Online Appendix to: "Revision Games"

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B Additional Discussion for Example 2 (Cournot Duopoly)

First, we offer a story to motivate this example. Two fishing boats depart from a harbor, and they must return when the fish market, located at the harbor, opens at 6:00am. They catch fish at a fishing ground that is far from the fish market. They have to depart the fishing ground at 5:00am to reach the market at 6:00am. Hence, 5:00am is the end of the revision game, and assume that the cost for the fishermen is associated with the transportation cost of the final catch (the fish at hand at 5:00am) to the market (consider the case where catching fish itself is easy and the cost of catching fish is negligible).

The fishermen wish to collude (i.e., to restrict their catch) so as to increase the price at the fish market. They start with a small amount of catch (the collusive quantity). They operate side by side, closely monitoring each other's behavior. A revision opportunity of their quantities corresponds to the arrival of a fish school, which follows a Poisson process.[1] When the Poisson arrival rate is $\lambda = 0.1$ and the time unit is a minute, a fish school comes every ten minutes on average. According to the optimal trigger strategy equilibrium, the fishermen do not touch fish schools until 4:49am. In the last eleven minutes, however, whenever a fish school visits them, they catch additional fish. If anyone deviates from this equilibrium plan, they catch a large amount (so that each fisherman's total amount becomes the Nash quantity) when the next fish school arrives. In this way, the fishermen can attain an expected payoff that is 97% of fully collusive profit in expectation as we will show below.

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[1]Pun not intended.

This story does not exactly match the revision-game model in that (i) the fishermen may be able to discard fish at any time in reality and (ii) catching additional fish may be infeasible in reality if the deadline is too close. We can formally show that introducing the possibility of discarding fish does not alter our equilibrium. The main reason is that the fishermen would like to catch more, if possible, under any equilibrium action profile of the optimal trigger strategy before time 0. The formal proof is given at the end of this section. For the latter point (ii), the issue can be addressed by the argument we present in Section 7.1.

Next, we calculate the expected payoff from the optimal trigger strategy equilibrium. It is computed as:

$$
\begin{aligned}
&\int_0^{t(x^*)} (a-2bx(t)-c)\,x(t)\lambda e^{-\lambda t}dt + e^{-\lambda t(x^*)}\,(a-2bx^*-c)\,x^* \\
&= \int_0^{t(x^*)} \left(a-2b\frac{a-c}{3b}\left(5-4e^{\frac{\lambda}{18}t}\right)-c\right)\frac{a-c}{3b}\left(5-4e^{\frac{\lambda}{18}t}\right)\lambda e^{-\lambda t}dt + e^{-\lambda t(x^*)}\left(a-2b\frac{a-c}{4b}-c\right)\frac{a-c}{4b} \\
&= \frac{(a-c)^2}{9b}\int_0^{t(x^*)} \left(-7+8e^{\frac{\lambda}{18}t}\right)\left(5-4e^{\frac{\lambda}{18}t}\right)\lambda e^{-\lambda t}dt + e^{-\lambda t(x^*)}\frac{(a-c)^2}{8b} \\
&= \frac{(a-c)^2}{9b}\left[35e^{-\lambda t}-72e^{-\frac{17}{18}\lambda t}+36e^{-\frac{8}{9}\lambda t}\right]_0^{t(x^*)} + e^{-\lambda t(x^*)}\frac{(a-c)^2}{8b} \\
&= \frac{(a-c)^2}{9b}\left(35\left(\frac{17}{16}\right)^{-18}-72\left(\frac{17}{16}\right)^{-17}+36\left(\frac{17}{16}\right)^{-16}+1\right)+\left(\frac{17}{16}\right)^{-18}\frac{(a-c)^2}{8b} \\
&= \frac{(a-c)^2}{8b}\left(\left(\frac{17}{16}\right)^{-18}\left(35\cdot\frac{8}{9}-64\cdot\frac{17}{16}+32\left(\frac{17}{16}\right)^2+1\right)+\frac{8}{9}\right) \\
&= \frac{(a-c)^2}{8b}\left(\left(\frac{17}{16}\right)^{-18}\left(-\frac{323}{9}+32\left(\frac{17}{16}\right)^2\right)+\frac{8}{9}\right).
\end{aligned}
$$

The collusive payoff is:

$$\left(a-2b\frac{a-c}{4b}-c\right)\frac{a-c}{4b}=\frac{(a-c)^2}{8b}.$$

Thus, the ratio of the former to the latter is:

$$\left(\frac{17}{16}\right)^{-18}\left(-\frac{323}{9}+32\left(\frac{17}{16}\right)^2\right)+\frac{8}{9}\simeq 0.96817\ldots$$

Model that allows discarding fish:

Assume that players can discard fish at any moment of time in $\{-t| -t = 0, -\Delta, -2\Delta, ...$ and $t < T\}$ for small $\Delta > 0$.[2] Consider the strategy profile of this game where (1) actions at the revision opportunities are the same as under the optimal trigger strategy of our original model, (2) players do not discard fish after any history before time 0, and (3) they take mutual best replies when they may discard fish at time 0.[3]

Let us elaborate on (3). When the profile of fish at hand is $(\overline{q}_1, \overline{q}_2)$ at time 0 (and a revision opportunity does not arise at time 0), players effectively play the Cournot game where their quantities are restricted to satisfy $q_i \leq \overline{q}_i$, $i = 1, 2$. Examination of the reaction curves in the restricted strategy spaces reveals that the unique Nash equilibrium of this fish discarding game at time 0 is (i) the Cournot Nash profile (q_1^N, q_2^N) if $q_i^N \leq \overline{q}_i$, $i = 1, 2$, (ii) $(\overline{q}_1, \overline{q}_2)$ if $R_i(q_{-i}) \geq \overline{q}_i$, $i = 1, 2$, where R_i is the reaction function of player i (they do not discard fish because they would like to catch more, if possible), and (iii) $(\overline{q}_i, R_{-i}(\overline{q}_i))$ if $q_i^N \geq \overline{q}_i$ and $R_{-i}(\overline{q}_i) \leq \overline{q}_{-i}$, $i = 1, 2$.

Given the above observation, one can check that if a player unilaterally discards fish at any moment before time 0 (after any history), her payoff at the fish discarding game at time 0 never increases. Hence, players have an incentive to follow the fish discarding rule specified by our strategies. Lastly, we show that players have an incentive to follow the revision rule specified by our strategies. When a revision opportunity arrives at $-t$, the equilibrium action $q(t)$ is no greater than the Cournot Nash quantity. Consider any unilateral deviation of player i at time t. If the deviation action is less than myopic best reply $R_i(q(t))$, it remains unchanged at the fish discarding game at time 0 (see case (ii) specified above). Alternatively, when player i deviates to a larger amount than the myopic best reply, the deviating action is later corrected towards the myopic best reply (case (iii) specified above). In either case, the player has no incentive to deviate because the trigger strategy profile constitutes an equilibrium in our original model (because the deviation to the myopic best reply is unprofitable under the optimal trigger strategy equilibrium profile in our model).

If a revision opportunity arrives (before time 0) when players are supposed to play the Cournot Nash equilibrium quantity, any unilateral deviation also remains to be

[2] This is to avoid technical issues associated with defining strategies in continuous time.

[3] In particular, actions in the revision opportunities depend only on what has happened on the previous revision opportunities (and the initial actions), as under the optimal trigger strategy.

unprofitable. This is because after any unilateral deviation, the opponent does not change his Nash action in the fish discarding game at time 0. Hence, the strategy profile we constructed under the possibility of discarding fish is an equilibrium, and by construction it achieves the same outcome as the optimal trigger strategy equilibrium of our original model.

C Appendix for Section 4

C.1 Proofs and a Calculation for Section 4.1

C.1.1 The Proofs for the Reduction Argument

We begin with the formal definitions of trigger strategy equilibria for the supply schedule game.

Formulation of the objective

As in Section 3, we define

$$X := \{x_i : [0,T] \to S \mid \pi_i(x_i(\cdot), x_i(\cdot)) \text{ is measurable}\}.$$

Given a pair $(x_1, x_2) \in X^2$ and a Nash equilibrium (s_1, s_2), we define seller i's **trigger strategy with plan** (x_1, x_2) **and a Nash equilibrium** (s_1, s_2), denoted by $\sigma_i((x_1, x_2), s_i)$, to be a strategy in which the following hold for each time $-t \in [-T, 0]$ such that there is an opportunity:

1. If each seller submits an order $x_i(\tau)$ for every $-\tau \in [-T, -t)$ at which there is an opportunity, then seller i submits the order $x_i(t)$.

2. Otherwise, each seller i submits the order s_i.

Let $\Sigma := \{(\sigma_i((x_1, x_2), s_i)_{i=1,2} | x_i \in X, s_i \in S \text{ for each } i = 1, 2\}$. We say that $(\sigma_i((x_1, x_2), s_i)_{i=1,2} \in \Sigma$ is **symmetric** if for every t, $q_1(x_1(t), x_2(t)) = q_2(x_1(t), x_2(t))$. That is, the realized supplies from the two sellers (after rationing) are the same with each other. Let $\bar{\Sigma} \subseteq \Sigma$ be the set of symmetric $(\sigma_i((x_1, x_2), s_i)_{i=1,2}$. As in Section 3, we can formulate the incentive compatibility condition, which defines subgame-perfect equilibria in trigger strategies. Let $\Sigma^* \subseteq \bar{\Sigma}$ be the set of subgame-perfect equilibria in $\bar{\Sigma}$. *Our objective is to find a strategy profile in Σ^* that generates the highest ex ante* *payoff to each seller.*[4] As in Section 3, there may exist multiple maximizers of the ex ante payoff to each seller. We will show that there is an *essentially unique* optimal plan of quantities, $q_i(x_1(t), x_2(t))$, $t \in [0, T]$ for each $i = 1, 2$, by which we mean that if both $(\sigma_i((x_1, x_2), s_i)_{i=1,2}, (\sigma_i((x_1', x_2'), s_i')_{i=1,2} \in \Sigma^*$ maximize the expected payoff, then $q_i(x_1(t), x_2(t)) = q_i(x_1'(t), x_2'(t))$ holds for each $i = 1, 2$ for almost all $t \in [0, T)$ and at $t = T$.[5]

Let (s_1^N, s_2^N) be a Nash equilibrium of the supply-schedule game such that $q_i(s_1^N, s_2^N) = \frac{a-c}{2b}$ for each $i = 1, 2$ and hence $\pi_i(s_1^N, s_2^N) = 0$ for each $i = 1, 2$. Such (s_1^N, s_2^N) exists by Lemma 1.

To prove that the reduction is possible, we present a series of lemmas. We first prove the following three lemmas (Lemmas 3, 4, and 5).

Lemma 3 (Severest punishment)**.** *Fix $(x_1, x_2) \in X^2$ and a Nash equilibrium $(s_1, s_2) \in S^2$, and suppose that $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Then, $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2} \in \Sigma^*$.*

Proof. Note that for any Nash equilibrium (s_1, s_2), $\pi_i(s_1, s_2) \geq 0 = \pi_i(s_1^N, s_2^N)$. This is because the order s_1' such that $s_1'(p) = 0$ for all p guarantees a payoff of zero against any supply scheme of the opponent, i.e., $\pi_1(s_1', s_2) = 0$ for every s_2, so the payoff under any Nash equilibrium must be no less than 0 (the same argument holds for seller 2). Seller 1's incentive compatibility condition at time $-t$ under $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ is:

$$e^{-\lambda t} \sup_{\tilde{s}_1 \in S} [\pi_1(\tilde{s}_1, x_2(t))] + \int_0^t \pi_i(s_1, s_2) \lambda e^{-\lambda \tau} d\tau \leq e^{-\lambda t} \pi_1(x_1(t), x_2(t)) + \int_0^t \pi_i(x_1(\tau), x_2(\tau)) \lambda e^{-\lambda \tau} d\tau.$$

This and $\pi_i(s_1, s_2) \geq \pi_i(s_1^N, s_2^N)$ imply:

$$e^{-\lambda t} \sup_{\tilde{s}_1 \in S} [\pi_1(\tilde{s}_1, x_2(t))] + \int_0^t \pi_i(s_1^N, s_2^N) \lambda e^{-\lambda \tau} d\tau \leq e^{-\lambda t} \pi_1(x_1(t), x_2(t)) + \int_0^t \pi_i(x_1(\tau), x_2(\tau)) \lambda e^{-\lambda \tau} d\tau,$$

which is the incentive compatibility condition at time $-t$ under $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2}$. Hence, $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2}$ is also a SPE. $\square$

[4] Note that, at this point, the existence of such a strategy profile is not obvious. The existence follows from the reduction argument that follows.

[5] Note that we do not require that the equality holds at $t = 0$ because there are multiple Nash equilibria in the component game.

In what follows, the vertical orders are going to be the key to the reduction. Let $\hat{s}[q] \in S$ for $q \geq 0$ be the order such that $\hat{s}[q](p) = q$ for every price $p \geq 0$.

Lemma 4 (Less than the Nash quantity)**.** *Fix $(x_1, x_2) \in X^2$ and a Nash equilibrium $(s_1, s_2) \in S^2$, and suppose that $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Let $\tilde{\mathcal{T}} \subseteq [0, T]$ be the set of times t such that $q_1(x_1(t), x_2(t)) = q_2(x_1(t), x_2(t)) > q^N$ for each $t \in \tilde{\mathcal{T}}$. If $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, then there exists $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$ that gives each seller a strictly greater ex ante payoff than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ such that $q_i(x_1'(t), x_2'(t)) \leq q^N$ for each $t \in [0, T]$.*

Proof. Take $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Consider a plan $(x_1', x_2') \in X^2$ defined by:

$$x_i'(t) = \begin{cases} x_i(t) & \text{if } t \notin \tilde{\mathcal{T}} \\ \hat{s}[q^N] & \text{if } t \in \tilde{\mathcal{T}} \end{cases} .$$

By definition, $q_i(x_1'(t), x_2'(t)) \leq q^N$ for each $t \in [0, T]$. We first show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$, and then show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ generates a strictly higher payoff to each seller than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$. We focus on seller 1 below. A symmetric argument holds for seller 2.

To show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$, first note that the incentive compatibility condition at time $-t$ for the subgame-perfect equilibrium $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$ can be written as

$$e^{-\lambda t} \sup_{\tilde{s}_1 \in S} \pi_1(\tilde{s}_1, x_2(t)) + \int_0^t \pi_1(s_1^N, s_2^N) \lambda e^{-\lambda \tau} d\tau \leq e^{-\lambda t} \pi_1(x_1(t), x_2(t)) + \int_0^t \pi_1(x_1(\tau), x_2(\tau)) \lambda e^{-\lambda \tau} d\tau. \tag{31}$$

Note that, for any $\tau \in [0, T]$,

$$\pi_1(x_1'(\tau), x_2'(\tau)) = \bar{\pi}_i\left(q^N, q^N\right) > \bar{\pi}_i(q_1(x_1(\tau), x_2(\tau)), q_2(x_1(\tau), x_2(\tau))) = \pi_1(x_1(\tau), x_2(\tau)). \tag{32}$$

(31) and (32) then imply that

$$e^{-\lambda t} \sup_{\tilde{s}_1 \in S} \pi_1(\tilde{s}_1, x_2(t)) + \int_0^t \pi_1(s_1^N, s_2^N) \lambda e^{-\lambda \tau} d\tau \leq e^{-\lambda t} \pi_1(x_1'(t), x_2'(t)) + \int_0^t \pi_1(x_1'(\tau), x_2'(\tau)) \lambda e^{-\lambda \tau} d\tau. \tag{33}$$

If $t \notin \tilde{\mathcal{T}}$, then $x_2(t)$ in the left-hand side of (33) can be replaced by $x_2'(t)$, which yields the incentive compatibility condition at time $-t$ for $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$.

To show that the incentive compatibility condition at time $-t$ with $t \in \tilde{\mathcal{T}}$ holds, (33) implies that it suffices to show that $\left(\hat{s}[q^N], \hat{s}[q^N]\right)$ is a Nash equilibrium of the supply-schedule game. To show this, take any deviation by seller 1, $s_1 \in S$, and let $\hat{p} = p(s_1, \hat{s}[q^N])$.

1. Suppose first that $\hat{p} < c$. Then, the maximized payoff is zero, which is no more than $\pi_1\left(\hat{s}[q^N], \hat{s}[q^N]\right)$.

2. Suppose that $\hat{p} > c$. In this case, by the "price first" rule, $q_2(s_1, \hat{s}[q^N]) \geq q^N$. This implies that $q_1(s_1, \hat{s}[q^N]) \leq D(\hat{p}) - q^N$, so 1's payoff is at most

$$\max\left\{0, (\hat{p} - c)\left(D(\hat{p}) - q^N\right)\right\}.$$

   Then, since (q^N, q^N) is a Nash equilibrium of the Cournot competition and $(\hat{p} - c)\left(D(\hat{p}) - q^N\right)$ is a payoff from a deviation in the Cournot competition, this upper bound is no more than $\pi_1\left(\hat{s}[q^N], \hat{s}[q^N]\right)$.

To show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ generates a strictly higher payoff to each seller than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$, recall (32). Thus, the difference of the ex ante payoffs can be calculated as:

$$\begin{aligned}
&\left[e^{-\lambda T}\pi_1(x_1'(T), x_2'(T)) + \int_0^T \pi_1(x_1'(t), x_2'(t))\lambda e^{-\lambda t}dt\right] \\
&- \left[e^{-\lambda T}\pi_1(x_1(T), x_2(T)) + \int_0^T \pi_1(x_1(t), x_2(t))\lambda e^{-\lambda t}dt\right] \\
&\geq e^{-\lambda T}\left(\pi_1(x_1'(T), x_2'(T)) - \pi_1(x_1(T), x_2(T))\right) \\
&+ \left(\int_{t\in\tilde{\mathcal{T}}} \left(\pi_1(x_1'(t), x_2'(t)) - \pi_1(x_1(t), x_2(t))\right)\lambda e^{-\lambda t}dt\right) \\
&> 0,
\end{aligned}$$

where the last inequality follows because $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, and (32) holds. This completes the proof. □

Lemma 5 (No demand rationing)**.** *Fix $(x_1, x_2) \in X^2$ and a Nash equilibrium (s_1, s_2), and suppose $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Let $\tilde{\mathcal{T}} \subseteq [0, T]$ be the set of times t such that $q_1(x_1(t), x_2(t)) + q_2(x_1(t), x_2(t)) < D(p(x_1(t), x_2(t)))$ and $p(x_1(t), x_2(t)) > c$ for each $t \in \tilde{\mathcal{T}}$. If $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, then there exists $(x_1', x_2') \in X$ such* *that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$ has a strictly greater ex ante payoff to each seller than under $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ and $q_1(x_1'(t), x_2'(t)) + q_2(x_1'(t), x_2'(t)) = D(p(x_1'(t), x_2'(t)))$ for each $t \in [0, T]$.*

Proof. Consider a plan $(x_1', x_2') \in X^2$ defined by

$$x_i'(t) = \begin{cases} x_i(t) & \text{if } t \notin \tilde{\mathcal{T}} \\ \tilde{x}_i(t) & \text{if } t \in \tilde{\mathcal{T}} \end{cases},$$

where

$$\tilde{x}_i(t)(p) = \begin{cases} x_i(t)(p) & \text{if } p < p(x_1(t), x_2(t)) \\ \frac{D(p(x_1(t), x_2(t)))}{2} & \text{if } p \geq p(x_1(t), x_2(t)) \end{cases}.$$

Then, by the definition of $q_i(\cdot, \cdot)$, $q_1(x_1'(t), x_2'(t)) + q_2(x_1'(t), x_2'(t)) = D(p(x_1'(t), x_2'(t)))$ holds for each $t \in [0, T]$.

To show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ generates a strictly higher payoff to each seller than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$, first note that for every $t \in \tilde{\mathcal{T}}$,

$$\begin{aligned} \pi_1(x_1'(t), x_2'(t)) &= (p(x_1(t), x_2(t)) - c)\frac{D(p(x_1(t), x_2(t)))}{2} \\ &> (p(x_1(t), x_2(t)) - c)q_i(x_1(t), x_2(t)) = \pi_1(x_1(t), x_2(t)), \end{aligned} \tag{34}$$

where the inequality comes from the assumption that $p(x_1(t), x_2(t)) > c$ and $q_1(x_1(t), x_2(t)) + q_2(x_1(t), x_2(t)) < D(p(x_1(t), x_2(t)))$. Thus, the difference of the ex ante payoffs can be calculated as:

$$\begin{aligned} &\left[e^{-\lambda T}\pi_1(x_1'(T), x_2'(T)) + \int_0^T \pi_1(x_1'(t), x_2'(t))\lambda e^{-\lambda t} dt\right] \\ &- \left[e^{-\lambda T}\pi_1(x_1(T), x_2(T)) + \int_0^T \pi_1(x_1(t), x_2(t))\lambda e^{-\lambda t} dt\right] \\ &\geq e^{-\lambda T}\left(\pi_1(x_1'(T), x_2'(T)) - \pi_1(x_1(T), x_2(T))\right) \\ &+ \left(\int_{t \in \tilde{\mathcal{T}}} \left(\pi_1(x_1'(t), x_2'(t)) - \pi_1(x_1(t), x_2(t))\right)\lambda e^{-\lambda t} dt\right) \\ &> 0, \end{aligned}$$

where the last inequality follows because $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, and (34) holds.

An analogous proof to the one for Lemma 4 can show that $(\sigma_i((x'_1, x'_2), s_i))_{i=1,2}$ is a SPE. This completes the proof. $\square$

Let

$$\Sigma^{**} = \Big\{(\sigma_i((x_1, x_2), s_i^N))_{i=1,2} \in \Sigma^* \mid q_i(x_1(t), x_2(t)) \le q^N \text{ and }$$
$$q_i(x_1(t), x_2(t)) = \frac{D(p(x_1(t), x_2(t)))}{2}, i = 1, 2\Big\}.$$

Note that $q_1(x_1(t), x_2(t)) = q_2(x_1(t), x_2(t)) \le q^N$ (cf. Lemma 4) implies that $p(x_1(t), x_2(t)) > c$ (cf. Lemma 5). Hence, if there is $(\sigma_i((x_1, x_2), s_i)_{i=1,2} \in \Sigma^{**}$ maximizing the ex ante payoff to each seller in Σ^{**}, then it maximizes the ex ante payoff in Σ^* as well. Moreover, for any maximizer of the ex ante payoff to each seller $(\sigma_i((x_1, x_2), s_i)_{i=1,2}$ in Σ^*, there exists a maximizer $(\sigma_i((x'_1, x'_2), s'_i)_{i=1,2}$ in Σ^* such that $(\sigma_i((x'_1, x'_2), s'_i)_{i=1,2} \in \Sigma^{**}$ and $q_i(x_1(t), x_2(t)) = q_i(x'_1(t), x'_2(t))$ for each i for almost all $t \in [0, T)$ and at $t = T$. The next lemma proves this point.

Lemma 6 (Restricting attention to Σ^{**})**.** *For any maximizer of the ex ante payoff to each seller $(\sigma_i((x_1, x_2), s_i)_{i=1,2}$ in Σ^*, there exists a maximizer $(\sigma_i((x'_1, x'_2), s'_i)_{i=1,2}$ in Σ^* such that $(\sigma_i((x'_1, x'_2), s'_i)_{i=1,2} \in \Sigma^{**}$ and $q_i(x_1(t), x_2(t)) = q_i(x'_1(t), x'_2(t))$ for each i for almost all $t \in [0, T)$ and at $t = T$.*

Proof. Fix $(\sigma_i((x_1, x_2), s_i)_{i=1,2} \in \Sigma^*$ maximizing the ex ante payoff to each seller in Σ^{**}. Let $\tilde{\mathcal{T}}$ be the set of times t such that $q_i(x_1(t), x_2(t)) > q^N$ for each $i = 1, 2$ or $q_1(x_1(t), x_2(t)) + q_2(x_1(t), x_2(t)) < D(p(x_1(t), x_2(t)))$. Lemmas 4 and 5 imply that $\tilde{\mathcal{T}}$ has measure zero and $T \notin \tilde{\mathcal{T}}$. Consider a profile of plans (x'_1, x'_2) such that

$$x'_i(t) = \begin{cases} x_i(t) & \text{if } t \notin \tilde{\mathcal{T}} \\ \hat{s}_i[q^N] & \text{if } t \in \tilde{\mathcal{T}} \end{cases}.$$

By definition, $(\sigma_i((x_1, x_2), s_i)_{i=1,2} \in \Sigma^{**}$ holds. Moreover, $q_i(x_1(t), x_2(t)) = q_i(x'_1(t), x'_2(t))$ for each i for almost all $t \in [0, T)$ and at $t = T$, and hence the ex ante payoffs under $(\sigma_i((x_1, x_2), s_i)_{i=1,2}$ and under $(\sigma_i((x'_1, x'_2), s_i)_{i=1,2}$ are the same, which implies that $(\sigma_i((x'_1, x'_2), s_i)_{i=1,2}$ maximizes the ex ante payoff in Σ^*.

An analogous proof to the one for Lemma 4 can show that $(\sigma_i((x'_1, x'_2), s_i))_{i=1,2}$ is a SPE. This completes the proof. $\square$

Overall, if there is an essentially unique plan of quantities among those induced by the maximizers in Σ^{**}, then there is an essentially unique plan of quantities among those induced by the maximizers in Σ^*.

Lemma 7 (Restricting attention to $\hat{S}$)**.** *Fix $(x_1, x_2) \in X^2$ and suppose that $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2} \in \Sigma^{**}$. Then, there exist $(\bar{x}_1, \bar{x}_2) \in X^2$ such that the following hold:*

1. *$\bar{x}_i(t) \in \hat{S}$ for each t and i.*
2. *For each t, $q_i(\bar{x}_1(t), \bar{x}_2(t)) = q_i(x_1(t), x_2(t))$ for each i and $p(\bar{x}_1(t), \bar{x}_2(t)) = p(x_1(t), x_2(t))$.*
3. *$(\sigma_i((\bar{x}_1, \bar{x}_2), s_i^N))_{i=1,2} \in \Sigma^{**}$.*

Proof. To show the lemma, we first prove the following claim:

Claim 1. *Fix an arbitrary $\bar{q} \in [0, q^N]$ and $(s_1, s_2) \in S^2$ such that $q_1(s_1, s_2) = q_2(s_1, s_2) = \bar{q}$ and $2\bar{q} = D(p(s_1, s_2))$. The following are true:*

1. *The price, quantities and profits under $(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$ are the same as those under (s_1, s_2). Formally, for each $i = 1, 2$, the following three equalities hold:*

$$p\left(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]\right) = p(s_1, s_2);$$

$$q_i(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]) = q_i(s_1, s_2);$$

$$\pi_i\left(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]\right) = \pi_i(s_1, s_2).$$

2. *The payoff after any deviation under $(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$ is no more than the one under (s_1, s_2). Formally, for each $s_1' \in S$ such that $\pi_1(s_1', \hat{s}_2[\bar{q}]) \geq \pi_1(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$, the following inequality holds for seller 1:*

$$\pi_1(s_1', \hat{s}_2[\bar{q}]) \leq \sup_{s_1'' \in S} \pi_1(s_1'', s_2).$$

The symmetric inequality holds for seller 2, too.

Proof of Claim 1.
Part 1: The equalities on quantities directly follow from the definition of $\hat{s}_i[\bar{q}]$.

Given the equalities on quantities and the definition of the $p(\cdot, \cdot)$ function, if $\bar{q} > 0$, the equality on prices holds because $\hat{s}_1[\bar{q}](p) + \hat{s}_2[\bar{q}](p) = 2\bar{q} \leq D(p)$ for all

$p \leq p(s_1, s_2)$ and $\hat{s}_1[\bar{q}](p) + \hat{s}_2[\bar{q}](p) = 2\bar{q} > D(p)$ for all $p > p(s_1, s_2)$ by the definition of $\hat{s}_i[\bar{q}]$ for each i and the assumption that D is strictly decreasing for p's such that $0 < D(p) < 2q^N$. If $\bar{q} = 0$, the equality on prices holds by the definition of the $p(\cdot, \cdot)$ function.

Finally, the equalities on profits follow because we have shown the equalities on quantities and prices.

Part 2: We prove the inequality for seller 1. A symmetric argument shows that the inequality for seller 2 holds, too.

Given $(s_1, s_2) \in S^2$, let

$$S^- = \{s_1' \in S | p(s_1', \hat{s}_2[\bar{q}]) < p(s_1, s_2)\},$$

$$S^0 = \{s_1' \in S | p(s_1', \hat{s}_2[\bar{q}]) = p(s_1, s_2)\},$$

$$S^+ = \{s_1' \in S | p(s_1', \hat{s}_2[\bar{q}]) > p(s_1, s_2)\}.$$

We will show that, for each element s_1' in each of S^- and S^0, there exists $s_1'' \in S$ such that $\pi_1(s_1', \hat{s}_2[\bar{q}]) \leq \pi_1(s_1'', s_2)$. Also, we show that $\pi_1(s_1', \hat{s}_2[\bar{q}]) < \pi_1(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$ for each $s_1' \in S^+$. Showing these claims suffice because $S^- \cup S^0 \cup S^+ = S$.

<u>Case 1:</u> Consider $s_1' \in S^-$ and let $\hat{p} = p(s_1', \hat{s}_2[\bar{q}]) \in [0, p(s_1, s_2))$.

Suppose first that $\hat{p} > c$. Take $s_1'' \in S$ such that $s_1''(p) = D(\hat{p}) - s_2(\hat{p})$ for all $p \in [0, \infty)$. Note that $p(s_1'', s_2) = \hat{p}$. This is because the definition of s_1'', the assumptions that s_2 is non-decreasing and D is strictly decreasing for prices below $p(s_1, s_2)$ imply that $s_1''(p) + s_2(p) \leq D(p)$ for all $p \leq \hat{p}$ and $s_1''(p) + s_2(p) > D(p)$ for all $p > \hat{p}$.

Then, we have:

$$\begin{aligned}
\pi_1(s_1'', s_2) &= (\hat{p} - c)\left(D\left(\hat{p}\right) - s_2(\hat{p})\right) \quad \text{(by the definition of } s_1''\text{)} \\
&\geq (\hat{p} - c)\left(D\left(\hat{p}\right) - q_2(s_1, s_2)\right) \quad \text{(by ``price first'' and } \hat{p} < p(s_1, s_2)\text{)} \\
&= (\hat{p} - c)\left(D\left(\hat{p}\right) - \bar{q}\right) \quad \text{(by the definition of } \bar{q}\text{)} \\
&= \pi_1(s_1', \hat{s}_2[\bar{q}]) \quad \text{(by ``price first'' and } \hat{p} > c > 0\text{)}.
\end{aligned}$$

If $\hat{p} \leq c$, then $\pi_1(s_1', \hat{s}_2[\bar{q}]) \leq 0$. Consider $s_1'' \in S$ such that $s_1''(p) = 0$ for all $p \in [0, \infty)$. Then, $\pi_1(s_1'', s_2) = 0$.

Overall, we have shown that for each $s_1' \in S^-$, there exists $s_1'' \in S$ such that $\pi_1(s_1', \hat{s}_2[\bar{q}]) \leq \pi_1(s_1'', s_2)$. Hence, we have that for each $s_1' \in S^-$, $\pi_1(s_1', \hat{s}_2[\bar{q}]) \leq \sup_{s_1'' \in S} \pi_1(s_1'', s_2)$.

Case 2: Consider $s_1' \in S^0$, that is, $p(s_1', \hat{s}_2[\bar{q}]) = p(s_1, s_2)$. First, note that

$$\begin{aligned}
\pi_1(s_1', \hat{s}_2[\bar{q}]) &\leq (p(s_1, s_2) - c)(D(p(s_1, s_2)) - \bar{q}) \quad \text{(by "price first" and the definition of } \bar{q}) \\
&= (p(s_1, s_2) - c)\bar{q}.
\end{aligned}$$

Second, note that

$$\sup_{s_1'' \in S^0} \pi_1(s_1'', s_2) \geq \pi_1(s_1, s_2) = (p(s_1, s_2) - c)q_1(s_1, s_2) = (p(s_1, s_2) - c)\bar{q}.$$

Combining, we have that for each $s_1' \in S^0$, $\pi_1(s_1', \hat{s}_2[q]) \leq \sup_{s_1'' \in S^0} \pi_1(s_1'', s_2)$.

Case 3: Consider $s_1' \in S^+$ and let $\hat{p} > p(s_1', \hat{s}_2[\bar{q}])$. Then, we have the following:

$$\begin{aligned}
\pi_1(s_1', \hat{s}_2[\bar{q}]) &\leq (\hat{p} - c)\,(D(\hat{p}) - \bar{q}) \quad \text{(by "price first")} \\
&= \bar{\pi}_1(D(\hat{p}) - \bar{q}, \bar{q}) \\
&\leq \bar{\pi}_1(\bar{q}, \bar{q}) \quad (\text{by } \bar{q} < \frac{a-c}{3b}) \\
&= \pi_1(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]).
\end{aligned}$$

Overall, we have shown the desired claim. $\square$

Having proved the claim, we now prove the lemma.

Fix $(x_1, x_2) \in X^2$ and suppose $(\sigma_i((x_1, x_2), (s_i^N)))_{i=1,2} \in \Sigma^{**}$. Seller 1's incentive compatibility constraint at time $-t$ implies the following:

$$e^{-\lambda t}\pi_i(x_1(t), x_2(t)) + \int_0^t \pi_i(x_1(\tau), x_2(\tau))\lambda e^{-\lambda\tau} d\tau \geq e^{-\lambda t}\pi_i(s_1', x_2(t)) + (1 - e^{-\lambda t})\pi_i(s_1^N, s_2^N)$$

for all $s_1' \in S_1$. By part 1 of Claim 1, we have

$$\begin{aligned}
&e^{-\lambda t}\pi_i(\hat{s}_1[q_1(x_1(t), x_2(t))], \hat{s}_2[q_2(x_1(t), x_2(t))]) + \int_0^t \pi_i(\hat{s}_1[q_1(x_1(\tau), x_2(\tau))], \hat{s}_2[q_2(x_1(\tau), x_2(\tau))])\lambda e^{-\lambda\tau} d\tau \\
&\qquad \geq e^{-\lambda t}\pi_i(s_1', x_2(t)) + (1 - e^{-\lambda t})\pi_i(s_1^N, s_2^N)
\end{aligned}$$

for all $s_1' \in S$.

Then, part 2 of Claim 1 implies that

$$e^{-\lambda t}\pi_i(\hat{s}_1[q_1(x_1(t), x_2(t))], \hat{s}_2[q_2(x_1(t), x_2(t))]) + \int_0^t \pi_i(\hat{s}_1[q_1(x_1(\tau), x_2(\tau))], \hat{s}_2[q_2(x_1(\tau), x_2(\tau))])\lambda e^{-\lambda\tau} d\tau$$

$$\geq e^{-\lambda t}\pi_i(s_1', \hat{s}_2[q_2(x_1(t), x_2(t))]) + (1 - e^{-\lambda t})\pi_i(s_1^N, s_2^N)$$

for all $s_1' \in S$.

The last inequality is the incentive comparability constraint for seller 1 under $(\sigma_i((\bar{x}_1, \bar{x}_2), s_i^N))_{i=1,2}$. A symmetric argument shows that the incentive comparability constraint holds for seller 2 as well. □

Lemma 7 implies that, as far as the plan of quantities is concerned, restricting attention to the following set of strategy profiles is without loss of generality:

$$\bar{\bar{\Sigma}} := \left\{(\sigma_i\left((x_1, x_2), s_1^N\right))_{i=1,2} \in \bar{\Sigma} \mid \exists q : [0, T] \to \mathbb{R}_+ \text{ s.t. } x_i(t) = \hat{s}_i[q(t)]\right\}.$$

Furthermore, note that a payoff of 0 can be attained in the semi-Cournot game in a Nash equilibrium in which each seller chooses $\emptyset$.

Hence, in order to prove that the reduction works, the only thing left is to show that the gain from an instantaneous deviation given any scheme $\hat{s}_i[q]$ of the opponent in the supply-schedule game is the same as the instantaneous gain from a deviation given any quantity q of the opponent in the semi-Cournot game when q is no more than the Nash quantity. The next lemma proves this.

Lemma 8. *For any* $q \leq \frac{a-c}{3b}$, $\sup_{s_i \in S} \pi_i(s_i, \hat{s}_i[q]) = \sup_{q' \in \mathbb{R}_+} \bar{\pi}_i(q', q)$.

Proof of Lemma 8. Fix $q \leq q^N$.

First, consider a deviation inducing a price strictly less than c. In the semi-Cournot competition, given seller 2's quantity q, any deviation that induces a price strictly less than c cannot be the optimal one for seller 1 since such deviations are strictly dominated by a deviation to set the zero quantity. In the supply-schedule game, given seller 2's supply schedule $\hat{s}_2[q]$, any deviation that induces a price strictly less than c cannot be the optimal one for seller 1 since $q < D(p)$ if $p < c$.

Second, consider a deviation to induce a price strictly above $p(\hat{s}_1[q], \hat{s}_2[q])$. In the semi-Cournot competition, $\arg\max_{q'} \bar{\pi}_i(q', q) \geq q^N$ if $q \leq q^N$, which implies that the induced price under the optimal deviation is no greater than $p(\hat{s}_1[q], \hat{s}_2[q])$. In the supply-schedule game, the "price first" rule implies that a deviation to any s_1 inducing a price $\tilde{p} \geq c > 0$ results in seller 1's realized supply that is no greater than $D(\tilde{p}) - q$. The same argument as in the semi-Cournot competition then implies that the induced price under the optimal deviation is no greater than $p(\hat{s}_1[q], \hat{s}_2[q])$ in the supply-schedule game as well.

Take $\hat{p} \in (c, p(\hat{s}_1[q], \hat{s}_2[q])]$. In the supply-schedule game, consider s_1' such that $p(s_1', \hat{s}_2[q]) = \hat{p}$. Then,

$$\begin{aligned}
\sup_{s_1' \in S \text{ s.t. } p(s_1', \hat{s}_2[q]) = \hat{p}} \pi_i(s_1', \hat{s}_2[q]) &= \pi_i(\tilde{s}_1, \hat{s}_2[q]) \\
&= (\hat{p} - c) \cdot (D(\hat{p}) - q) \\
&= \sup_{q' \in \mathbb{R}_+ \text{ s.t. } D(\hat{p}) = q' + q} \bar{\pi}_i(q', q),
\end{aligned}$$

where we define $\tilde{s}_1$ so that $\tilde{s}_1(p) = D(\hat{p}) - q$ for all p.

Thus,

$$\sup_{s_i \in S} \pi_i(s_i, \hat{s}_i[q]) = \sup_{\hat{p} \in (c, p(\hat{s}_1[q], \hat{s}_2[q])]} \left(\sup_{s_1' \in S \text{ s.t. } p(s_1', \hat{s}_2[q]) = \hat{p}} \pi_i(s_1', \hat{s}_2[q]) \right)$$

$$= \sup_{\hat{p} \in (c, p(\hat{s}_1[q], \hat{s}_2[q])]} \left(\sup_{q' \in \mathbb{R}_+ \text{ s.t. } D(\hat{p}) = q' + q} \bar{\pi}_i(q', q) \right) = \sup_{q' \in \mathbb{R}_+} \bar{\pi}_i(q', q).$$

A symmetric argument holds for seller 2. □

C.1.2 Proofs

Proof of Lemma 1. The "if" direction: Take any $(q_1, q_2) \in Q^N$. We prove that there exists a Nash equilibrium (s_1, s_2) with $(q_1(s_1, s_2), q_2(s_1, s_2)) = (q_1, q_2)$. To show this, fix an arbitrary $\bar{Q} \geq \frac{a}{b}$ and consider s_i for each $i = 1, 2$ such that

$$s_i(p) = \begin{cases} q_i & \text{if } p < a - b(q_1 + q_2) \\ \bar{Q} & \text{if } a - b(q_1 + q_2) \leq p \end{cases}.$$

Note that $p(s_1, s_2) = a - b(q_1 + q_2) \geq c$. Also, the "price first" rule implies that $q_1(s_1, s_2) = q_1$.

Consider s_1' such that $p(s_1', s_2) \leq p(s_1, s_2)$. Then, either (i) $p(s_1', s_2) = p(s_1, s_2)$ and $q_1(s_1', s_2) \leq q_1$, or (ii) $q_1(s_1', s_2) \geq q_1$. In case (i),

$$\pi_1(s_1', s_2) = (p(s_1', s_2) - c)q_1(s_1', s_2) \leq (p(s_1, s_2) - c)q_1(s_1, s_2) = \pi_1(s_1, s_2).$$

In case (ii),

$$\pi_1(s_1', s_2) \leq ((a - b(q_1(s_1', s_2) + q_2)) - c)\, q_1(s_1', s_2). \tag{35}$$

Since $((a - b(x + q_2)) - c)\, x$ is decreasing in x when $x \geq \frac{a-c-bq_2}{2b}$ and we have $\frac{a-c-bq_2}{2b} \leq q_1 \leq q_1(s_1', s_2)$ by assumption, the right hand side of (35) is no greater than

$$(p(s_1, s_2) - c)\, q_1,$$

which is equal to $\pi_1(s_1, s_2)$.

Finally, there is no s_1' such that $p(s_1', s_2) > p(s_1, s_2)$. Otherwise, there exists $p \in (p(s_1, s_2), p(s_1', s_2))$ such that $s_1'(p) + s_2(p) \leq D(p)$ by the definition of the $p(\cdot, \cdot)$ function, but this would imply $s_1'(p) + \bar{Q} \leq D(p)$, which violates the assumption on $\bar{Q}$ that $\bar{Q} \geq \frac{a}{b}$.

A symmetric argument holds for s_2, and this completes the proof for the "if" direction.

<u>The "only if" direction:</u>

First, take $(q_1, q_2) \in \mathbb{R}^2_+$ such that $q_1 < \frac{a-c-bq_2}{2b}$ and (s_1, s_2) such that $q_i(s_1, s_2) = q_i$ for each $i = 1, 2$. We prove that (s_1, s_2) is not a Nash equilibrium. Let

$$s_i'(p) = \begin{cases} \frac{a-c-bq_2}{2b} & \text{if } p < a - b(q_1 + q_2) \\ \bar{Q} & \text{if } a - b(q_1 + q_2) \leq p \end{cases}.$$

First, note that $p(s_1', s_2) > 0$. This is because, otherwise, $s_1'(\epsilon) + s_2(\epsilon) > D(\epsilon)$ must hold for any $\epsilon > 0$. However, since $q_1 < \frac{a-c-bq_2}{2b}$ implies $a - b(q_1 + q_2) > 0$, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$, $s_1'(\epsilon) + s_2(\epsilon) \leq \frac{a-c-bq_2}{2b} + q_2 = \frac{a-c+bq_2}{2b} = \frac{a-c}{2b} + \frac{q_2}{2} < \frac{a-c}{2b} + \frac{\frac{a-c}{b}}{2} = \frac{a-c}{b} < D(0)$. Since $D(\cdot)$ is continuous, it is true that there exists $\epsilon > 0$ such that $s_1'(\epsilon) + s_2(\epsilon) < D(\epsilon)$, a contradiction.

If $p(s_1', s_2) \in (0, p(s_1, s_2))$, then $q_2(s_1', s_2) \leq s_2(p(s_1', s_2)) \leq q_2$ holds, where the first inequality follows because the realized quantity must be no more than what s_2 specifies, and the second inequality follows because of the "price first" rule. Also, $q_1(s_1', s_2) = \frac{a-c-bq_2}{2b}$ because of the definition of s_1' and the "price first" rule. Hence, we have:

$$\begin{aligned}
\pi_1(s_1', s_2) &= (p(s_1', s_2) - c)q_1(s_1', s_2) \\
&= \left(a - b\left(\frac{a-c-bq_2}{2b} + q_2(s_1', s_2)\right) - c\right)\frac{a-c-bq_2}{2b} \\
&\geq \left(a - b\left(\frac{a-c-bq_2}{2b} + q_2\right) - c\right)\frac{a-c-bq_2}{2b} \\
&> (a - b\,(q_1 + q_2) - c)\,q_1 \text{ (because } q_1 \neq \frac{a-c-bq_2}{2b}) \\
&= \pi_1(s_1, s_2).
\end{aligned}$$

If $p(s_1', s_2) = p(s_1, s_2)$, then

$$\begin{aligned}
\pi_1(s_1', s_2) &= (p(s_1', s_2) - c)q_1(s_1', s_2) \\
&= (p(s_1, s_2) - c)q_1(s_1', s_2) \\
&> (p(s_1, s_2) - c)q_1(s_1, s_2) \\
&= \pi_1(s_1, s_2),
\end{aligned}$$

where the inequality follows from the definitions of (q_1, q_2) and s_1', and the "price first" rule.

Finally, by the definition of s_1', $p(s_1', s_2) \leq p(s_1, s_2)$. Overall, (s_1, s_2) is not a Nash equilibrium.

The case with $q_2 < \frac{a-c-bq_2}{2b}$ is perfectly symmetric.

Second, take $(q_1, q_2) \in \mathbb{R}_+^2$ such that $q_1 + q_2 > \frac{a-c}{b}$ and (s_1, s_2) such that $q_i(s_1, s_2) = q_i$ for each $i = 1, 2$. We prove that (s_1, s_2) is not a Nash equilibrium. To see this, without loss of generality let $q_1 > 0$, and observe

$$\pi_1(s_1, s_2) = (a - b(q_1 + q_2) - c)q_1 < (a - b\frac{a-c}{b} - c)q_1 = 0.$$

However, for s_1' such that $s_1'(p) = 0$ for all $p \in \mathbb{R}_+$, we have $\pi_1(s_1', s_2) = 0$. Hence, (s_1, s_2) is not a Nash equilibrium. $\square$

Proof of Proposition 2. Solving the differential equation given in the text, we obtain:

$$\begin{aligned}
&\frac{6b(3bq-(a-c))}{(a-c-bq)^2}dq = \lambda dt \\
&\iff 6\left(3\ln(a-c-bq) + \frac{2(a-c)}{a-c-bq}\right) = \lambda t + C,
\end{aligned}$$

where C is a constant. Given the initial condition $\lim_{t\downarrow 0} q(t) = \frac{a-c}{3b}$, we have

$$C = 18\left(\ln(a-c) + \ln\left(\frac{2}{3}\right) + 1\right).$$

The time at which the quantity reaches the collusive quantity $q^* = \frac{a-c}{4b}$ is $t^* = \frac{1}{\lambda}(36\ln(3) - 52\ln(2) - 2)$. Manipulating, we obtain the solution presented in the statement of the proposition.

Since this plan of quantities is essentially unique (in the sense defined in Section 4.1) in the revision game of the semi-Cournot competition, it follows that the plan of quantities induced by the strategy profile in Σ^{**} (and thus in Σ^*) is essentially unique in the revision game of the supply-schedule game. □

C.1.3 Calculating the Expected Payoff Bound

Let $\pi^* = \frac{(a-c)^2}{8b}$ be the payoff at q^*. The payoff at $q(0)$ is denoted $\pi^N = \frac{(a-c)^2}{9b}$.

The expected payoff can be bounded as follows:

$$\begin{aligned}
&e^{-\lambda t^*}(a-c-2bx^*)x^* + \int_0^{t^*}(a-c-2bx(\tau))x(\tau)\lambda e^{-\lambda\tau}d\tau \\
&\geq e^{-\lambda t^*}(a-c-2bx^*)x^* + (1-e^{-\lambda t^*})\frac{(a-c)^2}{9b} \\
&\geq e^{-(36\ln(3)-52\ln(2)-2)}\frac{(a-c)^2}{8b} + (1-e^{-(36\ln(3)-52\ln(2)-2)})\frac{(a-c)^2}{9b} \\
&= e^{-(36\ln(3)-52\ln(2)-2)}\pi^* + (1-e^{-(36\ln(3)-52\ln(2)-2)})\frac{8}{9}\pi^* \\
&= \left(e^{-(36\ln(3)-52\ln(2)-2)} + \frac{8}{9}(1-e^{-(36\ln(3)-52\ln(2)-2)})\right)\pi^* \\
&= \left(0.88683650092 + \frac{8}{9}(1-0.88683650092)\right)\pi^* \\
&= 0.98742627788\pi^*.
\end{aligned}$$

C.2 Voter Behavior, Proof, and the Detail for Remark 4 for Section 4.2

C.2.1 A Microfoundation of the Voter Behavior

The winning probability (13) is derived from the following behavior of voters. There is a unit mass of voters $i \in [0,1]$, where fraction θ is pro-THAAD and the others are anti-THAAD. We assume that the pro-THAAD voters constitute a majority, i.e.,

$$\theta \in (1/2, 1]. \tag{36}$$

Pro-THAAD voter i votes for A if

$$x_A + \widetilde{\eta}_A + \widetilde{\varepsilon}_{iA} > x_B + \widetilde{\eta}_B + \widetilde{\varepsilon}_{iB},$$

where $\widetilde{\eta}_A$ and $\widetilde{\eta}_B$ are candidate-specific shocks that are common to all voters, and $\widetilde{\varepsilon}_{iA}$ and $\widetilde{\varepsilon}_{iB}$ are voter-specific idiosyncratic shocks (there is no abstention). Pro-THAAD voter i votes for B under the symmetric condition.[6] Similarly, anti-THAAD voter i votes for A if

$$1 - x_A + \widetilde{\eta}_A + \widetilde{\varepsilon}_{iA} > 1 - x_B + \widetilde{\eta}_B + \widetilde{\varepsilon}_{iB},$$

and anti-THAAD voter i votes for B under the symmetric condition. We assume uniform distributions: $\widetilde{\eta}_B - \widetilde{\eta}_A \sim U[-c, c]$ and $\widetilde{\varepsilon}_{iB} - \widetilde{\varepsilon}_{iB} \sim U[-h, h]$. The latter is interpreted as "i.i.d." shocks across voters and we adopt the convention that the law of large numbers holds in the continuum population. We also assume that the voter-specific shocks are more important than the common shocks in the following sense:

$$h \geq 1 + c. \tag{37}$$

In the population of pro-THAAD voters, the fraction of voters voting for A given η_A and η_B is equal to

$$\begin{aligned}\Pr(x_A + \eta_A + \widetilde{\varepsilon}_{iA} > x_B + \eta_B + \widetilde{\varepsilon}_{iB}) &= \Pr(x_A - x_B + \eta_A - \eta_B > \widetilde{\varepsilon}_{iB} - \widetilde{\varepsilon}_{iA}) \\ &= \frac{1}{2} + \frac{x_A - x_B + \eta_A - \eta_B}{2h}.\end{aligned}$$

[6] The behavior of the voter i such that $x_A + \widetilde{\eta}_A + \widetilde{\varepsilon}_{iA} = x_B + \widetilde{\eta}_B + \widetilde{\varepsilon}_{iB}$ does not affect the analysis because the measure of such voters is zero.. The same remark applies to the anti-THAAD voter i satisfying an analogous equality.

Here we used our assumption that $\widetilde{\varepsilon}_{iB} - \widetilde{\varepsilon}_{iB} \sim U[-h, h]$ and (37), which ensures $-h \leq x_A - x_B + \eta_A - \eta_B \leq h$. The symmetric expression applies to the anti-THAAD voters, and therefore the total mass of A-voters given common shocks η_A and η_B is

$$\theta\left(\frac{1}{2} + \frac{x_A - x_B + \eta_A - \eta_B}{2h}\right) + (1-\theta)\left(\frac{1}{2} + \frac{(1-x_A) - (1-x_B) + \eta_A - \eta_B}{2h}\right)$$

$$= \frac{1}{2} + \frac{(2\theta - 1)(x_A - x_B) + \eta_A - \eta_B}{2h}.$$

Candidate A's winning probability, denoted $P_A(x_A, x_B)$, is the probability that the above fraction is more than $1/2$. This reduces to

$$P_A(x_A, x_B) = \Pr((2\theta - 1)(x_A - x_B) > \widetilde{\eta}_B - \widetilde{\eta}_A). \tag{38}$$

Recall that $\widetilde{\eta}_B - \widetilde{\eta}_A \sim U[-c, c]$ and assume that the support is large enough so that

$$2\theta - 1 \leq c. \tag{39}$$

Then, (38) is equal to

$$P_A(x_A, x_B) = \frac{1}{2} + \frac{(2\theta - 1)(x_A - x_B)}{2c}.$$

By defining $\delta := (2\theta - 1)/c$, we have obtained the proposed winning probability (13). Parameter restrictions (36) and (39) imply that $\delta \in (0, 1]$.

C.2.2 Proof of Proposition 3

Proof. The first-order condition is

$$0 = \frac{\partial \pi_A}{\partial x_A} = \frac{\delta}{2}((1 - x_A) + w) - \frac{1 + \delta(x_A - x_B)}{2} - \frac{\delta}{2}\gamma(1 - x_B). \tag{40}$$

This implies that, if x_A is a best response to x_B, then

$$x_A = \begin{cases} 0 & \text{if } x_B \leq \frac{\frac{1}{\delta} + \gamma - w - 1}{1+\gamma} \\ \frac{(1+\gamma)x_B - \gamma + w + \frac{\delta - 1}{\delta}}{2} \in (0, 1] & \text{if } x_B \in \left(\frac{\frac{1}{\delta} + \gamma - w - 1}{1+\gamma}, \frac{\frac{1}{\delta} + \gamma - w + 1}{1+\gamma}\right] \\ 1 & \text{if } x_B > \frac{\frac{1}{\delta} + \gamma - w + 1}{1+\gamma} \end{cases}. \tag{41}$$

The symmetric expression holds for B's best response to an arbitrary x_A. This enables us to compute the unique Nash equilibrium of the component game as in (15).

First, suppose that $w \leq \frac{(1-\delta)+\delta\gamma}{\delta}$. Then, the Nash equilibrium is the action profile that maximizes each candidate's payoff among symmetric action profiles. Thus, there is a unique optimal trigger strategy plan, and it is the one in which the Nash equilibrium action profile $(0,0)$ is played on (and off) the path.

Consequently, in what follows, we consider the case $\frac{(1-\delta)+\delta\gamma}{\delta} < w$. Under this assumption, we first solve for the optimal grim trigger strategy plan assuming that $\delta = 1$. Then, using the result for the case with $\delta = 1$, we solve for the optimal grim trigger strategy plan for the case with $\delta < 1$. We denote the optimal plan under parameter δ by $x^\delta(\cdot)$.

A. The case with $\delta = 1$:

First, we assume $\delta = 1$ and solve for x^1. Let us compute $d(x)$, $\pi(x)$ and π^N. By substituting x into x_A and x_B in (14), we have

$$\pi(x) = \frac{1}{2}\left((1+\gamma)(1-x) + w\right).$$

Thus, substituting (15) and $\delta = 1$ into this,

$$\pi^N = \begin{cases} \frac{1}{2}\left((1+\gamma)\frac{1-w}{1-\gamma} + w\right) & \text{if } w \leq \frac{1}{\delta} \\ \frac{w}{2} & \text{if } \frac{1}{\delta} < w \end{cases}.$$

A-1. The case with $\frac{(1-\delta)+\delta\gamma}{\delta} < w \leq \frac{1}{\delta}$:

Note that, in this case, $[0,1] \subseteq \left(\frac{\frac{1}{\delta}+\gamma-w-1}{1+\gamma}, \frac{\frac{1}{\delta}+\gamma-w+1}{1+\gamma}\right]$, and thus a best response to any $x_B \in [0,1]$ is $x_A = \frac{(1+\gamma)x_B - \gamma + w + \frac{\delta-1}{\delta}}{2}$. Substituting this into (14), setting $x = x^B$ and $\delta = 1$, and rearranging, for every $x \in [0,1]$,

$$d(x) + \pi(x) = \gamma(1-x) + \frac{1}{2}\left(\frac{(1-\gamma)(1-x)+w+1}{2}\right)^2.$$

Now, recall that our optimal plan is given by

$$\frac{dx^1}{dt} = \lambda\frac{d(x^1) + \pi(x^1) - \pi^N}{d'(x^1)}.$$

Hence, by substituting, we obtain

$$\frac{dx^1}{dt} = -\lambda \frac{(1-\gamma)^2(1-x^1) + (1-\gamma)w + 3 + 5\gamma}{2(1-\gamma)^2}.$$

This implies

$$\int \lambda dt = -\int \frac{2(1-\gamma)^2}{(1-\gamma)^2(1-x^1) + (1-\gamma)w + 3 + 5\gamma} dx^1,$$

which implies

$$\lambda t + C = 2\ln\left((1-\gamma)^2(1-x^1) + (1-\gamma)w + 3 + 5\gamma\right)$$

for some constant C. To solve for C, substitute (15) and $t = 0$ into this to get $C = 2\ln(4 + 4\gamma)$. Hence, we have

$$e^{\frac{\lambda}{2}t} = \frac{(1-\gamma)^2(1-x^1(t)) + (1-\gamma)w + 3 + 5\gamma}{4 + 4\gamma},$$

or

$$x^1(t) = \frac{-e^{\frac{\lambda}{2}t}(4+4\gamma) + ((1-\gamma)w + 4 + 3\gamma + \gamma^2)}{(1-\gamma)^2}. \tag{42}$$

A-2. The case with $\frac{1}{\delta} < w$:

Second, suppose that $\frac{1}{\delta} < w$. For each $w > \frac{1}{\delta}$, we have $\frac{\frac{1}{\delta}+\gamma-w+1}{1+\gamma} < 1$. Hence, (41) implies that there exists $\epsilon > 0$ such that for all $x \in (1-\epsilon, 1]$, action 1 is the unique best response to x. Thus, by substituting $x_A = 1$ into (14), setting $x_B = x$ and $\delta = 1$, we obtain

$$d(x) + \pi(x) = \frac{2-x}{2}w + \frac{x}{2}\gamma(1-x)$$

for $x \in (1-\epsilon, 1]$. This implies that $\frac{d(x)}{\pi(x)-\pi^N} = \frac{w-1+\gamma(x-1)}{1+\gamma}$ for all $x \in (1-\epsilon, 1]$. This converges to $\frac{w-1}{1+\gamma}$ as $x \to 1$, and it is strictly positive because $\frac{1}{\delta} < w$. Theorem 4 then implies that there is a unique trigger strategy equilibrium, and in that equilibrium, candidates play the Nash action all the time.

B. The case with general δ:

Now, we consider the case with $\delta < 1$ and solve for x^δ. To deal with this case, it is useful to introduce a change of a variable for each $i = A, B$ as follows:

$$y_i = 1 - \delta(1 - x_i).$$

Note that, since $x_i \in [0, 1]$, we have $y_i \in [1 - \delta, 1]$. Moreover,

$$y_i - y_j = \delta(x_i - x_j) \quad \text{and} \quad 1 - x_i = \frac{1}{\delta}(1 - y_i)$$

hold. Hence, the payoff in (14) can be rewritten as:

$$\begin{aligned}
\pi_A(x_A, x_B) = &\quad \frac{1 + (y_A - y_B)}{2}(\frac{1}{\delta}(1 - y_A) + w) + \frac{1 + (y_B - y_A)}{2} \cdot \gamma \cdot \frac{1}{\delta}(1 - y_B) \\
= &\quad \frac{1}{\delta}\left[\frac{1 + (y_A - y_B)}{2}((1 - y_A) + \delta w) + \frac{1 + (y_B - y_A)}{2} \cdot \gamma(1 - y_B)\right].
\end{aligned}$$

Note that this expression is proportional to (14) in which we substitute y_i into x_i for each $i = A, B$ and δw into w.[7] Hence, by (42), the optimal trigger strategy plan under general δ satisfies

$$y(t) = \begin{cases} \frac{-e^{\frac{\lambda}{2}t}(4+4\gamma)+((1-\gamma)\delta w+4+3\gamma+\gamma^2)}{(1-\gamma)^2} & \text{if } w \le \frac{1}{\delta} \\ 1 & \text{if } \frac{1}{\delta} < w \end{cases}.$$

Hence,

$$x^\delta(t) = \frac{y(t) - (1 - \delta)}{\delta} = \begin{cases} \frac{-e^{\frac{\lambda}{2}t}(4+4\gamma)+\delta(1-\gamma)(w+1-\gamma)+3+5\gamma}{\delta(1-\gamma)^2} = x^N - \frac{(e^{\frac{\lambda}{2}t}-1)(4+4\gamma)}{\delta(1-\gamma)^2} & \text{if } w \le \frac{1}{\delta} \\ 1 & \text{if } \frac{1}{\delta} < w \end{cases}.$$

Solving $x^\delta(t^*) = 0$, we obtain

$$t^* = \frac{2}{\lambda} \ln\left(\frac{\delta(1 - \gamma)(w + 1 - \gamma) + 3 + 5\gamma}{4 + 4\gamma}\right).$$

$\square$

[7] This and $\frac{dx_i}{dt}\delta = \frac{dy_i}{dt}$ imply the differential equation for general δ presented in Section 4.2.

C.3 Catering to the middle

If $\gamma = 0$ and $\delta = \frac{1}{2}$, the model can also be interpreted in the following manner: Suppose that the policy space is $[-1, 1]$. Candidate 1's bliss point is -1 and candidate 2's is 1. Candidate 1 is faithful to her party's identity ("left wing"), so she chooses a policy from $[-1, 0]$ while never wanting to choose a "right wing" policy in $(0, 1]$ possibly because of reputational concern. Symmetrically, candidate 2 chooses a policy from $[0, 1]$.[8] Each candidate's winning probability is determined by the standard Hotelling rule where the median voter is uniformly distributed over $[-1, 1]$, and a candidate's utility from the implemented policy is given by $\max\{1-z, 0\}$ where z is the distance from the bliss point to the implemented policy. Then, each candidate's payoff from any given policy profile $(y_A, y_B) \in [-1, 1]^2$ is given by (14) with substitutions that $x_A = 1 + y_A$ and $x_B = 1 - y_B$. Hence, our model nests such a model as a special case.

We note that, in this alternative model, the utility function for the implemented policy is convex. Such policy preferences are especially relevant for issues that provoke strongly opinionated reactions (e.g. same-sex marriage, abortion, gun control, and so forth). This is because, for these policy issues, it is natural to assume that one's utility arising from policy preferences sharply decreases as the implemented policy moves away from her bliss point.[9] The functional form $\max\{1 - z, 0\}$ is the simplest way to capture this possibility. This convexity allows for there to be a potential room for cooperation in the revision game as long as the Nash profile is not $(-1, 1)$.[10]

With this reformulation of the model, Proposition 3 shows that if the office motivation is not too large or too small, each candidate starts from announcing their most preferred policies (policy -1 for candidate 1 and policy 1 for candidate 2). They stick to such announcements until a certain time before the election day, and then begin catering to the middle towards the end of the campaign period.

[8] Technically, if each candidate can choose a policy from $[-1, 1]$, then there does not exist a pure Nash equilibrium in the component game because a best response does not necessarily exist. We conjecture that, if we allowed for candidates to choose their policies from $[-1, 1]$, there would exist a nontrivial equilibrium in which at each opportunity candidates mix across multiple policies. Kamada and Sugaya (2019) analyze a mixed strategy plan in the context of their model, and their simulation result shows quite complicated dynamics of mixing probabilities. For this reason, here we do not delve into such an analysis.

[9] See Osborne (1995) for a criticism on the use of concave utility functions for preferences over electoral policies. Kamada and Kojima (2014) discuss implications of convex voter utility functions.

[10] Note that, on the other hand, there would be no room for cooperation if the utility is concave as traditionally assumed in the political science literature.

D Appendix for Sections 5 and 7

D.1 Proofs for Section 5

D.1.1 Proof of Theorem 3

Proof. Take $\epsilon > 0$ for Assumption (*) and $k \in (0, 1)$ for condition (17) to hold.

Note first that, by condition (17) and Assumption (*)-3, we can find $\bar{a} \in (a^N, a^N + \epsilon]$ such that for all $a \in [a^N, \bar{a}]$, $(d(a))^k \le \pi(a) - \pi^N$ holds.

Next we introduce a generalized inverse of function d that is measurable. We will construct a non-trivial equilibrium plan from this function. Note that, by definition, $d(a) \ge 0$ for all a and $d(a) = 0$ means that a is a symmetric Nash equilibrium. Since we are assuming that a^N is the unique symmetric Nash equilibrium, $d > 0$ on $(a^N, \overline{a}]$. Our goal here is to find a measurable function $b : [0, d(\overline{a})] \to [a^N, \overline{a}]$ such that $d(b(\delta)) = \delta$ for each $\delta \in [0, d(\overline{a})]$. If d^{-1} exists in the given domain (i.e., if d is increasing on $[a^N, \bar{a}]$) then we let $b = d^{-1}$. More generally, we construct b as follows. First, define a function on $[a^N, \overline{a}]$ by

$$\overline{d}(a) := \max_{a' \in [a^N, a]} d(a').$$

This is well-defined because the function d is continuous on a compact set $[a^N, a]$. By construction, $\overline{d}$ is non-decreasing, and it is continuous by Berge's theorem of maximum.[11] By construction, $\overline{d}(0) = 0$ and $\overline{d}(\overline{a}) \ge d(\overline{a})$. Hence, the continuity of $\overline{d}$ implies that, for any $\delta \in [0, d(\overline{a})]$, there is some a_δ such that $\delta = \overline{d}(a_\delta)$. By the definition of $\overline{d}$, there must be some a^*_δ such that $\delta = \overline{d}(a_\delta) = d(a^*_\delta)$ (i.e., a^*_δ maximizes d on $[a^N, a_\delta]$). Define $b(\delta)$ to be a^*_δ.[12] By construction, b is an increasing function and therefore measurable.[13]

Now let $\hat{\epsilon} := \min\left\{ d(\bar{a})^{\frac{1-s}{2}}, \frac{\lambda(1-s)}{s+1} \right\}$. We are going to show that a trigger strategy

[11] The correspondence that maps a to $[a^N, a]$ is both upper and lower semicontinuous, and d is continuous. Hence, the conditions for Berge's theorem are satisfied.

[12] If a^*_δ is not unique, choose any one.

[13] Suppose b is not increasing and there are $\delta < \delta'$ such that $b(\delta) \ge b(\delta')$ $(\ge a^N)$. By the construction of b, there is some a_δ such that $b(\delta) \in \arg\max_{a' \in [a^N, a_\delta]} d(a')$. This implies that $d(b(\delta)) \ge d(a)$ for all $a \in [a^N, b(\delta)]$, and in particular for $a = b(\delta')$. Thus, we obtain $\delta = d(b(\delta)) \ge d(b(\delta')) = \delta'$, which contradicts our premise $\delta < \delta'$.

plan

$$x(t)=\begin{cases} b\left(t^{\frac{2}{1-k}}\right) & \text{if } t<\hat{\epsilon} \\ \\ b\left(\hat{\epsilon}^{\frac{2}{1-k}}\right) & \text{if } t\geq\hat{\epsilon} \end{cases} \tag{43}$$

satisfies the incentive constraint

$$\int_0^t \left(\pi(x(\tau))-\pi^N\right)\lambda e^{-\lambda\tau}d\tau \geq d(x(t))e^{-\lambda t} \tag{44}$$

for all $t\in[0,T]$. First, we show that the plan $x(t)$ is well-defined. Recall that $\hat{\epsilon}$ was defined to be less than $d(\bar{a})^{\frac{1-k}{2}}$, and therefore, for all $t\leq\hat{\epsilon}$, we have $t^{\frac{2}{1-k}}\leq\hat{\epsilon}^{\frac{2}{1-k}}\leq d(\bar{a})$. Hence, $t^{\frac{2}{1-k}}$ (for $t\leq\hat{\epsilon}$) is in the domain of b (i.e., $[0,d(\overline{a})]$), and therefore $x(t)$ given by (43) is indeed well-defined. Second, since b is measurable, the integral in the above incentive constraint is well-defined. Third, we show that the inequality in the incentive constraint (44) holds. To see this, first consider the case $t\leq\hat{\epsilon}$. We have

$$\begin{aligned} \int_0^t \left(\pi(x(\tau))-\pi^N\right)\lambda e^{-\lambda\tau}d\tau &\geq \int_0^t \left(d(x(\tau))\right)^k\lambda e^{-\lambda\tau}d\tau=\int_0^t \tau^{\frac{2k}{1-k}}\lambda e^{-\lambda\tau}d\tau \\ &> \lambda e^{-\lambda t}\int_0^t \tau^{\frac{2k}{1-k}}d\tau \\ &= \lambda e^{-\lambda t}\frac{1}{\frac{2k}{1-k}+1}t^{\frac{2k}{1-k}+1}=\left(\frac{\lambda(1-k)}{k+1}t^{-1}\right)t^{\frac{2}{1-k}}e^{-\lambda t} \\ &\geq t^{\frac{2}{1-k}}e^{-\lambda t}=d(x(t))e^{-\lambda t}. \end{aligned}$$

The first inequality follows from (i) $x(\tau)=b\left(\tau^{\frac{2}{1-k}}\right)\in[a^N,\bar{a}]$ (because the range of function b is $[a^N,\bar{a}]$) and (ii) $\pi(a)-\pi^N\geq(d(a))^k$ for all $a\in[a^N,\bar{a}]$ (as we have shown at the beginning of the proof). The last inequality follows from $t\leq\hat{\epsilon}\leq\frac{\lambda(1-k)}{k+1}$ (by the definition of $\hat{\epsilon}$).

Next, consider the case $t>\hat{\epsilon}$. We have

$$\begin{aligned} \int_0^t \left(\pi(x(\tau))-\pi^N\right)\lambda e^{-\lambda\tau}d\tau &\geq \int_0^{\hat{\epsilon}} \left(\pi(x(\tau))-\pi^N\right)\lambda e^{-\lambda\tau}d\tau \\ &\geq \hat{\epsilon}^{\frac{2}{1-k}}e^{-\lambda\hat{\epsilon}}\geq\hat{\epsilon}^{\frac{2}{1-k}}e^{-\lambda t}=d(x(t))e^{-\lambda t}. \end{aligned}$$

The first inequality follows from $\pi(x(\tau))-\pi^N=\pi(b(\hat{\epsilon}^{\frac{2}{1-k}}))-\pi^N\geq 0$ for all $\tau>\hat{\epsilon}$ because (i) the range of b is $[a^N,\overline{a}]$ and (ii) $\pi(a)-\pi^N\geq 0$ for all $a\in[a^N,\overline{a}]$ by

Assumption (*)-3. The second inequality follows from the third inequality for the case of $t \leq \hat{\epsilon}$.

Hence, the non-trivial plan (43) satisfies the incentive constraint (44) for all $t \in [0, T]$. This completes the proof. $\square$

D.1.2 Proof of Theorem 4

First, we introduce notation to define general strategies in the revision game. Let a history h_t at time $t \in [0, T)$ be a description of the current remaining time, the action profile at time $-T$, and a sequence of pairs of the remaining time and the action profile chosen at the past opportunities, as follows:

$$h_t = (t, a^T, (t^k, a^k)_{k=1}^n)$$

for some nonnegative integer n, where $t^k \in (t, T)$ for any k, and $t^{k-1} > t^k$ for any integer k no less than 2 and no more than n. Note that the description of h_t does not include the information about the action profile taken at time $-t$. Let H_t be the set of all such histories. The set of histories at time $-T$, H_T, is a singleton set consisting of a null history. Let $H = \bigcup_{t \in [0,T]} H_t$. Player i's (pure) strategy is defined as a mapping $\sigma_i : H \to A_i$. We define (pure-strategy) SPE in the standard manner.

Proof. First, let us introduce a few notations. Denote by "$\tilde{h}_t = h_t$" the event under which the history at time $-t$ is $h_t \in H_t$. We also denote by $h_t^+ = (h_t, a)$ a pair of a history at $-t$ and an action profile taken at $-t$. Denote by "$\tilde{h}_t^+ = (h_t, a)$" the event under which the history at time $-t$ is $h_t \in H_t$ and players take the action profile a.

Now, fix a SPE σ. Step 1 shows that, if players play an action profile a' under some history at some time $-t$ under σ, then $\pi_i(a') = \pi_i(a^N)$ holds for each player i. Then we show in Step 2 that only a^N can be played under any history under σ.

Step 1: Only the Nash payoff is possible under σ

Suppose that at time $-t \in [-T, 0]$, it is the case that for every time $-s > -t$, if an action profile $a' \in A$ is taken in some SPE then $\pi_i(a') = \pi_i(a^N)$ holds. We will show that for any i, $\pi_i(\sigma(h_{t+\epsilon})) = \pi_i(a^N)$ for any $h_{t+\epsilon} \in H_{t+\epsilon}$ if $\epsilon > 0$ is sufficiently small.

Step 1-1: Defining $\mathcal{C}$, $\bar{\mathcal{D}}$, and $\underline{\mathcal{D}}$

Fix $\epsilon \geq 0$ and take an arbitrary history $h_{t+\epsilon} \in H_{t+\epsilon}$. Let $\mathcal{C}$ be the continuation payoff from following σ_i at history $h_{t+\epsilon}$, and $\bar{\mathcal{D}}$ be the supremum continuation payoff from a deviation. Note that $\mathcal{C}$ can be written as follows:

$$\mathcal{C} = \underbrace{(1-e^{-\lambda t})}_{\text{Prob of at least one arrival in } (-t,0]} \times \pi_i^N + \underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in } (-t,0]} \times$$

$$\left[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in } (-t-\epsilon,t]} \pi_i(\sigma(h_{t+\epsilon})) + \int_0^\epsilon \underbrace{\mathbb{E}_\sigma[\pi_i(\sigma(h_{t+s}))|\tilde{h}^+_{t+\epsilon} = (h_{t+\epsilon}, \sigma(h_{t+\epsilon}))]}_{\text{Payoff when the final arrival is at } -t-s} \lambda e^{-\lambda s} ds \right].$$

The incentive compatibility condition for player i at history $h_{t+\epsilon}$ can be expressed as

$$\bar{\mathcal{D}} \leq \mathcal{C}.$$

In Step 1-2, we show that this incentive compatibility condition implies

$$\underline{\mathcal{D}} \leq \mathcal{C},$$

where

$$\underline{\mathcal{D}} := \underbrace{(1-e^{-\lambda t})}_{\text{Prob of at least one arrival in } (-t,0]} \times \pi_i^N +$$

$$\underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in } (-t,0]} \times \left[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in } (-t-\epsilon,-t]} \times (\pi_i(\sigma(h_{t+\epsilon})) + d(\sigma(h_{t+\epsilon})) \right.$$

$$\left. + \int_0^\epsilon \underbrace{\inf_{a \in A} \mathbb{E}_\sigma[\pi_i(\sigma(h_{t+s}))|\tilde{h}^+_{t+\epsilon} = (h_{t+\epsilon}, a)]}_{\text{Infimum payoff when the final arrival is at } -t-s} \lambda e^{-\lambda s} ds \right].$$

Step 1-2: Showing $\bar{\mathcal{D}} \leq \mathcal{C} \implies \underline{\mathcal{D}} \leq \mathcal{C}$

To prove that $\bar{\mathcal{D}} \leq \mathcal{C}$ implies $\underline{\mathcal{D}} \leq \mathcal{C}$, it suffices to prove that $\underline{\mathcal{D}} \leq \bar{\mathcal{D}}$. To see why this inequality holds, define i's expected continuation payoff from deviating to $a_i \in A_i$ at history $h_{t+\epsilon}$ and then following σ_i thereafter:

$$\mathcal{D}(a_i) := \underbrace{(1-e^{-\lambda t})}_{\text{Prob of at least one arrival in } (-t,0]} \times \pi_i^N +$$

$$\underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in } (-t,0]} \times \left[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in } (-t-\epsilon,-t]} \times (\pi_i(a_i, \sigma(h_{t+\epsilon}))) \right.$$

$$\left. + \int_0^\epsilon \underbrace{\mathbb{E}_\sigma[\pi_i(\sigma(h_{t+s}))|\tilde{h}^+_{t+\epsilon} = (h_{t+\epsilon}, (a_i, \sigma_{-i}(h_{t+\epsilon})))]}_{\text{Payoff when the final arrival is at } -t-s} \lambda e^{-\lambda s} ds \right]$$

By the definition of $d_i(\cdot)$, there must exist a sequence $\{a_i^k\}_{k=1}^\infty$ such that $\pi_i(a_i^k, \sigma_{-i}(h_{t+\epsilon})) \to \pi_i(\sigma(h_{t+\epsilon})) + d_i(\sigma(h_{t+\epsilon}))$ as $k \to \infty$. Hence, for any $\xi > 0$, there exists $K_\xi < \infty$ such that for all $k > K_\xi$, $\pi_i(a_i^k, \sigma_{-i}(h_{t+\epsilon})) \geq \pi_i(\sigma(h_{t+\epsilon})) + d_i(\sigma(h_{t+\epsilon})) - \xi$.

Therefore, for any $\xi > 0$, $k > K_\xi$, we have

$$\mathcal{D}(a_i^k) \geq \underbrace{(1 - e^{-\lambda t})}_{\text{Prob of at least one arrival in } (-t,0]} \times \pi_i^N +$$

$$\underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in } (-t,0]} \times \left[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in } (-t-\epsilon,-t]} \times (\pi_i(\sigma(h_{t+\epsilon})) + d_i(\sigma(h_{t+\epsilon})) - \xi) \right.$$

$$\left. + \int_0^\epsilon \underbrace{\mathbb{E}_\sigma[\pi_i(\sigma(h_{t+s}))|\tilde{h}^+_{t+\epsilon} = (h_{t+\epsilon}, (a_i^k, \sigma_{-i}(h_{t+\epsilon})))]}_{\text{Payoff when the final arrival is at } -t-s} \lambda e^{-\lambda s} ds \right]. \tag{45}$$

By the definition of $\underline{\mathcal{D}}$, the right-hand side of (45) is no less than $\underline{\mathcal{D}} - e^{-\lambda(t+\epsilon)}\xi$. Hence, we have

$$\underline{\mathcal{D}} - e^{-\lambda(t+\epsilon)}\xi \leq \mathcal{D}(a_i^k) \tag{46}$$

for any $\xi > 0$ and $k > K_\xi$.

Note also that, for any k, deviating to a_i^k and following σ_i thereafter is a feasible deviation. Thus, for any k, we have

$$\mathcal{D}(a_i^k) \leq \bar{\mathcal{D}}. \tag{47}$$

Conditions (46) and (47) imply:

$$\underline{\mathcal{D}} - e^{-\lambda(t+\epsilon)}\xi \leq \bar{\mathcal{D}}$$

for any $\xi > 0$. Thus, we obtain

$$\underline{\mathcal{D}} \leq \bar{\mathcal{D}}.$$

Hence, the incentive compatibility condition ($\bar{\mathcal{D}} \leq \mathcal{C}$) implies $\underline{\mathcal{D}} \leq \mathcal{C}$.

Step 1-3: Bounding $|\pi_i(\sigma(h_{t+\epsilon})) - \pi^N|$

Now, manipulating $\underline{\mathcal{D}} \leq \mathcal{C}$, we obtain:

$$d_i(a(h_{t+\epsilon}, \sigma)) \leq e^{\lambda\epsilon} \int_0^\epsilon \left(\mathbb{E}_\sigma[\pi_i(\sigma(h_{t+s}))|\tilde{h}_{t+\epsilon} = h_{t+\epsilon}] - \inf_{a \in A} \mathbb{E}_\sigma[\pi_i(\sigma(h_{t+s}))|\tilde{h}^+_{t+\epsilon} = (h_{t+\epsilon}, a)] \right) \lambda e^{-\lambda s} ds. \tag{48}$$

If $|\pi_i(\sigma(h_{t+s})) - \pi^N| \leq M$ for all $s \in [0, \epsilon]$, (48) implies

$$d_i(\sigma(h_{t+\epsilon})) \leq e^{\lambda\epsilon} \int_0^\epsilon 2M\lambda e^{-\lambda s} ds,$$

where the right-hand side is no more than $2M\lambda\epsilon e^{\lambda\epsilon}$. This and condition (18) imply:

$$|\pi_i(\sigma(h_{t+\epsilon})) - \pi^N| \leq \frac{2\lambda}{m}\epsilon e^{\lambda\epsilon} M,$$

where

$$m = \inf_{a \in A \setminus \{a^N\}} \frac{d_i(a)}{|\pi_i(a) - \pi_i^N|} > 0$$

is a positive number implied by condition (18). The same argument can be used to show that for any $s \in [0, \epsilon]$, for any $h_{t+s} \in H_{t+s}$,

$$|\pi_i(\sigma(h_{t+s})) - \pi^N| \leq \frac{2\lambda}{m} s e^{\lambda s} M,$$

where the right-hand side is no more than $\frac{2\lambda}{m}\epsilon e^{\lambda\epsilon}M$. Hence, we conclude that if $|\pi_i(\sigma(h_{t+s})) - \pi^N| \leq M$ for all $s \in [0, \epsilon]$, then $|\pi_i(\sigma(h_{t+s})) - \pi^N| \leq \frac{2\lambda}{m}\epsilon e^{\lambda\epsilon}M$ for all $s \in [0, \epsilon]$.

Since $|\pi_i(\sigma(h_{t+s})) - \pi^N| \leq \bar{\pi}_i - \underline{\pi}_i$ for all $s \in [0, \epsilon]$, this implies that for any positive integer n, we have

$$|\pi_i(\sigma(h_{t+\epsilon})) - \pi^N| \leq \left(\frac{2\lambda}{m}\epsilon e^{\lambda\epsilon}\right)^n (\bar{\pi}_i - \underline{\pi}_i).$$

Notice that there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon})$, $\frac{2\lambda}{m}\epsilon e^{\lambda} < 1$ holds. Hence, there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon})$, the only action profile $\sigma(h_{t+\epsilon})$ that satisfies the above equality for all n is $\pi_i(\sigma(h_{t+\epsilon})) = \pi_i(a^N)$.

Hence, for every time $-t \in [-T, 0]$, in any SPE, if an action profile a' is taken under some history at $-t$, then for each player i, we have $\pi_i(a') = \pi_i(a^N)$.

Step 2: Only Nash action is possible under σ

Now, suppose that under σ, there exists some t and $h_t \in H_t$ such that $\sigma(h_t) \neq a^N$. Then, player i's incentive compatibility condition at h_t can be written as follows:

$$e^{-\lambda t}\left(\pi_i(\sigma(h_t)) + d_i(\sigma(h_t))\right) + (1 - e^{-\lambda t})\pi_i^N \leq \pi^N,$$

which is equivalent to $d_i(\sigma(h_t)) \leq 0$. However, since a^N is a unique Nash equilibrium and $\sigma(h_t) \neq a^N$, there exists i such that $d_i(\sigma(h_t)) > 0$. This is a contradiction. Hence, we conclude that for any $t \in [0, T]$, for any $h_t \in H_t$, we have $\sigma(h_t) = a^N$. $\square$

D.2 Details for Section 7.1

D.2.1 Soft Deadline

The precise statements of the claims about introducing a soft deadline in Section 7.1 are as follows:

Proposition 8. *The following hold for any $T \geq 0$ and $\gamma \in (0, 1]$ in the soft deadline game.*

1. *Under (17), for any $\epsilon > 0$, cooperation can be sustained by a trigger strategy equilibrium.*
2. *Under (18), there exists $\bar{\epsilon} < 1$ such that for all $\epsilon < \bar{\epsilon}$, there is a unique trigger strategy equilibrium, and it has the plan $x(t) = a^N$ for each t.*

Proof. For any trigger strategy equilibrium plan $x(\cdot)$, define action a^x by an arbitrary action in A that satisfies the following equation:

$$\pi(a^x) = \sup_{t \geq 0} \pi(x(t)). \tag{49}$$

This is well defined because the action space is compact and π is continuous. Plan $x(\cdot)$ satisfies the following incentive constraint at any time $t \geq 0$,

$$\pi(x(t))P + \sum_{s=1}^{\infty} \pi(x(t+s))R_s \geq (\pi(x(t)) + d(x(t))\, P + \pi^N(1-P), \tag{50}$$

where P is the probability of no revision in the future and R_s is the probability that the last revision opportunity arrives s periods later. By definition, $\sum_{s=1}^{\infty} R_s = 1 - P$, and therefore (49) implies that the left-hand side of (50) is less than or equal to $\pi(a^x)$. By considering the right-hand side of (50) for a convergent subsequence $t = t_1, t_2, ...$ where $\lim_{n\to\infty} \pi(x(t_n)) = \pi(a^x)$ and $\lim_{n\to\infty} d(x(t_n)) = d(a^x)$ (both convergence can simultaneously hold because d is continuous as well), we obtain

$$\pi(a^x) \geq (\pi(a^x) + d(a^x))\, P + \pi^N(1-P), \tag{51}$$

which can be arranged as

$$\frac{d(a^x)}{\pi(a^x) - \pi^N} \leq \frac{1}{P} - 1 \tag{52}$$

if $\pi(a^x) > \pi^N$. This is a necessary condition for any $x(\cdot)$ to be sustained by a trigger strategy equilibrium. Conversely, if this condition is satisfied and $x(\cdot)$ is a constant plan $\forall t \geq 0\ x(t) = a$, then $x(\cdot)$ constitutes an equilibrium in the subgame after period 0.[14]

Now consider condition (17). For any $\epsilon > 0$, there is a positive probability of a revision in the future, and therefore P, the probability of no revision in the future, is strictly less than 1, so that $\frac{1}{P} - 1 > 0$. Note that condition (17) implies $\lim_{a\to a^N} \frac{d(a)}{\pi(a)-\pi^N} = 0$. Hence, under (17), there exists $\varepsilon > 0$ such that for any $x(\cdot)$ such that $a^x \neq a^N$ and a^x is in the ε-neighborhood of a^N, condition (52) holds. Note that such $x(\cdot)$ exists because there is $a \in A \setminus \{a^N\}$ in the ε-neighborhood of a^N such that $\pi(a) > \pi^N$ by assumption. Hence, we conclude that cooperation can be sustained from time 0 on (this also implies that (weakly more) cooperation can be sustained in all periods before).

Next, consider condition (18). Note that P, the probability of no revision in the future, is equal to 1 if $\epsilon = 0$ (i.e., the game ends immediately), and P is continuous in

[14] This is because (51) is the incentive constraint for the constant plan.

ϵ.[15] Therefore, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$, the right-hand side of (52) is strictly less than the infimum in condition (18), which means that (52) is violated for any $x(\cdot)$ such that $\pi(a^x) > \pi^N$. Hence, in any trigger strategy equilibrium, players play a^N from period 0 on. By backward induction, a^N is also played in any period. $\square$

D.2.2 ε-Willingness to Punish

Recall that we consider a model in which time is discrete $-t = -T, ..., -1, 0$, and a revision opportunity arrives with probability $\gamma = 0.1$ in each period. In the continuous time model with Poisson arrival rate λ, a revision opportunity arrives approximately with probability $\lambda\Delta$ in a small time interval Δ. To compare the discrete- and continuous-time models, we assume $\lambda = 1$ and the length of one period in the discrete time models is $\Delta = 0.1$ (so that $\gamma = \lambda\Delta = 0.1$).

In the continuous time models with selfish players ($\varepsilon = 0$), the optimal plan in Model 1 turns out to be $x(t) = t$ until it hits the optimal action 1.[16] In Model 2, the optimal plan exhibits no cooperation: $x(t) = 0$ for all t. Those predictions of the revision games are depicted by the thick curves in Figure 5.[17]

The discrete time models with ε-incentive to punish can be numerically solved backwards, and the solutions are depicted by thin curves in the figure (the discrete points are interpolated). As we can see, substantial cooperation is sustained even under very small incentive to punish (ε) in Model 1. In contrast, in Model 2, cooperation becomes very hard to sustain as the incentive to punish decreases.

Let us summarize the implications of our analysis above. In realistic situations, the assumption of the revision game that there is always a positive probability that another revision is possible may not be satisfied. In such a situation, however, if players have small incentive to punish a deviator, the predictions of the revision game survive: substantial cooperation is possible if and only if cooperation is sustained in the revision game.

[15]This can be verified by $P = \sum_{t=0}^{\infty}(1-\epsilon)(1-\gamma)^t\epsilon^t$.

[16]This is the solution to the differential equation $\frac{dx}{dt} = \lambda\frac{d(x)+\pi(x)-\pi^N}{d'(x)} = \frac{x^2+(2x-x^2)}{2x} = 1$.

[17]In the figure, t corresponds to the remaining time in the continuous-time models (revision games).

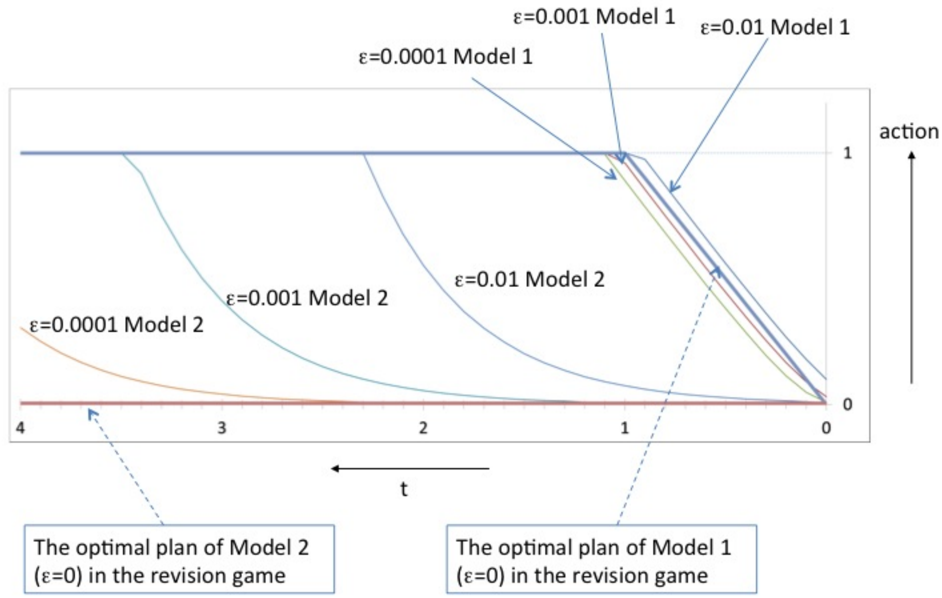

Figure 5: Finite Time Condition

D.3 Detailed Discussion for Section 7.2

Revision games have some similarities to stochastic games (e.g., Dutta (1995) and Hörner, Sugaya, Takahashi and Vielle (2011)) and to repeated games with a decreasing discount factor (Bernheim and Dasgupta, 1995). We compare those models in the framework of a general stochastic-game model, where players obtain flow payoffs in *each* period $n = 0, 1, 2, ...$ Although a payoff accrues only once (at the deadline) in the revision game, it can be interpreted as a special case of this general stochastic game. This observation facilitates a clear-cut comparison.

In the general stochastic-game model, the flow playoff to player i in period $n = 0, 1, ...$ is

$$u_i(a(n), s(n))$$

where $a(n)$ and $s(n) \in S$ are the action profile and state in period n, respectively. The probability distribution of $s(n + 1)$ depends on $s(n)$ and $a(n)$. The game is terminated at the end of period n with probability $r(s(n), a(n))$. The flow payoff in period n is discounted by $D(s(n), n)$. Therefore, if the game is terminated at the end of period N, the realized payoff is

$$\sum_{n=0}^{N} u_i(a(n), s(n)) D(s(n), n).$$

Dynamic games that have some similarities to revision games are special cases of this general model.

- The first stochastic-game model proposed by Shapley (1953): $D(s(n), n) \equiv 1$ (discounting comes from random termination).

- Scholastic games where the folk theorems have been proved (Dutta (1995) and Hörner, Sugaya, Takahashi and Vielle (2011)): $D(s(n), n) = \delta^n$ and $r(s(n), a(n)) \equiv 0$ (no random termination).

- Repeated games with a decreasing discount factor (Bernheim and Dasgupta, 1995): S is a singleton (there is no state variable), $D(n) = \delta(0) \times \cdots \times \delta(n)$ ($\lim_{n\to\infty} \delta(n) = 0$) and $r(a(n)) \equiv 0$ (no random termination).

We will show that the revision game is also strategically equivalent to a special case of this model, which we call "Game F."[18] To transform the revision game into Game F, first interpret the beginning of the revision game as period $n = 0$, and interpret the n^{th} arrival of a Poisson revision opportunity as period $n > 0$. Define $s(n)$, the state in period n, as the remaining time in the revision game (state space is $S = [0, T]$), and define the termination probability at a given remaining time (=state) by the probability of no future revision opportunity at such a remaining time in the revision game.[19] Each player i obtains in each period n discounted flow payoff

$$\pi_i(a(n)) e^{-\lambda s(n)}.$$

Let us now show that the revision game and Game F share the same expected payoff. We present such an analysis for the case in which the current action depends only on the current state (the remaining time).[20] In general, if payoff $f(t)$ accrues

[18] The "F" stands for flow playoffs.

[19] This provides the following state transition and termination probability. The initial state is $s(0) = T$. Given $s(n)$, the probability of $s(n+1) \in [s(n), T]$ is zero, the density of $s(n+1) \in [0, s(n))$ is $e^{-\lambda(s(n+1)-s(n))}\lambda$, and the probability of termination of the game is $e^{-\lambda s(n)}$.

[20] A more general case is given in Appendix D.3.1.

each time a Poisson arrival happens, the expected sum of realized payoffs is

$$\int_0^T f(t)\lambda dt.$$

Heuristically, this is true because the probability of one Poisson arrival in a small time interval Δ is approximately equal to $\lambda\Delta$. If we apply this formula for $f(t) = \pi_i(a(t))e^{-\lambda t}$ (Game F's payoff) where $a(t)$ denotes the action taken at state t, we can see that the expected payoff in Game F is

$$\pi_i(a(T))e^{-\lambda T} + \int_0^T \pi_i(a(t))e^{-\lambda t}\lambda dt, \tag{53}$$

which is exactly the same as that in the revision game.

The equivalence between the revision game and Game F enables us to directly compare the revision game with related dynamic games. First, the mechanism to sustain cooperation in all those models are the same: the deviation gain today is wiped out by the reduction of the future payoff. Secondly, discounting of the future payoff is obtained by random termination of the game. This is the sole mechanism of discounting in Shapley (1953), and it also partially accounts for the discounting in Game F ($\simeq$ the revision game). Milnor and Shapley (1957) provided a stochastic game model of gamblers' ruin where a payoff accrues only when the game is terminated (i.e., one of the players is ruined). The timing of payoff realization of this game is similar to that of the revision game in that a payoff accrues only once.[21]

Thirdly, the revision game ($\simeq$ Game F) is not a special case of the stochastic games where the folk theorems have been proved, such as Dutta (1995) and Hörner, Sugaya, Takahashi and Vielle (2011). In the latter, it is crucial that the state transition is *irreducible*: any state s is reachable from any state s' in a finite number of periods with probability one. This is clearly not the case in Game F, where the state, which corresponds to the remaining time in the revision game, never increases. Hence, the revision game belongs to the class of stochastic games (non-irreducible ones) for which

[21] The discussion paper version of this paper (Kamada and Kandori, 2017a) and Sherstyuk, Tarui and Saijo (2013) observe that the discounted repeated game is strategically equivalent to the game where (i) players prepare actions of the stage game in each period $t = 0, 1, ..,$ (ii) the game is terminated at the end of each period with some probability $r > 0$, and (iii) players obtain the stage game payoff in the last period only. Kamada and Kandori (2017a) call this game the stationary revision game.

the set of equilibria has not been fully characterized.

Fourthly, there is a similarity and difference between the revision game and the repeated-game model with a decreasing discount factor (Bernheim and Dasgupta, 1995). When the current state is $s \in [0, T]$ in Game F, by the same argument to derive formula (53), the continuation playoff is

$$\int_0^s \pi_i(a(t))e^{-\lambda t}\lambda dt. \tag{54}$$

In contrast, the continuation payoff in the Bernheim-Dasgupta model in period m is

$$\sum_{n=m+1}^{\infty} \pi_i(a(n)) \prod_{k=m+1}^{n} \delta(k). \tag{55}$$

Those formulae clarify the similarity and difference between Game F ($\simeq$ the revision game) and the Bernheim-Dasgupta model.

- In both models, the magnitude of continuation payoff decreases as time passes by: both (54) and (55) tend to zero as $s \to 0$ (s corresponds to the remaining time in the revision game) and $m \to \infty$ (because $\lim_{k\to\infty} \delta(k) = 0$) if $|\pi(a)|$ is uniformly bounded.

- In the continuation payoffs, smaller weights are attached to the future flow payoffs in the Bernheim-Dasgupta model, while the opposite is true in Game F ($\simeq$ the revision game).

To see the latter point, note that, in Game F, the discounted flow payoff at state s (= the remaining time in the revision game) is

$$\pi_i(a(s))e^{-\lambda s},$$

whose coefficient of discounting $e^{-\lambda s}$ *increases* as time passes by (as s decreases). This fact translates into different efficiency properties of those games: the optimal trigger strategy equilibrium attains approximate efficiency in the Bernheim-Dasgupta model, while this is not true in the revision game.[22] In summary, the revision game has some

[22] The action distribution induced by the optimal trigger strategy equilibrium (Proposition 1) fails to attach probability close to 1 to the efficient action.

similarities to the Bernheim-Dasgupta model, but the former is not a reformulation of the latter in a continuous time/state framework.

D.3.1 The Non-Markov Case

We formally show that the revision game and Game F share the same strategy spaces and payoffs. At remaining time/state s, players in those games observe a sequence of past events and the current state

$$((T, a(T)), (s(1), a(1)), ..., (s(n), a(n)), s),$$

where $T > s(1) > \cdots > s(n) > s$ and n is an arbitrary positive integer. This records that players were called upon to move when the remaining time/state was $T, s(1), \ldots, s(n)$, and it also shows what action profiles were chosen in those occasions. Let H_s be the set of all those histories at s. A (pure) strategy of player i, denoted σ_i, both in the revision game and in Game F, is a mapping from histories to current actions $\sigma_i : \bigcup_{s=0}^{T} H_s \to A$, where H_T is a singleton set of the initial dummy history to determine the initial action.

A strategy profile σ and Poisson arrival rate λ determines the probability measure $P_s^{\sigma,\lambda}$ over the set of possible histories at s, H_s. Since a Poisson arrival at time $-s$ is independent of the past history, $P_s^{\sigma,\lambda}$ is also equal to the probability measure of the past history conditional on a Poisson arrival at time $-s$. Hence, the expected payoff associated with the action profile chosen at s is expressed as

$$\int_{H_s} \pi_i(\sigma(h))dP_s^{\sigma,\lambda}(h) =: \pi_i(s).$$

Given this definition, by the same argument as in Appendix D.3, the expected payoff to player i, both in the revision game and in Game F, is given by

$$\pi_i(T)e^{-\lambda T} + \int_0^T \pi_i(s)e^{-\lambda s}\lambda ds.$$

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