gamma_\beta(1 - p)(1 - \varepsilon_b) \geq \gamma_\beta(1 - p)(1 - \varepsilon_b)^N$. But this holds because $N \geq 1$. Hence, k is taking a best response.

If $k \leq K - 1$, then her payoff from the deviation is

$$\gamma_\alpha P(\alpha) + \gamma_\beta(1 - P(\alpha))\tilde{Y},$$

where $\tilde{Y} := \text{Prob}(L - 1 \text{ or more signals out of } N - 1 - k)$.

Thus, the payoff from playing A is no less than the payoff from playing B if and only if

$$\gamma_\alpha P(\alpha)(1 - (1 - \varepsilon_a)^{N-k}) + \gamma_\beta(1 - P(\alpha)) \geq \gamma_\alpha P(\alpha) + \gamma_\beta(1 - P(\alpha))\tilde{Y},$$

or

$$P(\alpha) \leq \frac{\gamma_\beta(1 - \tilde{Y})}{\gamma_\beta(1 - \tilde{Y}) + \gamma_\alpha(1 - \varepsilon_a)^{N-k}}.$$

Since $P(\alpha) = \frac{p(1 - \varepsilon_a)^k}{p(1 - \varepsilon_a)^k + (1 - p)(1 - \varepsilon_b)}$, this is equivalent to:¹

$$\frac{p(1 - \varepsilon_a)^k}{(1 - p)(1 - \varepsilon_b)} \leq \frac{\gamma_\beta(1 - \tilde{Y})}{\gamma_\alpha(1 - \varepsilon_a)^{N-k}},$$

or

$$\frac{p}{1 - p} \frac{(1 - \varepsilon_a)^{N-1}}{(1 - \varepsilon_b)} \frac{\gamma_\alpha}{\gamma_\beta} \leq 1 - \tilde{Y}. \tag{11}$$

Now, note that \tilde{Y} is equal to Y defined in the proof of Proposition 3 where we set $m = 1$ and $i = k + 1$, and the roles of A and B (and thus the roles of K and L , and of ε_a and ε_b) are reversed. Since that proof shows $(1 - \varepsilon_a)^{L+m-i} \leq 1 - Y$, we have

$$(1 - \varepsilon_b)^{K+1-(k+1)} \leq 1 - \tilde{Y},$$

¹See footnote 23 of the main text for this derivation.

or

$$(1 - \varepsilon_b)^{K-k} \leq 1 - \tilde{Y}.$$

Since $k \geq 1$ and $K \leq N$, we have $(1 - \varepsilon_b)^{N-1} \leq (1 - \varepsilon_b)^{K-k} \leq 1 - \tilde{Y}$, which implies $\frac{p}{1-p} \frac{(1-\varepsilon_a)^N}{1-\varepsilon_b} \frac{\gamma_\alpha}{\gamma_\beta} \leq 1 - \tilde{Y}$ because we assumed $\gamma_\alpha p(1 - \varepsilon_a)^N \leq \gamma_\beta(1 - p)(1 - \varepsilon_b)^N$. This shows that k is taking a best response when $k \leq K - 1$.

Finally, since the payoff is 0 under $\bar{\sigma}$ if and only if the state is α and no one receives a signal, the expected payoff is $\gamma_\alpha p + \gamma_\beta(1 - p) - \gamma_\alpha p(1 - \varepsilon_a)^N$.

Part 2:

Fix any $f \in F$. Let $f(\emptyset, \dots, \emptyset)(A) = q$. Then, conditional on the signal realization $(\emptyset, \dots, \emptyset)$, the voters' payoff is $\gamma_\alpha q$ under state α and $\gamma_\beta(1 - q)$ under state β . The probability that the signal profile $(\emptyset, \dots, \emptyset)$ realizes is $(1 - \varepsilon_a)^N$ under state α and $(1 - \varepsilon_b)^N$ under state β . Hence, the ex ante expected payoff is at most

$$\begin{aligned} & \gamma_\alpha p(1 - (1 - \varepsilon_a)^N) + \gamma_\beta(1 - p)(1 - (1 - \varepsilon_b)^N) + \max_{q \in [0,1]} (\gamma_\alpha p q(1 - \varepsilon_a)^N + \gamma_\beta(1 - p)(1 - q)(1 - \varepsilon_b)^N) \\ &= (\gamma_\alpha p + \gamma_\beta(1 - p) - \gamma_\alpha p(1 - \varepsilon_a)^N - \gamma_\beta(1 - p)(1 - \varepsilon_b)^N) + \max\{\gamma_\alpha p(1 - \varepsilon_a)^N, \gamma_\beta(1 - p)(1 - \varepsilon_b)^N\} \\ &= \gamma_\alpha p + \gamma_\beta(1 - p) + \max\{\gamma_\alpha p(1 - \varepsilon_a)^N - \gamma_\alpha p(1 - \varepsilon_a)^N - \gamma_\beta(1 - p)(1 - \varepsilon_b)^N, \\ & \quad \gamma_\beta(1 - p)(1 - \varepsilon_b)^N - \gamma_\alpha p(1 - \varepsilon_a)^N - \gamma_\beta(1 - p)(1 - \varepsilon_b)^N\} \\ &= \gamma_\alpha p + \gamma_\beta(1 - p) + \max\{-\gamma_\beta(1 - p)(1 - \varepsilon_b)^N, -\gamma_\alpha p(1 - \varepsilon_a)^N\} \\ &= \gamma_\alpha p + \gamma_\beta(1 - p) - \min\{\gamma_\alpha p(1 - \varepsilon_a)^N, \gamma_\beta(1 - p)(1 - \varepsilon_b)^N\} \\ &= \bar{R}. \end{aligned}$$

Part 3:

We only provide the proof for part (3b). The proof for part (3a) is symmetric.

Consider the following strategy profile, which we denote by $\underline{\sigma}$.

- Voter $k = 1, \dots, K - 1$: If all the actions observed so far are A , then play A irrespective of the signal. Otherwise, play a consistent strategy in which A is played if no signal is received.
- Voter $k = K, \dots, N - 1$: If all the actions observed so far are n times of A followed by $k - 1 - n$ times of B , then play B irrespective of the signal. Otherwise, play a consistent strategy in which A is played if no signal is received.

- Voters $k = N$: Play a consistent strategy in which A is played if no signal is received.

Note that this strategy profile has the ex ante payoff of $p + (1 - p)\varepsilon_b$. We show that, if $n \geq 2$, $\underline{\sigma}$ constitutes an equilibrium with a belief that, after any deviation, any voter who has not received a signal assigns probability 1 to α (such a belief can easily be shown to be consistent in the sense of Kreps and Wilson (1982)).

Consider voter k . Fix any history of actions by the previous voters. First, suppose that k receives signal a . Then, since the outcome will be A if k follows $\underline{\sigma}_k$, playing $\underline{\sigma}_k$ induces the expected payoff of 1, which is the highest possible payoff in this game. Hence, $\underline{\sigma}_k$ is a best response.

Second, suppose that k receives signal b .

- Suppose that the voters so far have followed $\underline{\sigma}$.
 - If $1 \leq k \leq K - 1$, then if she follows $\hat{\sigma}_k$ then her payoff is ε_b . If instead she plays B then at least $L - 1$ voters from the set of subsequent voters have to receive signal b in order for k to expect the payoff of 1, and otherwise she receives the payoff of 0. Hence, her payoff is $O(\varepsilon_b^{L-1})$. Since $L - 1 \geq 2$, this implies that there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon_b < \bar{\varepsilon}$, following $\underline{\sigma}_k$ is a best response.
 - If $K \leq k \leq N - 1$, then playing A ends the game with outcome A , so it induces the expected payoff of 0. Hence, playing B is a best response.
 - If $k = N$, then playing B induces the payoff of 1 with probability 1, so it is a best response to play B .
- Suppose that there is a voter who has deviated from $\underline{\sigma}$. Then, the play by the subsequent voters will not be affected by the action taken by voter k . Hence, it is a best response for voter k to provide additional vote for B . Therefore, playing B is a best response.

Third, suppose that voter k did not receive a signal.

- Suppose that the voters so far have followed $\underline{\sigma}$. Then, the posterior belief on α is $\frac{1}{2}$.

- Suppose $k \leq K - 1$. If k follows $\underline{\sigma}_k$, then her payoff is $\frac{1}{2} + \frac{1}{2}\varepsilon_b$. If she does not follow $\underline{\sigma}_k$, then her payoff is $\frac{1}{2} + \frac{1}{2}O(\varepsilon_b^{L-1})$. Since $L - 1 \geq 2$, following $\underline{\sigma}_k$ is a best response.
 - Suppose $K \leq k \leq N - 1$. If k follows $\underline{\sigma}_k$, then her payoff is $\frac{1}{2} + \frac{1}{2}\varepsilon_b$. If she does not follow $\underline{\sigma}_k$, then her payoff is $\frac{1}{2}$. Hence, following $\underline{\sigma}_k$ is a best response.
 - Suppose $k = N$. Then, k is indifferent between the two actions. So following $\underline{\sigma}_k$ is a best response.
- Suppose that there is a voter who has deviated from $\underline{\sigma}$ before k 's move. Then, the play by the subsequent voters will not be affected by the action taken by voter k . Hence it is a best response for k to provide additional vote for A because k 's belief assigns probability 1 to α .

This completes the proof of part 3.

Part 4:

The proof we present below is analogous to the one in the symmetric case (the proof of Proposition 2).

Fix a pure strategy equilibrium σ . Take the default sequence of σ . Suppose first that the last voter on this default sequence, whom we denote voter k^* , plays A .

First, we show the following lemma.

Lemma 4. *Suppose that the votes by voters $1, \dots, k^* - 1$ have followed the default sequence of σ . Then, voter k^* votes for B if and only if she receives signal b .*

Proof of Lemma 4. Consider the history in which voters $1, \dots, k^* - 1$ have followed the default sequence of σ . Given the assumption that k^* plays A given no signal, it suffices to check the cases when she receives a signal.

1. First, we show that k^* plays B if she receives signal b . We show this using induction on k . To do this, fix $k' \in \{k^*, \dots, N\}$. Suppose, as an induction hypothesis, that for every $k > k'$, voter k plays B if he receives signal b .² Suppose that voter k' receives signal b . Given this signal, her posterior probability on β is 1. If k' votes for A , then her expected payoff is 0. If k' votes for B , then there

²Note that this hypothesis is vacuously true when $k' = N$.

is a strictly positive probability that all the subsequent voters receive signal b and, in that case, the game ends with outcome B by the induction hypothesis. Hence, her expected payoff is strictly positive. Thus, playing B is a unique best response for k' , and hence, voter k' plays B if she receives signal b . This shows that voter k^* plays B if she receives signal b .

2. Next, we show that k^* plays A if she receives signal a . To see this, suppose to the contrary that k^* plays B if she receives signal a . In this case, the outcome must be A with probability 1 after k^* plays B because otherwise, playing B would give k^* a payoff strictly less than 1, while she would get the payoff of 1 if she played A , making her choice B suboptimal. This implies that the outcome will be A if k^* chooses B when the state is α , no matter what her signal is. Now, suppose that k^* did not receive a signal. Let $P(\alpha)$ be the posterior of k^* at such an information set. Then, the expected payoff of voter k^* is $P(\alpha)$ if she plays A . If she instead plays B , then the outcome will be A with probability 1 if the state is α as we have concluded. The argument in item 1 above implies that if the state is β and all the subsequent voters receive signal b , which happens with probability $\varepsilon^{2n+1-k^*} > 0$, then the outcome is B . Hence, the expected payoff of voter k^* is at least $P(\alpha) + (1 - P(\alpha))\varepsilon^{2n+1-k^*}$ if she plays B , and this is strictly greater than $P(\alpha)$ because $1 - P(\alpha) > 0$ and $\varepsilon > 0$. Hence, playing A is suboptimal for k^* when she does not receive a signal, which contradicts the assumption that she votes for A given no signal. Thus, k^* plays A if she receives signal a .

□

Given Lemma 4, once k^* plays B , the subsequent voters assign posterior probability 1 to state β , so the outcome will be B with probability 1.

Lemma 5. *Under σ , if voter 1 receives signal a , then she expects that the outcome will be A with probability 1.*

Proof of Lemma 5. We use induction. Fix $k \leq k^*$. Suppose as an induction hypothesis that for every $k' \in \{k+1, \dots, k^*\}$, if the actions by voters $1, \dots, k' - 1$ have followed the default sequence and k' receives signal a , then the outcome will be A with probability 1. Then, suppose that the actions by voters $1, \dots, k-1$ have followed the default sequence and k receives signal a . If k plays the action specified in the

default sequence, then either (i) no subsequent voters receive a signal, or (ii) there is at least one subsequent voter who receives signal a . In case (i), by the definition of the default sequence, the outcome will be A with probability 1. In case (ii), the first voter who receives signal a expects that the outcome will be A , which follows from the induction hypothesis. Hence, the outcome will be A in either case. Therefore, k 's expected payoff is 1 if k plays the action specified in the default sequence. Since k under σ must be doing at least as good as playing any action, this implies that, under σ , k expects the payoff of 1, and hence she expects that the outcome will be A with probability 1. This completes the induction argument. Therefore, we have shown that, if voter 1 receives signal a , then she expects that the outcome will be A with probability 1. \square

Lemma 6. *Under σ , if any voter receives signal b when the votes so far followed the default sequence, then she expects that the outcome will be B with probability at least ε_b .*

Proof of Lemma 6. We use induction. Fix $k \leq k^* - 1$. Suppose as an induction hypothesis that for every $k' \in \{k+1, \dots, k^* - 1\}$, if the actions by voters $1, \dots, k' - 1$ have followed the default sequence and k' receives signal b , then the outcome will be B with probability at least ε_b . Then, suppose that the actions by voters $1, \dots, k - 1$ have followed the default sequence and k receives signal b . If k plays the action specified in the default sequence, then either (i) no subsequent voters in $\{k+1, \dots, k^* - 1\}$ receive a signal, or (ii) there is at least one subsequent voter in $\{k+1, \dots, k^* - 1\}$ who receives signal b . In case (i), by Lemma 4, voter k^* will play B if she receives signal b , and thus there is probability ε_b that the outcome will be B as we concluded after Lemma 4. In case (ii), the first voter who receives signal b expects that the outcome will be B with a probability of at least ε_b , which follows from the induction hypothesis. Hence, the outcome will be B with at least ε_b probability in either case. Therefore, k 's expected payoff under σ must be at least ε_b if she plays the action specified in the default sequence. Since k under σ must be doing at least as good as playing any action, this implies that, under σ , k expects a payoff of at least ε_b , and hence she expects that the outcome will be B with probability at least ε_b . This completes the induction argument. Therefore, we have shown that, if any voter receives signal b , then she expects that the outcome will be B with probability at least ε_b . \square

Finally, consider voter 1. If the state is α , then either (i) no voters in $\{1, \dots, k^*\}$

receive a signal, or (ii) there is at least one voter in $\{1, \dots, k^*\}$ who receives signal a . In case (i), the outcome will be A with probability 1 by the choice of the default sequence. In case (ii), we have shown that the outcome will be A with probability 1. Hence, voter 1's expected payoff is 1 conditional on this event.

If the state is β , then either (i) no voters in $\{1, \dots, k^* - 1\}$ receive a signal, or (ii) there is at least one voter in $\{1, \dots, k^* - 1\}$ who receives signal b . In case (i), voter k^* will have seen the history that has followed the default sequence. If she receives no signal, which happens with probability $1 - \varepsilon_b$, then she plays A by assumption, and thus the outcome will be A with probability 1. If she receives signal b , which happens with probability ε_b , then the outcome will be B with probability 1 as we concluded after Lemma 4. Overall, voter 1 expects the probability that the outcome will be B to be ε_b conditional on (i). In case (ii), take the first voter who has received signal b . We have shown that this voter assigns a probability of at least ε_b that the outcome will be B . Thus, conditional on the event that (ii) realizes, voter 1 expects that the probability that the outcome will be B is at least ε_b . Therefore, if the state is β , the outcome will be B with at least ε_b probability.

Overall, since the prior on the states α and β are p and $1 - p$, respectively, voter 1's expected payoff is at least

$$p \cdot 1 + (1 - p) \cdot \varepsilon_b = p + (1 - p)\varepsilon_b.$$

Now, if the last voter on the default sequence plays B , a symmetric proof shows that voter 1's expected payoff is at least $p\varepsilon_a + 1 - p$. Overall, the minimum equilibrium payoff is $\min\{p + (1 - p)\varepsilon_b, p\varepsilon_a + 1 - p\}$.

Since $p + (1 - p)\varepsilon_b < p\varepsilon_a + 1 - p$ if and only if $p(1 - \varepsilon_a) < (1 - p)(1 - \varepsilon_b)$, the proof is complete. \square

D Proof of Proposition 3

Fix the general voting model with $(p, \varepsilon_a, \varepsilon_b, K, L, \gamma_\alpha, \gamma_\beta)$ and a sequence $S \in \mathcal{S}$ that ends with A . To prove the three parts of the proposition, we show the following two claims:

- (A) If $\frac{1-p}{p} \frac{(1-\varepsilon_b)^K}{(1-\varepsilon_a)^L} \frac{\gamma_\beta}{\gamma_\alpha} \leq 1$, then there is a pure consistent strategy equilibrium σ such that S is the default sequence of σ .

- (B) If $\frac{1-p}{p} \frac{(1-\varepsilon_b)^K}{(1-\varepsilon_a)^L} \frac{\gamma_\beta}{\gamma_\alpha} > 1$, then there is no pure consistent strategy equilibrium σ such that S is the default sequence of σ .

By symmetry, these are enough to show the three parts of the proposition.

Proof for claim (A). Fix $k \in \{\min\{K, L\} + 1, \min\{K, L\} + 2, \dots, N\}$ and take a sequence of actions $S^* = (X_1, \dots, X_k) \in \{A, B\}^k$ such that X_k appears exactly K times if $X_k = A$ and L times if $X_k = B$. We consider the case when $X_k = A$. The case for $X_k = B$ is symmetric.

Consider the consistent strategy profile σ satisfying the following conditions:

1. For any $i = 1, \dots, k$, take any sequence $S = (X_1, \dots, X_{i-1}) \in \{A, B\}^{i-1}$. That is, S coincides with S^* before voter i . Then, i chooses the action specified in S^* given no signal, i.e.,

$$\sigma_i(X_1, \dots, X_{i-1}, \emptyset)(X_i) = 1.$$

2. For any $i = 1, \dots, N$ and $i' = 1, \dots, \min\{k, i - 1\}$, take any sequence $S = (X_1, \dots, X_{i'-1}, X'_{i'}, \dots, X'_{i-1}) \in \{A, B\}^{i-1}$ such that $X'_{i'} \neq X_{i'}$. That is, i' is the first voter who does not follow the sequence (X_1, \dots, X_k) . Then, i chooses the same action as $X'_{i'}$ given no signal, i.e.,

$$\sigma_i(S, \emptyset)(X'_{i'}) = 1.$$

Now we check incentives. First, voters who have received a signal take a best response given condition 2 above.

Second, under any histories described in condition 2, by letting i have a belief that the state is α if $X'_{i'} = A$ and β if $X'_{i'} = B$, it is straightforward to see that i is taking a best response.

Third, under any histories described in condition 1, suppose that there have been m times of A 's before i . Voter i 's posterior on α is then

$$P(\alpha) = \frac{p(1 - \varepsilon_a)^{i-m}}{p(1 - \varepsilon_a)^{i-m} + (1 - p)(1 - \varepsilon_b)^{m+1}},$$

and her posterior on β is

$$P(\beta) = 1 - P(\alpha) = \frac{(1-p)(1-\varepsilon_b)^{m+1}}{p(1-\varepsilon_a)^{i-m} + (1-p)(1-\varepsilon_b)^{m+1}}.$$

We consider the following two (exhaustive) cases.

1. Suppose that $X_i = A$.

If i plays A , her payoff is:

$$\gamma_\alpha P(\alpha) + \gamma_\beta(1 - P(\alpha))(1 - (1 - \varepsilon_b)^{K-1-m}).$$

Suppose now that i plays B . If the state is β , then outcome B realizes with probability 1 given that the voters follow σ . If the state is α , the outcome becomes A if and only if at least $K - m$ subsequent voters receive signal a . Thus, her payoff is

$$\gamma_\beta(1 - P(\alpha)) + \gamma_\alpha P(\alpha) \cdot Y,$$

where $Y := \text{Prob}(K - m \text{ or more "a" signals out of } N - i)$.

Thus, the payoff from playing A is no less than the payoff from B if and only if:

$$\gamma_\alpha P(\alpha) + \gamma_\beta(1 - P(\alpha))(1 - (1 - \varepsilon_b)^{K-1-m}) \geq \gamma_\beta(1 - P(\alpha)) + \gamma_\alpha P(\alpha)Y \iff$$

$$P(\alpha)[\gamma_\alpha 1 + \gamma_\beta(1 - \varepsilon_b)^{K-1-m} - \gamma_\alpha Y] \geq \gamma_\beta(1 - \varepsilon_b)^{K-1-m} \iff P(\alpha) \geq \frac{(1 - \varepsilon_b)^{K-1-m}}{(1 - \varepsilon_b)^{K-1-m} + \frac{\gamma_\alpha}{\gamma_\beta}(1 - Y)}.$$

Now, notice that

$$\begin{aligned} P(\alpha) &= \frac{p(1 - \varepsilon_a)^{i-m}}{p(1 - \varepsilon_a)^{i-m} + (1-p)(1 - \varepsilon_b)^{m+1}} \\ &= \frac{(1 - \varepsilon_b)^{K-1-m}}{(1 - \varepsilon_b)^{K-1-m} + \frac{1-p}{p} \frac{(1 - \varepsilon_b)^K}{(1 - \varepsilon_a)^{i-m}}}. \end{aligned}$$

Hence, the payoff from playing A is no less than the payoff from B if and only if:

$$\frac{1-p}{p} \frac{(1 - \varepsilon_b)^K}{(1 - \varepsilon_a)^{i-m}} \leq \frac{\gamma_\alpha}{\gamma_\beta}(1 - Y).$$

Now, recall that we have $\frac{1-p}{p} \frac{(1-\varepsilon_b)^K}{(1-\varepsilon_a)^L} \frac{\gamma_\beta}{\gamma_\alpha} \leq 1$ by assumption. Thus, it suffices to show that

$$\frac{(1-\varepsilon_a)^L}{(1-\varepsilon_a)^{i-m}} \leq 1 - Y,$$

or

$$(1-\varepsilon_a)^{L+m-i} \leq \text{Prob}(K-1-m \text{ or less "a" signals out of } N-i).$$

Now, let $Z_1 := L + m - i$ and $Z_2 := N - i$. Notice that $Z_1 \leq Z_2$ because $Z_2 - Z_1 = N - L - m = K - 1 - m \geq 0$. The payoff from playing A is no less than the payoff from B if and only if:

$$(1-\varepsilon_a)^{Z_1} \leq \text{Prob}(Z_2 - Z_1 \text{ or less "a" signals out of } Z_2). \quad (12)$$

Note that

$$\begin{aligned} & \text{Prob}(Z_2 - Z_1 \text{ or less "a" signals out of } Z_2) \\ & \geq \text{Prob}(\text{The last } Z_1 \text{ voters do not receive an "a" signal}) = (1-\varepsilon_a)^{Z_1}. \end{aligned}$$

Thus, eq. (12) indeed holds, and therefore, the payoff from playing A is no less than the payoff from B .

2. Suppose that $X_i = B$.

First, consider the case when i plays B .

Suppose the state is α , which happens with probability $P(\alpha)$. In this case, i 's playing B ensures that there is probability 1 that A will be chosen as the outcome.

Suppose the state is β , which happens with probability $1 - P(\alpha)$. Since there have been m times of A right after i plays B and there are K times of A in the given sequence, there will be $K - m$ times of A at which the voter, upon receiving signal b , can change the outcome to B . The probability that at least one voter out of $K - m$ voters receives signal b is $1 - (1 - \varepsilon_b)^{K-m}$.

Thus, overall, the payoff from i 's voting for B is

$$\gamma_\alpha P(\alpha) + \gamma_\beta (1 - P(\alpha)) (1 - (1 - \varepsilon_b)^{K-m}).$$

Second, consider the case when i plays A .

If the state is α , then outcome A realizes with probability 1 given that the voters follow σ . If the state is β , the outcome becomes B if and only if at least $L - (i - (m + 1))$ subsequent voters receive signal b . Thus, her payoff is

$$\gamma_\alpha P(\alpha) + \gamma_\beta (1 - P(\alpha)) \hat{Y},$$

where $\hat{Y} := \text{Prob}(L - (i - (m + 1)) \text{ or more "b" signals out of } N - i)$.

Thus, the payoff from playing B is no less than the payoff from A if and only if:

$$\begin{aligned} \gamma_\alpha P(\alpha) + \gamma_\beta (1 - P(\alpha)) (1 - (1 - \varepsilon_b)^{K-m}) &\geq \gamma_\alpha P(\alpha) + \gamma_\beta (1 - P(\alpha)) \hat{Y} \iff \\ (1 - \varepsilon_b)^{K-m} &\leq 1 - \hat{Y}, \end{aligned}$$

or

$$(1 - \varepsilon_b)^{K-m} \leq \text{Prob}(L - (i - m) \text{ or less "b" signals out of } N - i).$$

Now, let $\hat{Z}_1 := K - m$ and $\hat{Z}_2 := N + 1 - i$. Notice that $\hat{Z}_1 \leq \hat{Z}_2$ because $\hat{Z}_2 - \hat{Z}_1 = N + 1 - i - (L - i + (m + 1)) = K - (m + 1) \geq 0$. $\hat{Z}_2 - \hat{Z}_1 = N + 1 - i - (K - m) = L - (i - m) \geq 0$.³ The payoff from playing B is no less than the payoff from A if and only if:

$$(1 - \varepsilon_b)^{\hat{Z}_1} \leq \text{Prob}(\hat{Z}_2 - \hat{Z}_1 \text{ or less "b" signals out of } \hat{Z}_2 - 1). \quad (13)$$

Note that

$$\begin{aligned} &\text{Prob}(\hat{Z}_2 - \hat{Z}_1 \text{ or less "b" signals out of } \hat{Z}_2 - 1) \\ &> \text{Prob}(\hat{Z}_2 - \hat{Z}_1 \text{ or less "b" signals out of } \hat{Z}_2) \\ &\geq \text{Prob}(\text{The last } \hat{Z}_1 \text{ voters do not receive an "b" signal}) \\ &= (1 - \varepsilon_b)^{\hat{Z}_1}. \end{aligned}$$

³There have been $i - m - 1$ times of B before i , and this number is no greater than $L - 1$, that is, $i - m - i \leq L - 1$. This leads to $L - (i - m) \geq 0$.

Thus, eq. (13) indeed holds, and therefore, the payoff from playing B is no less than the payoff from A .

Overall, playing X_i is a best response for voter i . \square

Proof for claim (B). The proof closely follows that of Theorem 5. Fix a pure consistent strategy equilibrium. Suppose that if no voter gets a signal, on the path of play, k is the voter who ends the game. Suppose for contradiction that k plays A on the default sequence.

Voter k has seen $K - 1$ times of action A and $k - K$ times of action B . The posterior on α is thus

$$\begin{aligned} P(\alpha) &= \frac{p(1 - \varepsilon_a)^{k-K+1}}{p(1 - \varepsilon_a)^{k-K+1} + (1 - p)(1 - \varepsilon_b)^K} = \frac{1}{1 + \frac{1-p}{p} \frac{(1-\varepsilon_b)^K}{(1-\varepsilon_a)^{k-K+1}}} \\ &= \frac{1}{1 + \frac{1-p}{p} \left(\frac{1-\varepsilon_b}{1-\varepsilon_a} \right)^K (1 - \varepsilon_a)^{2K-1-k}}. \end{aligned}$$

If voter k plays A , then the game ends and her expected payoff becomes $\gamma_\alpha P(\alpha)$.

If voter k votes for B instead, then by Bayes rule each subsequent voter who receives her turn to vote and receives no signal assigns probability 1 to state β . This implies that she votes for B because voting for A results in the expected payoff of 0 while voting for B ensures a strictly positive payoff due to the event in which all the subsequent voters receive signal b .

Let q be the probability that voter $K + L - 1$ assigns to A when he does not receive a signal. Then, k 's payoff if she plays B is

$$\gamma_\alpha P(\alpha) [1 - (1 - q)(1 - \varepsilon_a)^{K+L-1-k}] + \gamma_\beta (1 - P(\alpha)) [\varepsilon_b + (1 - \varepsilon_b)(1 - q)].$$

Let this be $f(q)$. Note that

$$\begin{aligned}
f(0) &= \gamma_\alpha P(\alpha) [1 - (1 - \varepsilon_a)^{K+L-1-k}] + \gamma_\beta (1 - P(\alpha)) = 1 \cdot \gamma_\beta - P(\alpha) ((\gamma_\beta - \gamma_\alpha) + \gamma_\alpha (1 - \varepsilon_a)^{K+L-1-k}) \\
&= 1 \cdot \gamma_\beta - \frac{(\gamma_\beta - \gamma_\alpha) + (1 - \varepsilon_a)^{K+L-1-k}}{1 + \frac{1-p}{p} \left(\frac{1-\varepsilon_b}{1-\varepsilon_a} \right)^K (1 - \varepsilon_a)^{2K-1-k}} \\
&= \frac{1 \cdot \gamma_\alpha + (1 - \varepsilon_a)^{K-1-k} \left[\gamma_\beta \frac{1-p}{p} \left(\frac{1-\varepsilon_b}{1-\varepsilon_a} \right)^K (1 - \varepsilon_a)^K - \gamma_\alpha (1 - \varepsilon_a)^L \right]}{1 + \frac{1-p}{p} \left(\frac{1-\varepsilon_b}{1-\varepsilon_a} \right)^K (1 - \varepsilon_a)^{2K-1-k}} \\
&> \frac{1 \cdot \gamma_\alpha}{1 + \frac{1-p}{p} \left(\frac{1-\varepsilon_b}{1-\varepsilon_a} \right)^K (1 - \varepsilon_a)^{2K-1-k}} = \gamma_\alpha P(\alpha),
\end{aligned}$$

where the last inequality is due to:

$$\gamma_\beta \frac{1-p}{p} \left(\frac{1-\varepsilon_b}{1-\varepsilon_a} \right)^K (1 - \varepsilon_a)^K - \gamma_\alpha (1 - \varepsilon_a)^L > 0 \iff \frac{1-p}{p} \frac{(1-\varepsilon_b)^K}{(1-\varepsilon_a)^L} \frac{\gamma_\beta}{\gamma_\alpha} > 1.$$

Also,

$$f(1) = \gamma_\alpha P(\alpha) + \gamma_\beta (1 - P(\alpha)) \varepsilon_b > \gamma_\alpha P(\alpha).$$

So $f(q) > \gamma_\alpha P(\alpha)$ for both $q = 0, 1$. This means that k would be better off playing B than A , contradicting the assumption that k plays A in the fixed equilibrium. The proof is complete. \square

E Examples for the Abstention Model

Example 3. Consider the strategy profile in which each voter uses a strategy that is consistent in the abstention model, votes for Φ if she receives signal ϕ and all past actions have been Φ , votes for A if she receives signal ϕ and action A was taken in the past when all previous voters have taken Φ , and vote for B if she receives signal ϕ and action B was taken in the past when all previous voters have taken Φ . If voter k receives signal ϕ and action A was taken in the past when all previous voters have taken Φ , then k 's belief assigns probability 1 to state α . Symmetrically, if voter k receives signal ϕ and action B was taken in the past when all previous voters have taken Φ , then k 's belief assigns probability 1 to state β . We show that this is an equilibrium.

First, suppose that voter k receives signal a . Consider the case in which all past actions have been Φ or action A was taken in the past when all previous voters have taken Φ . In this case, if k chooses A , then her payoff is 1 with probability 1. Since 1 is the highest feasible payoff, choosing A is indeed a best response. Consider the case in which action B was taken in the past when all previous voters have taken Φ . In this case, there is a positive probability that k 's action affects the outcome of the game, and this probability is independent of k 's action. Moreover, when k 's action affects the outcome of the game, it is her best response to vote for A . Thus, it is a best response for the voters receiving signal a to play A regardless of the history.

A symmetric argument shows that it is a best response for the voters receiving signal b to play B regardless of the history.

Suppose that voter k receives signal ϕ . Consider the case in which all the past actions are Φ . If $k = N$, then she is indifferent among all actions, so playing Φ is a best response. So suppose $k < N$. If k plays Φ , then her payoff is

$$(1 - \varepsilon)^{N-k} \frac{1}{2} + (1 - (1 - \varepsilon)^{N-k}) \cdot 1 = 1 - \frac{1}{2}(1 - \varepsilon)^{N-k}$$

because if no one receives the signal then the payoff is $\frac{1}{2}$, while if there is at least one voter receiving a signal then the payoff is 1. If she plays A , then her payoff is at most

$$1 - \left[(1 - \varepsilon)^{N-k} + \frac{1}{2}(N - k)\varepsilon(1 - \varepsilon)^{N-k-1} \right]$$

because the cases in which her payoff is not 1 include the situations in which the state is β and there is only zero or one voter who receives a signal after k . It is easy to see by inspection that the payoff from playing Φ is larger than the upper bound of the payoff from playing A . In the same way, the payoff from playing Φ is larger than the payoff from playing B . Hence, playing Φ is a best response.

Finally, suppose that voter k receives signal ϕ and consider the case in which there is a past action that was not Φ . Suppose without loss that the first such action was A . In this case, k assigns probability 1 to state α . By playing A , she can guarantee the payoff of 1, which is the highest payoff, so this is a best response. \square

Example 4. Suppose $N = 4$. Consider the following strategy profile.

- Voter 1 votes for Φ if her signal is a , A if her signal is ϕ , and B if her signal is b .

- Voter 2 plays a strategy that is consistent in the abstention model. If he receives signal ϕ , then he plays
 - Φ if 1 played A ;
 - A if 1 played Φ ;
 - B if 1 played B .
- Voter 3 plays a strategy that is consistent in the abstention model. If she receives signal ϕ , then she plays
 - Φ if the action sequence was (A, Φ) ;
 - A if the action sequence was (Φ, \cdot) or (A, A) ;
 - B if the action sequence was (B, \cdot) or (A, B) .
- Voter 4 plays a strategy that is consistent in the abstention model. If he receives signal ϕ , then he plays
 - Φ if the action sequence was (A, Φ, Φ) ;
 - A if the action sequence was (Φ, \cdot, \cdot) or (A, A, B) ;
 - B if the action sequence was (B, \cdot, \cdot) or (A, B, \cdot) or (A, \cdot, B) .

Voters 3 and 4 have information sets that can be reached with probability zero when they do not receive a signal. In such a case, they assign probability 1 to state α under the history in which the above specification says they play A , and they assign probability 1 to state β under the history in which the above specification says they play B . We show that this is an equilibrium when $\varepsilon > 0$ is sufficiently small.

Let us check the incentives. The incentives of voters 2, 3, and 4 are straightforward so we check the incentive of voter 1. If she receives signal a or b , it is again straightforward that following the given strategy is a best response. So consider the case in which she receives signal ϕ .

In this case, if she plays A , then her payoff is 1 if the state is α while, if the state is β , her payoff is at least $\frac{1}{2} \cdot 3\varepsilon(1 - \varepsilon)^2$ because if at least one of the subsequent voters receives signal b then there is probability $1/2$ that the payoff is 1.

If she plays Φ , then her payoff is 1 if the state is α . If the state is β , the payoff is $O(\varepsilon^2)$ because the only cases in which the payoff is 1 is when at least two subsequent voters receive signal b .

If she plays B , then her payoff is 1 if the state is β . If the state is α , the payoff is $O(\varepsilon^2)$ because the only cases in which the payoff is positive is when at least two subsequent voters receive signal a .

Overall, when voter 1 receives signal ϕ , it is a best response to play A when $\varepsilon > 0$ is sufficiently small. \square

In both examples, the ex ante payoff is

$$(1 - \varepsilon)^N \frac{1}{2} + (1 - (1 - \varepsilon)^N) \cdot 1.$$

Note that this is the highest payoff achievable by any social choice function.