Revision Games*

Part I: Theory

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Abstract

We show that players can cooperate even when they interact only once, if they prepare and revise their actions before they actually interact. We propose a class of games called *revision games*, to formalize such a situation. In a revision game, players start with initially prepared actions, followed by a sequence of random revision opportunities. In the course of revisions players closely monitor each other's behavior. It is shown that players can cooperate, and that their behavior under the optimal equilibrium is described by a simple differential equation. In the case of quantity competition, players can achieve 97% of fully collusive profit.

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1 Introduction

The purpose of this paper is two-fold. First, we present a previously unidentified mechanism to sustain cooperation. In particular, we show that cooperation can be sustained in a non-repeated interaction, if players prepare and revise their actions before they interact. Second, we propose a class of stylized models, which we call *revision games*, to formulate and analyze such a situation.

In social or economic problems, agents often prepare and revise their actions in advance before they interact. For example, when researchers compete for a research grant, they prepare and revise their research proposals before the deadline for submission. Likewise, although quantity competition is usually treated as a simultaneous-move game, in reality firms prepare their outputs in advance, and they often revise their production plans over time. In the case of price competition, firms may quote prices before the opening of a market, and they may revise their prices after observing each other's quotes. We show that in such a situation players can cooperate (or collude) even though they interact only once, if their prepared actions cannot be hidden.

A revision game starts at time $-T$ and ends at time 0 (time is continuous). Players prepare actions at the beginning, and then they obtain revision opportunities according to a Poisson process. Prepared actions are mutually observable, and the actions chosen at the last revision opportunity are played at time 0. In the class of well-behaved *smooth* payoff functions, we show that players can cooperate by a version of the trigger strategy defined over the revision process. The optimal revision plan has a tractable characterization: it is given by a simple differential equation. We apply this result to various problems including quantity competition, price competition and electoral campaigns, and show that players can often achieve substantial level of cooperation.[1]

Let us explain how revision games work, when applied to the Cournot duopoly (quantity competition) with a linear demand curve and an identical and constant marginal cost. Players' behavior in the revision game is represented by a *revision plan* $x(t)$. This means that, when a revision opportunity arrives at time $-t$, they are supposed to adjust their quantity to $x(t)$. The *trigger strategy* (in revision games)

[1]The present paper analyzes the Cournot market. All other applications can be found in a companion paper, Kamada and Kandori (2017).

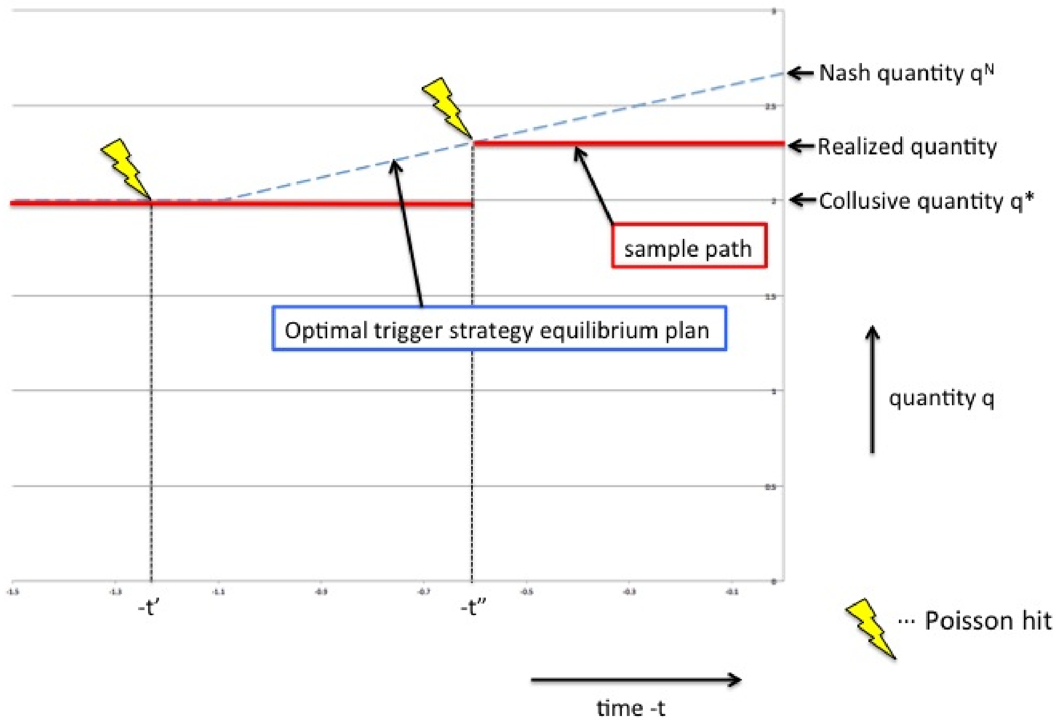

Figure 1: The optimal trigger strategy equilibrium plan and a sample path for the revision game of Cournot duopoly. Arrival rate $\lambda = 1$

stipulates that, if anyone fails to follow the revision plan, they choose the static Nash equilibrium quantity at all *future* revision opportunities. We will show that the optimal revision plan supported by the trigger strategy is given by a simple differential equation

$$\frac{dx}{dt} = \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)},$$

where $d(x)$ and $\pi(x)$ are the gain from deviation and the payoff when each player chooses quantity x, π^N is the Nash equilibrium payoff, and λ is the Poisson arrival rate. A closed-form solution $x(t)$ is obtained and it is depicted as a dashed curve of Figure 1. This optimal trigger strategy equilibrium plan starts at the collusive quantity and then follows the differential equation. Finally it reaches the Nash equilibrium quantity at the end of the revision game. In contrast, the bold segments depict a typical *realized path* of quantity. Whenever a revision opportunity arrives, the prepared quantity is adjusted according to the revision plan. At time $-t'$, the first revision opportunity arrives, but players do not revise their quantity. Closer to the deadline at time $-t$", a revision opportunity comes again, and the players increase their quantity to x''. Then they encounter no more revision opportunity, and the amount shipped into the market is x'' (thus, the final outcome is random and determined by when the last revision opportunity arrives). The expected profit turns out to be substantial: it is 97% of the fully collusive level.

Why can we sustain cooperation in revision games? When a revision opportunity arrives near the end of a revision game, there is still a very small but positive probability that another revision is possible in the remaining time. This means that, if a player cheats now, he has some (small) probability of being punished in the future. Hence players can cooperate a little bit near the end of the game. Using this as a foothold, players can cooperate more, when a revision opportunity arrives before. By a repeated application of this mechanism, players can cooperate substantially when they are far away from the end of the revision game.

A motivating story would be helpful to understand a situation that could possibly be represented by the quantity revision game. Two fishing boats depart from a harbor early in the morning, and they must return when the fish market, located near the harbor, opens at 6:00 am (this is the end of the revision game). They wish to collude (i.e., to restrict their catch) so as to increase the price at the fish market. They start with a small amount of catch (the collusive quantity). They operate side by side, closely monitoring each other's behavior. A revision opportunity of their quantities corresponds to the arrival of a fish school, which follows a Poisson process.[2] When the Poisson arrival rate is $\lambda = 0.1$ and the time unit is a minute, a fish school comes every ten minutes on average. According to the optimal trigger strategy equilibrium, the fishermen do not touch fish schools until 5:49AM. In the last eleven minutes, however, whenever a fish school visits them, they catch additional fish. If anyone deviates from this equilibrium plan, they catch a large amount (so that the total amount becomes the Nash quantity) when the next fish school arrives. In this way, the fishermen can happily obtain 97% of fully collusive profit.

Cooperation cannot always be sustained in revision games, and we derive a *necessary and sufficient* condition for cooperation. In particular, we show that the possibility of cooperation in revision games hinges on the well-known effect recognized

[2]Pun not intended.

by Akerlof and Yellen (1985). They observed that deviating from an optimal action entails a negligible effect on one's own payoff (because of the first order condition of optimality) but typically affects others' payoffs substantially. We will show that this is essential: cooperation is sustained if and only if the Akerlof-Yellen effect is present in an appropriate sense (see Section 6.1).

1.1 Related literature

Several papers examine revision processes that are different from ours. Kalai (1981) and Bhaskar (1989) examine models where players *can always react* to their opponents' revisions, and show that a unique and efficient outcome is obtained. In contrast, players in our revision games may not obtain a chance to react, thus full cooperation cannot be obtained. In Caruana and Einav (2008a, b) revision is possible at *any moment of time* with some *switching costs*. Thus, players have an incentive to act like the Stackelberg leader, by using the switching cost as a commitment device. Their second paper (2008b) considers the Cournot duopoly, and shows that the firms end up producing *more* than the Nash quantity in their equilibrium. Hence, unlike in our model, the outcome is *less* cooperative than the static Nash equilibrium.

Some existing works examine random revision opportunities. Vives (1995, 2001) present infinite-horizon discrete-time models, where in each period n, with probability γ_n the stage game payoff in that period is realized and the game ends immediately. The probability γ_n is nondecreasing in n. This includes the "stationary" revision game that we discuss in Section 2 as a special case (by assuming γ_n is a constant).[3] Those papers consider a continuum of agents (and a single large agent in Vives (2001)), and therefore there does not exist the kind of strategic interaction that is the main focus of our paper. Ambrus and Lu (2015) analyze a multilateral bargaining problem in a continuous-time finite-horizon setting where opportunities of proposals arrive via Poisson processes. Although their model is similar to ours, they focus on (the unique) Markov perfect equilibrium, which corresponds, in our model, to the trivial equilibrium where the Nash equilibrium action is always played.

Revision games are similar to repeated games, in the sense that the possibility of future punishment provides incentives to cooperate. We elaborate on this issue in Section 2 and Section 6.2. In this respect, the paper by Chou and Geanakoplos (1988)

[3]For the case of non-constant γ_n, Vives's model is closely related to Bernheim and Dasgupta (1995) that we discuss shortly. See Section 6.2 for detail.

is the most closely related to ours. They consider T-period repeated games in which a player can commit to a ε-optimal action in the final period, and prove that, if the stage game has a "smooth" payoff function, the folk theorem obtains as $T \to \infty$. That is, a little bit of commitment at the last period provides a basis for substantial cooperation in the entire game. Our paper shows that a similar mechanism can operate in revision processes, without the need for suboptimal behavior at the deadline. Bernheim and Dasgupta (1995) consider discrete-time, infinite-horizon repeated games in which a time-dependent discount factor tends to zero over time, and show that cooperation can be sustained if the discount factor falls sufficiently slowly. Our model is not isomorphic to theirs, as is explained in Section 6.2.

Ambrus et al. (2014) analyze a model of eBay-style auctions in which bidders have chances to submit their bids at Poisson opportunities before the deadline. They show that there are equilibria in which bidders gradually increase their bids. Their equilibria are built on the existence of multiple best replies in the eBay-style auction, and the mechanism behind the revision behavior in their model is different from ours.

Finally, various follow-up papers have been written while we were circulating earlier versions of the present paper. We discuss those papers in the companion paper (Kamada and Kandori, 2017).

The rest of the paper is organized as follows: The next section presents revision games without a deadline to help the readers build up some intuition. The main model (with a deadline) is presented in Section 3. In Section 4, we first prove existence, essential uniqueness and differentiability of the optimal plan. Then we derive the differential equation that characterizes the optimal plan. Section 5 provides a leading example of quantity competition. Section 6 examines the conditions for cooperation and compares our model to a repeated-game model with a decreasing discount factor. Section 7 concludes. The companion paper (Kamada and Kandori, 2017) presents various applications and examine the robustness of the results.

2 An Example (Two Samurai): Stationary Revision Games

The purpose of this paper is to analyze a class of games where (i) a normal-form game is played only once, (ii) players must prepare their actions in advance, (iii) the opportunities to revise prepared actions arrive randomly over time, and (iv) prepared actions are observable. We refer to those games as *revision games*. In this section, we start with a simple case, where the problem is stationary in the sense that in each period $t = 0, 1, 2, ...$, there is a fixed, positive probability p with which a given normal-form game is played. We refer to this class of revision games as *stationary revision games*. This class will turn out to be isomorphic to a familiar class of games, and it helps to build some intuition on how revision games work. The point we make is a simple one, so we just present an example.

Suppose that a rural village faces an attack of bandits. In each period $t = 0, 1, 2, ...$ the bandits attack the village with probability $p \in (0, 1)$ around midnight. They attack only once. The villagers hired two samurai, $i = 1, 2$, and they must prepare to defend the village (to show up at the village gate around midnight) or not (to hide away and watch the gate from a distance). Hence in each period they observe each other's prepared actions. When the bandits attack, the samurai receive the following payoffs.

	Defend	Hide
Defend	2, 2	-1, 3
Hide	3, -1	0, 0

This is a Prisoner's Dilemma game. Now consider player i's expected payoff. We denote player i's payoff by $\pi_i(t)$, when the bandits' attack occurs at time t. We also assume that players have a common discount factor $\delta \in (0, 1)$. Player i's expected payoff is

$$\begin{aligned} & p\pi_i(0) + \delta(1-p)p\pi_i(1) + \delta^2(1-p)^2 p\pi_i(2) + \cdots \\ = \; & p\sum_{t=0}^{\infty} \overline{\delta}^t \pi_i(t), \end{aligned}$$

where $\overline{\delta} := \delta(1-p)$. Theorefore, *stationary revision games are isomorphic to infinitely repeated games*, and cooperation can be sustained in a subgame perfect equilibrium if δ is high and p is small.[4] Even though the prisoner's dilemma game is played only once, players manage to cooperate. A mechanism to sustain cooperation is the trigger strategy, which works as follows. As long as the samurai have been showing up at the gate, they continue to do so (to prepare to defend the village). If anyone hides away, however, they stop showing up at the gate in the future.

The next section deals with our main model, where there is a fixed deadline to prepare actions. We will show that some cooperation can be sustained in such games (revision games with a deadline), and the basic mechanism to sustain cooperation is essentially the same as in the model in the present section.

3 Revision Games with a Deadline - The Main Model

Consider a normal-form game with players $i = 1, ..., N$. Player i's action and payoff are denoted by $a_i \in A_i$ and $\pi_i(a_1, ..., a_N)$, respectively. This game is played at time 0, but players have to prepare their actions in advance, and they also have some stochastic opportunities to revise their prepared actions. Hence, technically the game under consideration is a dynamic game with preparation and revisions of actions, where the normal-form game π is played at the end of the dynamic game (time 0). To distinguish the entire dynamic game and the game π played at the deadline, the former is referred to as a *revision game* and π is referred to as the *component game*. In what follows a "revision game" refers to the one with a prespecified deadline.

Specifically, we consider two specifications. In both cases, time is continuous, $-t \in [-T, 0]$ with $T > 0$. At time $-T$, each player i simultaneously chooses an action from A_i. In time interval $(-T, 0]$, revision opportunities arrive stochastically, according to a process defined shortly. There is no cost of revision. At time 0, the payoffs $\pi(a') = (\pi_1(a'), \ldots, \pi_N(a'))$ materialize, where a'_i is i's action in the last revision opportunity.

1. *Synchronous revision game*: There is a single Poisson process with arrival rate $\lambda > 0$ defined over the time interval $(-T, 0]$. At each arrival, each player i simultaneously chooses an action from A_i.

[4] Sherstyuk, Tarui, and Saijo (2013) independently make the same observation about the equivalence of repeated games and stationary revision games.

2. *Asynchronous revision game*: For each player i, there is a Poisson process with arrival rate $\lambda_i > 0$ defined over the time interval $(-T, 0]$. At each arrival, i chooses an action from A_i.

In the present paper, we analyze the first case. The second case is addressed in the companion paper (Kamada and Kandori (2017)), and a more general treatment can be found in Kamada and Kandori (2012). We assume that players observe all the past events in the revision game, and analyze subgame perfect equilibria.

4 The Optimal Trigger Strategy Equilibrium

In this section, we develop a general theory for the case of synchronous moves. We restrict our attention to two players with one-dimensional continuous action space. The assumption of two players is just for simplicity: our results easily extend to the case of N players. The assumption of continuous actions, in contrast, is crucial, and its role is discussed in great depth in the companion paper (Kamada and Kandori, 2017).

Consider a general two-person symmetric component game with action $a_i \in A_i$ and payoff function π_i, $i = 1, 2$. A player's action space A_i is a convex subset (an interval) in $\mathbb{R}$: Examples include $A_i = [\underline{a}_i, \overline{a}_i]$ or $[0, \infty)$. Symmetry means $A_1 = A_2 =: A$ and $\pi_1(a, a') = \pi_2(a', a)$ for all $a, a' \in A$.

Here we confine our attention to symmetric revision-game equilibrium that uses the "trigger strategy." A symmetric **trigger strategy** is characterized by its equilibrium revision plan $x : [0, T] \to A$. Players start with initial action $x(T)$, and when a revision opportunity arrives at time $-t$, they choose action $x(t)$. Note that we adopt a convention to measure the time backwards: t refers to the *remaining time* in the revision game. If any player fails to follow the revision plan, then both players choose the Nash equilibrium action of the component game (which we will assume to be unique) in all future revision opportunities.

Below we characterize the optimal symmetric trigger strategy equilibria, which maximizes the sum of two players' payoffs. First, we provide a heuristic explanation. Then we provide rigorous statements and proofs in the following subsections.

A Heuristic Description: First, the expected payoff of the revision game associated with the trigger strategy equilibrium plan x is defined by

$$V(x) := \pi(x(T))e^{-\lambda T} + \int_0^T \pi(x(t))\lambda e^{-\lambda t} dt. \tag{1}$$

The coefficient of the first term on the right hand side, $e^{-\lambda T}$, is the Poisson probability that no revision is possible in the entire duration of the revision game (T). In that case, the initially chosen action $x(T)$ is implemented. The integral in (1) is interpreted as follows. Note that λ is a density of a Poisson arrival at any moment of time and $e^{-\lambda t}$ is the probability that no revision is possible in the remaining time t. Therefore, $\lambda e^{-\lambda t}$ is *the density of the last revision opportunity.* Overall, the integral in (1) represents the expected payoff when at least one revision opportunity arrives in the revision game.

The *incentive constraint* at time t for the trigger strategy equilibrium is

$$\text{(IC(t)):}\ d(x(t))e^{-\lambda t} \leq \int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds, \tag{2}$$

where $d(x(t))$ represents the gain from deviation ($d(a) := \max_{a_1} \pi_1(a_1, a) - \pi_1(a, a)$) and $\pi^N = \pi_i(a^N, a^N)$ is the symmetric Nash equilibrium payoff. This is the most important condition in our analysis, and it is interpreted as follows. With probability $e^{-\lambda t}$, there is no revision opportunity in the remaining time t, and in that case the player receives the gain from deviation at the current revision opportunity $d(x(t))$. If at least one revision opportunity arrives in the remaining time, the realized payoff (given by the action profile in the last revision opportunity) is decreased. In particular, when s is the last revision opportunity, the realized payoff is decreased from $\pi(x(s))$ to π^N. As we explained above, the density of the last revision opportunity is $\lambda e^{-\lambda s}$, and the associated expected loss is given by the integral on the right hand side of (2).

It can be shown that the optimal trigger strategy equilibrium path satisfies the incentive constraints with equalities:

$$d(x(t))e^{-\lambda t} = \int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds, \text{ for all } t \tag{3}$$

and in particular this implies an obvious requirement $x(0) = a^N$ (if a revision opportunity arrives at the deadline, players should choose the Nash action). By differentiating both sides of the binding incentive constraint (3), we obtain the differential equation

$$\frac{dx}{dt} = \frac{\lambda \left(d(x) + \pi(x) - \pi^N \right)}{d'(x)},$$

and hence the optimal path is a solution to this differential equation with $x(0) = a^N$.

The heuristic description above needs to be made precise in a number of points. First, we need to find a set of conditions under which the above procedure works. In particular, the existence and differentiability of the optimal path should be derived, rather than assumed. Also, proving the binding incentive constraint (3) turns out to be non-trivial. Those issues are addressed in the next subsection. Secondly, one may expect that cooperation is always sustained, but this turns our to be false. The sustainability cooperation requires a certain condition, and we derive a necessary and sufficient condition in the second subsection. The third subsection states the main result, and the fourth considers an important special case of smooth games. Finally, the last subsection discusses the properties of the expected equilibrium payoffs.

4.1 Existence, Differentiability, and the Binding Incentive Constraints

In what follows we present a set of regularity conditions that ensure the existence of the optimal trigger strategy equilibrium and its characterization by means of a simple differential equation. Let the payoff function for symmetric actions be defined as

$$\pi(a) := \pi_1(a, a) = \pi_2(a, a).$$

We assume the following properties.

- **A1:** A unique pure symmetric Nash equilibrium action a^N and the unique best symmetric action $a^* := \arg\max_{a \in A} \pi(a)$ exist, and they are distinct.
- **A2:** If $a^N < a^*$, the symmetric payoff $\pi(a)$ is strictly increasing for $a < a^*$ (a symmetric condition holds if $a^* < a^N$).

For simplicity of exposition, except for Section 5 (the Cournot duopoly example), we will focus on the case $a^N < a^*$.

- **A3:** $\pi_1(a_1, a_2)$ is continuous. Furthermore, $\max_{a_1} \pi_1(a_1, a_2)$ exists for all a_2, and therefore we can define the *gain from deviation* at a symmetric profile (a, a) by

$$d(a) := \max_{a_1} \pi_1(a_1, a) - \pi_1(a, a). \tag{4}$$

- **A4:** If $a^N < a^*$, $d(a)$ is strictly increasing on $[a^N, a^*]$ and non-decreasing for $a^* < a$ (symmetric conditions hold if $a^* < a^N$).

Assumptions A1 and A3 are innocuous technical assumptions (the requirement that actions are continuous variables, however, is one of the crucial conditions for our result). A2 and A4 are monotonicity conditions that simplify our analysis. Assumption A2 requires that the symmetric payoff $\pi(a)$ monotonically decreases as we move away from the optimal action a^* (in the relevant region for our analysis). Assumption A4 says that the gain from deviation monotonically increases as we move away from the Nash equilibrium (again in the relevant region for our analysis).

Assumptions A1-A4 guarantee the existence of the optimal trigger strategy equilibrium path that is continuous and satisfies the binding incentive constraint. To state this result, we need to introduce some concepts and notation. As we have explained, the expected payoff at the beginning of the game (i.e., at time $-T$) associated with the trigger strategy equilibrium path x is defined by

$$V(x) := \pi(x(T))e^{-\lambda T} + \int_0^T \pi(x(t))\lambda e^{-\lambda t} dt. \tag{5}$$

We define *the set of feasible plans* X as the set of paths over which expected payoff (5) can be defined (i.e., $\pi(x(\cdot))$ is Lebesgue measurable):

$$X := \{x : [0, T] \to A \mid \pi \circ x \text{ is measurable}\}.$$

Given a feasible plan $x \in X$, the (trigger strategy) *incentive constraint* at time t is

$$\text{(IC(t)):}\ d(x(t))e^{-\lambda t} \leq \int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds, \tag{6}$$

The set of *trigger strategy equilibrium plans* is formally defined as

$$X^* := \{x \in X \mid \text{IC(t) holds for all } t \in [0, T]\}.$$

A plan that achieves the highest ex ante expected payoff within X^* is referred to as *an optimal trigger strategy equilibrium plan*.

The following is a key technical result that rigorously establishes the existence of the optimal trigger strategy equilibrium and its two crucial properties: the optimal plan is *continuous* and satisfies the *binding incentive constraints*.

Theorem 1 *Under Assumptions A1 - A4, there is an optimal trigger strategy equilibrium plan $\overline{x}(t)$ ($V(\overline{x}) = \max_{x \in X^*} V(x)$) that is continuous for all t and satisfies the binding incentive constraint when $\overline{x}(t) \neq a^*$:*

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{7}$$

Furthermore, $\overline{x}(t) \in [a^N, a^*]$ for all t if $a^N < a^*$ (and a symmetric condition holds if $a^* < a^N$).

The proof is given in Appendix A. Note that the binding incentive constraint at the deadline $t = 0$ implies $d(\overline{x}(0)) = 0$ (there is no gain from deviation). Thus, Theorem 1 shows that $\overline{x}(0)$ should be a (symmetric) Nash equilibrium action a^N.

Let us make a small technical remark about the multiplicity of the optimal plans. Recall that $\overline{x}(t)$ is a particular optimal trigger strategy equilibrium plan with the binding incentive constraint (the one that is described in Theorem 1). There are, however, other optimal plans. One can see that

$$x(t) := \begin{cases} a^N & \text{if } t \text{ is in a measure zero set in } (0, T) \\ \\ \overline{x}(t) & \text{otherwise} \end{cases}.$$

is also a trigger strategy equilibrium plan that satisfies the incentive constraint (6) and achieves the same expected payoff as $\overline{x}(t)$ does, because the probability that revision opportunities happen in the measure-zero set is zero. Hence, the above plan is also optimal. However, it is easy to show that there is *essentially a unique optimal plan*.

Proposition 1 *The optimal plan is essentially unique: If $y(t)$ is an optimal trigger strategy equilibrium plan, then $y(t) = \overline{x}(t)$ almost everywhere, where $\overline{x}(t)$ is the optimal plan that satisfies the binding incentive constraint (7).*

The proof is given in Appendix B. Hereafter, the continuous optimal plan $\overline{x}(t)$ that satisfies the binding incentive constraint is referred to as *the optimal plan.*

Next we present an additional assumption that guarantees the differentiability of the optimal plan $\overline{x}(t)$.

- **A5:** The gain from deviation d (defined by (4)) is differentiable, and $d' > 0$ on $(a^N, a^*]$ if $a^N < a^*$ (a symmetric condition holds if $a^* < a^N$).

Below is an important remark about the derivative of the gain from deviation at the Nash action, $d'(a^N)$.

Remark 1 *By definition, the gain from deviation $d(a)$ is minimized (equal to zero) at the Nash action a^N. If a^N is an interior point and $d(a)$ is smooth, then the first order condition for the minimization implies*

$$d'(a^N) = 0.$$

A number of applications satisfy the above condition, and we need to pay special attention to this fact when solving for the differential equation in the next Lemma (namely, the denominator on the right hand side of differential equation (8) vanishes at the Nash action a^N).

Next we derive a differential equation to characterize the optimal plan. Heuristically, the differential equation is derived by differentiating both sides of the binding incentive constraint (7) that was established by Theorem 1. In the following lemma, we make this argument precise, by deriving (rather than assuming) the differentiability of the optimal plan. We present this result for the case $a^N < a^*$.

Lemma 1 *Under Assumptions A1-A5, the optimal plan $\overline{x}(t)$ is differentiable when $\overline{x}(t) \neq a^N, a^*$, and satisfies differential equation*

$$\frac{dx}{dt} = \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} := f(x) \tag{8}$$

for $x \in (a^N, a^*)$.

Proof. Note that $\overline{x}(t) \in [a^N, a^*]$ (by Theorem 1). By Assumption A5, d has an inverse function on $(a^N, a^*]$, denoted by d^{-1}. Thus, if $\overline{x}(t) \in (a^N, a^*)$, the binding incentive constraint implies

$$\overline{x}(t) = d^{-1}\left(e^{\lambda t}\int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}ds\right). \tag{9}$$

The crucial step to show the differentiability of $\overline{x}(t)$ is to note the differentiability of integral $\int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}ds$ with respect to t. Specifically, the continuity of $\overline{x}$, established by Theorem 1, implies that $\left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}$ is continuous, and the fundamental theorem of calculus shows that $\int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}ds$ is differentiable with respect to t (with the derivative $\left(\pi(\overline{x}(t)) - \pi^N\right)\lambda e^{-\lambda t}$). Also, A5 implies that d^{-1} is differentiable with derivative $1/d'(a)$ (note that A5 guarantees $d'(a) \neq 0$ for $a \in (a^N, a^*]$). Therefore the right hand side of (9) is differentiable with respect to t, and differentiating both sides of (9), we obtain the differential equation (8) when $\overline{x}(t) \in (a^N, a^*)$. ■

4.2 FTC: A Necessary and Sufficient Condition for Cooperation

One may expect that cooperation can always be sustained in the revision game under our Assumptions A1-A5. This turns out to be false. Sustainability of cooperation requires some conditions. This conceptually important point is closely related to the following technical issue. The optimal plan appears to be obtained by solving the differential equation (8) with the boundary condition $\overline{x}(0) = a^N$ (and in this way we can find if cooperation is possible or not). There are, however, important caveats.

- **Caveat 1:** In a well-behaved smooth game, the derivative of the gain from deviation is equal to zero at the Nash action ($d'(a^N) = 0$), as we have explained in Remark 1. In such a case, the differential equation $dx/dt = \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$ is *not* defined at the Nash equilibrium point a^N (because the denominator $d'(x)$ vanishes at the Nash action). Hence we cannot solve the differential equation with the boundary condition $\overline{x}(0) = a^N$.

- **Caveat 2:** When $d'(a^N) \neq 0$, the differential equation *is* defined at the Nash action. However, it always has a *trivial solution* $x(t) = a^N$ for all t. (Note that

$x(t) = a^N$ satisfies $dx/dt = 0$ and $d(a^N) + \pi(a^N) - \pi^N = 0 + \pi^N - \pi^N = 0$.) In such a case, cooperation is possible only if the differential equation with the boundary condition $\overline{x}(0) = a^N$ admits *multiple solutions*.

In what follows, we will address those concerns. The caveats above ultimately stem from the following observation. The differential equation (8) is derived by (differentiating) the binding incentive constraint (it must be satisfied by the optimal trigger strategy equilibrium plan, according to Theorem 1);

$$d(x(t))e^{-\lambda t} = \int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds \quad \text{for } x(t) \neq a^*, \tag{10}$$

but this integral equation always has a *trivial solution* $x(t) = a^N$ for all t. To show that cooperation is possible, we need to establish that this integral equation has another solution that sustains cooperation.

Our conditions A1 - A5 alone do not guarantee that the integral equation (10) has a non-trivial solution. In some cases, cooperation cannot be satisfied in a revision game, even if A1-A5 are satisfied. Consider the following example.

Example 1 *Linear Exchange Game*[5]*: Two players $i = 1, 2$ exchange goods. Player i chooses a quantity (or quality) $a_i \in [0, 1]$ of the goods she provides to the other player. The cost of effort of player i is equal to $a_i/2$. In total, player i's payoff is equal to $\pi_i = a_{-i} - a_i/2$.*

The Nash and optimal actions, the symmetric payoff, and the gain from deviation are $a^N = 0$ (the dominant strategy), $a^* = 1$, $\pi(a) = a - a/2 = a/2$, and $d(a) = a/2$ (the optimal deviation is to produce zero and save the cost $a/2$, respectively). This example satisfies A1 - A5, but the only trigger strategy equilibrium is the trivial one that always plays the Nash action. The reason is as follows. This example has a feature that the derivative of the gain from deviation $d'(a)$ is non-zero at the Nash action $a^N = 0$ (d' is always equal to 1/2), so that Caveat 2 applies. The differential equation is

$$dx/dt = \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} = 2x,$$

and this differential equation with boundary condition $x(0) = 0$ has a *unique*, trivial solution $x(t) = 0 (= a^N)$ for all t. Hence it is the only plan that satisfies the binding

[5] This is also known as the linear public good provision game.

incentive constraint. Therefore, by Lemma 1, this is the optimal (and only) trigger strategy equilibrium.

Conceptually, cooperation is impossible in this example because the gain from deviation $d(a) = a/2$ and the benefit $\pi(a) - \pi^N = a/2$ are of the same order of magnitude near the Nash equilibrium action. For cooperation to be sustained, the former should be much smaller than the latter around the Nash action. The full discussion is given in Section 6.1.

How can we identify the condition under which a non-trivial optimal plan exists? Because of Caveat 1, we do not directly solve the differential equation (8) with boundary condition $x(0) = a^N$. Instead, we employ the following procedure. Consider the case $a^N < a^*$. Note that, by Theorem 1, the optimal plan lies in a certain interval; $\overline{x}(t) \in [a^N, a^*]$ for all t.

- **Step 1:** Solve the differential equation (8) with a boundary condition with an arbitrary non-Nash action; $x(0) = a^0 \in (a^N, a^*]$. A standard regularity condition assures that the differential equation is well-defined and has a *unique* solution on $(a^N, a^*]$ (this regularity condition will be stated in A6).

- **Step 2:** Ask if the solution $x(t)$ can approach the Nash equilibrium a^N in a finite amount of time, when we follow the differential equation *before* time 0 (i.e., if $x(t) \to a^N$ as $t \to -t^*$ for some $t^* < \infty$).[6] This is the Finite Time Condition (FTC) that we will introduce. If this condition is satisfied, shift the origin of time by t^* to construct a new plan $y(t) := x(t - t^*)$. This plan starts with the Nash action $y(0) = a^N$, is continuous, and follows the differential equation (8) for $t > 0$. Thus it satisfies the binding trigger strategy incentive constraint. It is non-trivial, because it reaches $a^0 \in (a^N, a^*]$ at $t = t^*$.

Hence, the necessary and sufficient condition for the existence of non-trivial plan is the FTC (assuming that the innocuous regularity condition (A6) is satisfied). FTC is a clear-cut condition that is easy to check. However, it does not provide much intuition when and why cooperation can be sustained. In Section 6.1, we discuss economically meaningful conditions for and against FTC.

Our procedure can be illustrated by Figure 2. Panel (a) describes the case where cooperation is possible. Solving the differential equation $dx/dt = \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$

[6] Here we extend the domain of paths from $[0, T]$ to $\mathbb{R}$. The same reservation applies to other places where the domain of a path includes negative numbers.

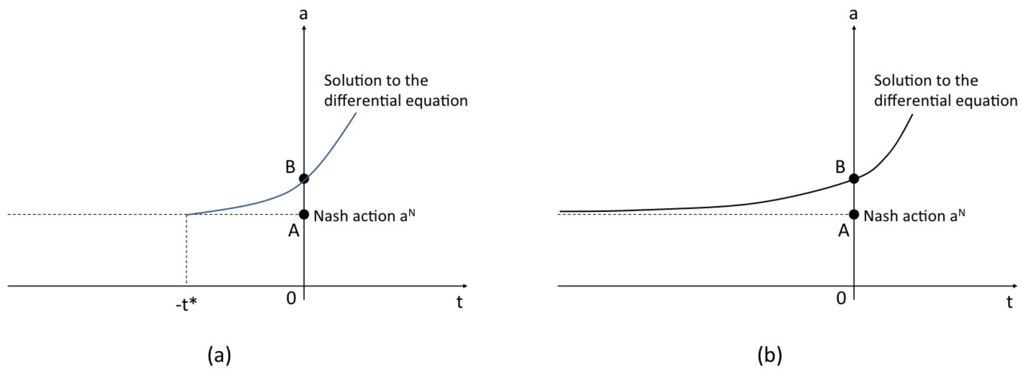

Figure 2: Illustration of the Finite Time Condition: the condition holds and cooperation is possible in Panel (a), while it does not hold and cooperation is impossible in Panel (b).

with the boundary condition at the Nash action (at point A) is problematic, because of Caveats 1 and 2. Instead, we solve the differential equation with an arbitrary boundary point B and ask if the solution can reach the Nash action within a finite amount of time (i.e., if the FTC is satisfied). It takes a finite amount of time in Panel (a), and cooperation can be sustained by the trigger strategies. In contrast, Panel (b) describes the case where cooperation cannot be sustained (because the solution takes an infinite amount of time to approach the Nash action: the FTC fails). Let us now implement our procedure. First, let us assume a standard regularity condition to guarantee that the solutions to the differential equation (8) exists and is unique when the boundary condition is a suitable non-Nash action.

- **A6:** Function $f(x) := \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$ is Lipschitz continuous on $[a^N+\varepsilon, a^*]$ for any $\varepsilon \in (0, a^*]$, if $a^N < a^*$[7] (a symmetric requirement holds if $a^* < a^N$).

Lipschitz continuity of f is easy to check, and is satisfied in a wide range of applications. For example, in the smooth games that we analyze in Section 4.4, A6 is automatically satisfied.

The following theorem identifies the necessary and sufficient condition for cooperation to be sustained by the trigger strategy.

[7] A function $f(x)$ is Lipschitz continuous on $[a^N+\varepsilon, a^*]$ if there exists a finite number $K \geq 0$ such that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq K$ for all $x \neq y$ in $[a^N+\varepsilon, a^*]$

Theorem 2 *Under Assumptions A1-A6, the optimal trigger strategy equilibrium plan* $\overline{x}(t)$ *is non-trivial (i.e.,* $\overline{x}(t) \neq a^N$ *for some* t*) if and only if*[8]

$$\text{(Finite Time Condition)} \quad \lim_{a \downarrow a^N} \int_a^{a^*} \frac{1}{f(x)} dx < \infty \tag{11}$$

is satisfied, where $f(x) = \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$.

The proof is given in Appendix C. Here we present basic intuition. The differential equation $dx/dt = f(x)$ implies that $\int_a^{a^*} \frac{1}{f(x)} dx = \int_a^{a^*} \frac{dt}{dx} dx$ represents the time to reach a^* from a, following the solution to the differential equation. Hence, the Finite Time Condition (11) means that it is possible to find a finite upper bound of the time for a solution to the differential equation $dx/dt = f(x)$ to travel from any point arbitrarily close to the Nash action a^N to the optimal action a^*.

4.3 Summary: the Main Result

Our main result is summarized as the following corollary to the key technical theorems in the previous subsections.

Corollary 1 *Under Assumptions A1 - A6, cooperation is sustained by the trigger strategy, if and only if the Finite Time Condition (11) is satisfied. There exists an essentially unique optimal trigger strategy equilibrium plan* $\overline{x}(t)$ *and it satisfied the following properties: (i) it is continuous in* t *and departs* a^N *at* $t = 0$ *(i.e.,* $\overline{x}(t) = a^N$ *if and only if* $t = 0$*), (ii) for* $t > 0$*, it solves differential equation*

$$\frac{dx}{dt} = \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} =: f(x) \tag{12}$$

until $\overline{x}(t)$ *hits the optimal action* a^**, and (iii) if* $\overline{x}(t)$ *hits the optimal action* a^* *it stays there (i.e.,* $\bar{x}(t') = a^*$ *for some* $t' \leq T$ *implies* $\bar{x}(t'') = a^*$ *for all* $t'' \in [t', T]$*). Furthermore, if the time horizon* T *is large enough,* $\overline{x}(t)$ *always hits the optimal action* a^* *at*

$$t(a^*) := \lim_{a \downarrow a^N} \int_a^{a^*} \frac{1}{f(x)} dx. \tag{13}$$

[8]Here we consider the case $a^N < a^*$. When $a^N > a^*$. $\lim_{a \downarrow a^N}$ in the Finite Time Condition should be replaced with $\lim_{a \uparrow a^N}$.

The proof is given in Appendix D. The optimal trigger strategy equilibrium has the following feature. Recall that $\overline{x}(t)$ is the action to be taken at time $-t$. If the time horizon is long enough (i.e., if $T \geq t(a^*)$), players start with the best action a^*, and they do not revise their actions until time $-t(a^*)$. After that, if a revision opportunity arrives, they choose an action $\overline{x}(t)$, which is closer to the Nash action. The closer the revision opportunity $-t$ is to the deadline, the closer the revised action $\overline{x}(t)$ is to the Nash equilibrium. At the deadline, the actions at the last revision opportunity are implemented. The realized actions are stochastic and the expected payoff is calculated in Section 4.5.

The necessary and sufficient condition for cooperation, the Finite Time Condition (11), is easy to check, but it is not as clear, intuitively speaking, what it requires. In Section 6.1, we show that the Finite Time Condition is satisfied when *the cost of cooperation tends to zero faster than the benefit of cooperation does, as action tends to the Nash equilibrium.* In the next subsection, we present a fairly general class of games where this requirement is naturally satisfied. It is the class of games with smooth payoff functions.

4.4 Smooth Games

In this subsection, we show that cooperation can be sustained in revision games when the component game has well-behaved smooth payoff functions. This is a fairly general class, and a leading example is the Cournot quantity competition. The conditions for smooth games are the following. Recall that we are focusing on two-player symmetric games, so that the component game is characterized by the symmetric payoff function $\pi_1(a_1, a_2)$.

- **S1:** The payoff function $\pi_1(a_1, a_2)$ is twice continuously differentiable.
- **S2:** There is a unique best reply $BR(a)$ for any action a, and the first and second order conditions are satisfied at the best reply:

$$\frac{\partial \pi_1(BR(a), a)}{\partial a_1} = 0, \quad \frac{\partial^2 \pi_1(BR(a), a)}{\partial {a_1}^2} < 0$$

- **S3:** $\pi'(a^N) > 0$ if $a^N < a^*$ and $\pi'(a^N) < 0$ if $a^* < a^N$.

Conditions S1 and S2 are standard smoothness requirements. S3 adds a fairly minor modification to our former condition A2, which states that $\pi(a)$ strictly increases if we change the action from the Nash action a^N to the optimal action a^*. Also note that S1 and a part of S2 (the existence of a best reply) directly imply A3. It also turns out that the Lipschitz condition A6 is also derived from S1-S3. Now we can state one of our main results.

Proposition 2 *If conditions A1, A2, A4, A5 and S1-S3 are satisfied, cooperation is sustained in the revision games. The optimal trigger strategy equilibrium plan is given as in Corollary 1.*

The proof is given in Appendix E, and here we provide its sketch. Note that the Finite Time Condition is

$$\lim_{a \downarrow a^N} \int_a^{a^*} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx < \infty,$$

and it is satisfied if either (i) the integrand $\frac{d'(x)}{d(x)+\pi(x)-\pi^N}$ tends to a finite number as $x \downarrow a^N$ or (ii) $\frac{d'(x)}{d(x)+\pi(x)-\pi^N}$ tends to ∞ sufficiently slowly as $x \downarrow a^N$. Appendix E shows that condition (i) is satisfied in the smooth games (and therefore cooperation can be sustained).

4.5 Expected Payoffs and Arrival Rate Invariance

In this subsection, we examine the expected payoff associated with the optimal trigger strategy equilibrium. One might expect that the probability distribution of the realized actions, and hence the expected payoffs, may depend on the arrival rate λ of revision opportunities. Frequent arrival of revision opportunities might make it easier (or more difficult) to cooperate. The next proposition, which is actually nothing but a simple observation, shows that this is not the case. To state the proposition, recall that the first time to hit the optimal action is denoted $t(a^*)$ (see (13)). To explicitly show its dependence on arrival rate λ, let us now denote it by $t_\lambda(a^*)$.

Proposition 3 ***(Arrival Rate Invariance)*** *Suppose that Assumptions A1 - A6 and the Finite Time Condition (11) are satisfied. Let $t_\lambda(a^*)$ be the (first) time to reach the optimal symmetric action as defined by (13) in Corollary 1. Then, as long* *as $t_\lambda(a^*) \le T$, the probability distribution of the action profile at period 0 under the optimal trigger strategy equilibrium is independent of the Poisson arrival rate λ.*

Proof. Consider a model with λ such that $t_\lambda(a^*) \le T$ and call it Model 1. Change the time scale so that one unit of time in Model 1 corresponds to λ units in the new model. Under the new time scale, the model has arrival rate 1 and time horizon λT. Call it Model 2. Models 1 and 2 represent the same revision game under different time scales. In Model 1, under the optimal trigger strategy equilibrium, no revision is made in $[-T, -t_\lambda(a^*)]$. Thus in Model 2, the same is true in $[-\lambda T, -t_1(a^*)]$. Deleting this inactive time interval from Model 2 does not change the probability distribution of the realized action profile. Hence, revision games with *any* arrival rate λ such that $t_\lambda(a^*) \le T$ are outcome equivalent to the one with arrival rate 1 and $T = t_1(a^*)$. ■

Note that the fact that payoffs realize only at the deadline $t = 0$ plays a crucial role in this proposition (otherwise, the expected payoffs would be affected by the arrival rate and the discount factor). Proposition 3 shows the following attractive feature of revision games: we can obtain a unique prediction that does not depend on the fine detail, namely the arrival rate λ of the revision opportunities. In particular, even if λ is sufficiently high (so that there are many chances to revise actions right before the deadline), the expected outcome in the component game is the same as in the case of low λ.

The above proposition shows that the probability distribution of the realized action profile can be obtained by focusing on the case $\lambda = 1$. Let $x_1(t)$ be the optimal trigger strategy equilibrium plan under $\lambda = 1$. To calculate the distribution, the time for $x_1(t)$ to hit $a \in [a^N, a^*]$, denoted by $t_1(a)$, turns out to be useful. Note that $t_1(a)$ is given by

$$t_1(a) := \lim_{a' \downarrow a^N} \int_{a'}^{a} \frac{dt}{dx} dx = \lim_{a' \downarrow a^N} \int_{a'}^{a} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx.$$

Now consider the density of realized action $x_1(t) \le a$. The density is $\lambda e^{-\lambda t} = e^{-t}$, which is the product of

- $\lambda = 1$ (the density of revision opportunity at time t) and
- $e^{-\lambda t} = e^{-t}$ (the probability that the revised action at time t, $x(t)$, will never be revised again).

Therefore, the cumulative distribution function of realized action a, denoted by $F(a)$, is given by

$$\begin{aligned} F(a) &= \int_{\{t|x_1(t)\leq a\}} e^{-t}dt \\ &= \int_0^{t_1(a)} e^{-t}dt = 1 - e^{-t_1(a)}, \end{aligned}$$

for $a \in [a^N, a^*)$. For $a \geq a^*$, $F(a) = 1$, because the realized action cannot be more than a^*. This implies that, at a^*, the distribution function $F(a)$ jumps by $e^{-t_1(a^*)}$. The jump means that a probability mass of $e^{-t_1(a^*)}$ is attached to action a^*. This is the probability that no revision opportunity arises after time $-t_1(a^*)$ under Poisson arrival rate $\lambda = 1$. Below we summarize our arguments.

Proposition 4 *Suppose that Assumptions A1 - A6 and the Finite Time Condition (11) are satisfied. Suppose also that the time horizon is long enough so that the efficient action a^* is chosen at the beginning of the revision game, under the optimal trigger strategy equilibrium. When $a^N < a^*$, the cumulative distribution function of the symmetric action realized at $t = 0$ is given by*

$$F(a) = \begin{cases} 0 & \text{if } a < a^N \\ 1 - e^{-t_1(a)} & \text{if } a^N \leq a < a^* \\ 1 & \text{if } a^* \leq a \end{cases},$$

where $t_1(a)$ is given by

$$t_1(a) := \lim_{a' \downarrow a^N} \int_{a'}^{a} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx. \tag{14}$$

When $a^* < a^N$, it is given by

$$F(a) = \begin{cases} 0 & \text{if } a < a^* \\ e^{-t_1(a)} & \text{if } a^* \leq a \leq a^N \\ 1 & \text{if } a^N < a \end{cases}.$$

5 Quantity Competition: Substantial Collusion is Possible via Revisions

In this section, we consider a Cournot duopoly game. The revision game in this case describes the situation where two firms gradually adjust their quantities before the market is open and they closely monitor each other's process of quantity adjustment. We will show that the firms can achieve almost 97% of the fully collusive profit.

Each firm $i = 1, 2$ with constant (and identical) marginal cost $c > 0$ chooses the quantity x_i, and the price P is determined by a linear demand curve $P = a - b(x_1 + x_2)$ with $a > c > 0$ and $b > 0$. Hence the (component game) payoff function for player i is $\pi_i = (a - b(x_i + x_{-i}) - c)\, x_i$. This is an example of smooth games in Section 4.4. The optimal trigger strategy equilibrium plan $x(t)$ is a solution for the following differential equation:[9]

$$\begin{aligned} \frac{dx}{dt} &= \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} \\ &= \frac{\lambda}{18}\left(x - 5\frac{a-c}{3b}\right). \end{aligned}$$

The differential equation has a simple solution

$$\overline{x}(t) = \frac{a-c}{3b}(5 - 4e^{\frac{\lambda}{18}t}).$$

This plan departs from the Cournot Nash quantity $x^N = \frac{a-c}{3b}$ at $t = 0$, and hits the optimal quantity $x^* = \frac{a-c}{4b}$ at $t^* = t(x^*) = \frac{18}{\lambda}\ln\left(\frac{17}{16}\right)$. Therefore, we have obtained the following.

Proposition 5 *In the revision game of the Cournot duopoly game, the optimal trigger strategy equilibrium plan, $\bar{q}(t)$, is characterized by*

$$\overline{x}(t) = \begin{cases} x^N\left(5 - 4e^{\frac{\lambda}{18}t}\right) & \text{if } t < t^* \\ x^* & \text{if } t^* \leq t \end{cases},$$

where $x^N = \frac{a-c}{3b}$ and $x^* = \frac{a-c}{4b}$ are Nash and the optimal quantities, respectively, and

[9] This follows from $d(q) = \frac{(a-c-3bq)^2}{4b}$, $\pi(q) = (a - c - 2bq)q$, and $\pi^N = \frac{(a-c)^2}{9b}$.

$t^* = \frac{18}{\lambda} \ln\left(\frac{17}{16}\right)$.

When the firms collude, they produce less than the Nash quantity. Thus the optimal plan is non-decreasing over time, starting with a small collusive quantity and gradually increasing towards the Nash quantity (recall that t in the proposition above refers to the remaining time in the revision game, so it refers to time $-t$). Figure 1 in the Introduction shows the shape of the optimal plan and a realized path. The Introduction also presents a real-life situation that might reasonably be formulated as the quantity competition revision game.

Next, we consider the welfare implication of the revision game of the Cournot duopoly. By Proposition 4, we can compute the equilibrium expected payoff, and it turns out that a surprisingly high degree of collusion can be achieved in this game. The next corollary says that, when two firms gradually adjust their quantities before the market is open (and if they closely monitor each other's process of quantity adjustment), then they can achieve almost 97% of the fully collusive profit. Those numbers are independent of the position and the slope of the demand curve (a and b) and the marginal cost c.

Corollary 2 *In the revision game of the Cournot duopoly game, there exists T' such that for all $T > T'$, the expected payoff under the optimal trigger strategy equilibrium is more than* 0.968 *of the fully collusive payoff.*

The proof is given in Appendix F.

6 Discussion

In this section, we first discuss a conceptually important issue of identifying conditions for sustainability of cooperation in revision games. Our analysis in Section 4.2 shows a clear-cut necessary and sufficient condition (the Finite Time Condition), but this condition itself is not easy to interpret. We will provide more intuitive conditions, which relate the relative magnitudes of gain from deviation and the benefit of cooperation near the Nash equilibrium. Secondly, we examine the relationship between the revision games and infinitely repeated games with time-varying discount factors.

6.1 What Determines the Possibility of Cooperation?

Our main result (Theorem 2) shows that the Finite Time Condition (FTC)

$$t(a^*) := \lim_{a \downarrow a^N} \int_a^{a^*} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx < \infty$$

is necessary and sufficient for a trigger strategy equilibrium to sustain cooperation. In this subsection, we will show that FTC is (almost) equivalent to the following (informally stated) intuitive property:

- **[Convergence Condition]** As the symmetric action profile (x, x) converges to the Nash equilibrium, the gain from deviation $(d(x))$ tends to zero faster than the benefit of cooperation $(\pi(x) - \pi^N)$ does.

It is intuitive that this condition would guarantee cooperation. Recall that in the revision game, cooperation is sustained in much the same way as in the repeated game: instantaneous gain from deviation is outweighed by the destruction of the future benefit of cooperation. When the Convergence Condition is satisfied, near the Nash action, the gain from deviation is much smaller than the benefit of cooperation. Hence, a little bit of cooperation is sustained near the deadline of the revision game. Using this as a foothold, more substantial cooperation is sustained as we move further away from the deadline.

Note also that the Convergence Condition is satisfied in smooth games. Since Nash equilibrium admits no gain from deviation, $d(x)$ is minimized at the Nash action a^N (and equal to zero). If d is smooth, the minimization implies $d'(a^N) = 0$, and therefore the gain from deviation is almost identically equal to zero near the Nash action. In contrast, this is not the case for the benefit of cooperation $(\pi(x) - \pi^N)$. Those facts, taken together, imply that $d(x)$ tends to zero faster than $\pi(x) - \pi^N$ as x tends to the Nash action. This is nothing but the Akerloff-Yellen effect: a little bit of cooperation (i.e., taking an action near the Nash equilibrium) entails a small cost, while it provides a benefit that is an order of magnitude larger.

A straightforward formulation of the Convergence Condition would be

$$\lim_{x \downarrow a^N} \frac{d(x)}{\pi(x) - \pi^N} = 0. \tag{15}$$

Recall that, except in Section 5, we are focusing on the case where players wish to increase their action relative to the Nash action (i.e., $a^N < a^*$). Hence what is relevant is the behavior of $\frac{d(x)}{\pi(x)-\pi^N}$ for $x \geq a^N$. This is why we have "$\lim_{x \downarrow a^N}$" in the above condition (15). Note also that, under our Assumption A2,

$$\frac{d(x)}{\pi(x) - \pi^N} \geq 0 \text{ for } x \geq a^N.$$

Hence, "Not (15)" is equivalent to

$$\lim_{x \downarrow a^N} \inf \frac{d(x)}{\pi(x) - \pi^N} > 0.$$

We will show that this condition implies no cooperation. On the other hand, condition (15) itself is not quite strong enough for the sustainability of cooperation.[10] Hence we will slightly strengthen condition (15). Note that, when $d(x)$ is small (less than 1), for any $k \in (0, 1)$, $d(x)^k > d(x)$. Our condition for cooperation is that this larger value $d(x)^k$ tends to zero faster than the benefit of cooperation $\pi(x) - \pi^N$ does:

$$\lim_{x \downarrow a^N} \frac{d(x)^k}{\pi(x) - \pi^N} = 0 \text{ for some } k \in (0, 1).$$

This is stronger than the limit condition (15) (i.e., it implies (15)), because $d(x)^k \geq d(x)$ and $\frac{d(x)}{\pi(x)-\pi^N} \geq 0$ imply $\frac{d(x)^k}{\pi(x)-\pi^N} \geq \frac{d(x)}{\pi(x)-\pi^N} \geq 0$. Moreover, the above condition is almost equal to the limit condition (15), because the constant k can be any number arbitrarily close to 1 (for example, $k = 0.999$). We will show that cooperation is sustained under this condition.

Theorem 3 *Suppose A1-A6 hold.*

1. *If* $\lim_{x \downarrow a^N} \inf \frac{d(x)}{\pi(x)-\pi^N} > 0$*, the Finite Time Condition fails and the unique trigger strategy equilibrium is to play the Nash action all the time:* $x(t) \equiv a^N$*.*

2. *If* $\lim_{x \downarrow a^N} \frac{d(x)^k}{\pi(x)-\pi^N} = 0$ *for some* $k \in (0, 1)$*, then the Finite Time Condition holds and cooperation can be sustained by the trigger strategy equilibria.*

The proof is given in Appendix G.

[10] To be more precise, we have been unable to show that (15) implies the Finite Time Condition.

6.2 Comparison with Infinite Repeated Games with Decreasing Discount Factors

Bernheim and Dasgupta (1995) (referred to as BD hereafter) consider an infinitely repeated game with time-dependent discount factor. Since their model and ours have similar features, let us compare the two models in detail. Their model is in discrete time and the objective function is

$$\pi(a_0) + \sum_{s=1}^{\infty} \pi(a_s) \prod_{\tau=1}^{s} \delta(\tau),$$

where the time-dependent discount factor $\delta(\tau)$ shrinks over time ($\delta(\tau) \to 0$ as $\tau \to \infty$).[11] A straightforward extension in continuous time, which is most readily comparable to the revision game, would be to assume (1) opportunities to play the stage game arrive according to a Poisson process, (2) payoff accrues only at the Poisson arrival time and (3) instantaneous discount rate $\rho(s)$ diverges over time ($\rho(s) \to \infty$ as $s \to \infty$). The continuation payoff in the repeated game at a Poisson arrival time s' is given by

$$\pi(a_{s'}) + \int_{s'}^{\infty} \pi(a_s) e^{\left(\int_{s'}^{s} -[\rho(\tau)] d\tau\right)} \lambda ds, \text{ and } 0 < \rho(s) \to \infty \text{ as } s \to \infty. \tag{16}$$

Let us represent the time in the revision game by $t \in [0, T)$ (i.e., time runs from 0 to T). We will map the repeated game time $s \in [0, \infty)$ to the revision game time s by a strictly increasing, continuously differentiable function $t = f(s)$ such that $f(0) = 0$ and $f(s) \to T$ as $s \to \infty$. The revision-game expected payoff at an arrival time t' is

$$e^{-\lambda(T-t')} \pi(a_{t'}) + \int_{t'}^{T} \pi(a_t) \lambda e^{-\lambda(T-t)} dt.$$

[11] The models in Vives (1995, 2001), which we discussed in the Introduction, have similar structure. Let $\gamma_\tau := 1 - \delta(\tau)$ be the probability that the game ends in period τ. The expected payoff in period 0 in the papers by Vives is given by

$$(1 - \delta(0)) u(a_0) + \sum_{s=1}^{\infty} (1 - \delta(s)) u(a_s) \prod_{\tau=1}^{s} \delta(\tau).$$

This is similar to the payoff in BD, and our argument in this subsection also applies to Vives (1995) and (2001).

Dividing both sides by $e^{-\lambda(T-t')}$, we can see that this is proportional to

$$\pi(a_{t'}) + \int_{t'}^{T} \pi(a_t)\lambda e^{\lambda(t-t')}dt. \tag{17}$$

Using the alternative time parameter $s \in [0, \infty)$ and defining s' by $t' = f(s')$, the above normalized continuation payoff in the revision game is expressed as

$$\pi(a_{f(s')}) + \int_{s'}^{\infty} \pi(a_{f(s)})\lambda e^{\lambda(f(s)-f(s'))}\frac{dt}{ds}ds$$

$$= \pi(a_{f(s')}) + \int_{s'}^{\infty} \pi(a_{f(s)})e^{\left(\int_{s'}^{s} -[-\lambda f'(\tau)]d\tau\right)}\left[\lambda f'(s)\right] ds \tag{18}$$

This representation of the revision game continuation payoff shows that the revision game is isomorphic to the continuous time infinite horizon repeated game with the following three properties.

1. The stage game arrives at time-dependent Poisson arrival rate $\lambda(s) = \lambda f'(s)$ (since $f'(s) \to 0$ as $s \to \infty$, the arrival rate converges to zero).
2. The stage payoff accrues only at the Poisson arrival time.
3. The instantaneous discount rate at s is *negative* and is given by $-\lambda f'(s)$.

Note that the stage game arrives increasingly infrequently over time (item 1), and the future payoff is *more* important than the present one (item 3). Those properties are not present in BD's payoff representation (16), and can be understood as follows. Recall the revision game continuation payoff (17). It has the following two properties.

- **A** If one compares two future times $s < s'$ in the continuation payoff, more weight is attached to the payoff at the latter time s'.
- **B** However, the total future payoff becomes less important over time.[12]

The revision game has seemingly contradicting features that the future is *more* important in one sense (A), while the opposite is true in another sense (B). Properties A and B of the revision game translate into properties 3 and 1 of the equivalent

[12] Note that future discounted payoff $\int_{t'}^{T} \pi(a_t)\lambda e^{\lambda(t-t')}dt$ decreases over time, because the integrating interval $[t, T]$ shrinks.

repeated game, respectively. BD and our model share a common feature B, but property A is unique to our model. Therefore, although BD and our model have some similarities, the two models are not isomorphic to each other. Indeed, Appendix H proves that it is impossible to rewrite (18) in such a way that (i) the stage game arrives with a continuous arrival-rate function $\tilde{\lambda}(s)$ (that may not be equal to $[\lambda f'(s)]$) and (ii) the continuous instantaneous time-dependent discount rate diverges as time tends to infinity.

7 Concluding Remarks

We analyzed a new class of games that we call "revision games," a situation where players prepare their actions in advance in a game. After the initial preparation, they have some opportunities to revise their actions, which arrive stochastically. Prepared actions are assumed to be mutually observable. We showed that players can achieve a certain level of cooperation in such a class of games. Cooperation is possible if the gain from deviation is a smaller order of magnitude than the benefit of cooperation near the Nash equilibrium. We characterized the equilibrium by a simple differential equation and applied it to analyze the Cournot duopoly game. Our paper sheds new light on the possibility of cooperation. In particular, cooperation and collusion might be possible in a non-repeated interaction, when players prepare and revise their actions before their actual interaction. Part II of the project (Kamada and Kandori, 2017) shows various applications, and the robustness of the results is also examined in depth.

References

[1] Akerlof, G. A. and J. L. Yellen (1985): "Can Small Deviations from Rationality Make Significant Differences to Economic Equilibria?," *American Economic Review*, 75, 708-720.

[2] Ambrus, A., J. Burns, and Y. Ishii (2014): "Gradual bidding in ebay-like auctions," mimeo.

[3] Ambrus, A. and S. E. Lu (2015): "A Continuous-Time Model of Multilateral Bargaining," *American Economic Journal: Microeconomics*, 7, 208-249.

[4] Bernheim, B. D. and A. Dasgupta (1995): "Repeated Games with Asymptotically Finite Horizons," *Journal of Economic Theory*, 67(1): pp.129-152

[5] Bhaskar, V. (1989): "Quick Responses in Duopoly Ensure Monopoly Pricing," *Economics Letters*, 29: 103-107.

[6] Caruana, G. and L. Einav (2008a): "A Theory of Endogenous Commitment," *Review of Economic Studies*, 75: 99-116.

[7] Caruana, G. and L. Einav (2008b): "Production Targets," *RAND Journal of Economics*, 39(4): 990-1217.

[8] Chou, C.-F. and J. Geanakoplos (1988): "The Power of Commitment" Cowles Foundation Discussion Paper No. 885

[9] Kalai, E. (1981): "Preplay Negotiations and the Prisoner's Dilemma", *Mathematical Social Sciences*, 1: 375-379.

[10] Kamada Y. and M. Kandori (2012): "Asynchronous Revision Games," mimeo.

[11] Kamada Y. and M. Kandori (2017): "Revision Games, Part II: Applications and Robustness", mimeo.

[12] Sherstyuk, K., N. Tarui, and T. Saijo (2013): "Payment Schemes in Infinite-Horizon Experimental Games", *Experimental Economics*, 16(1): 125-153.

[13] Vives, X. (1995): "The Speed of Information Revelation in a Financial Market Mechanism," *Journal of Economic Theory*, 67: 178-204.

[14] Vives, X. (2001): "Strategic Behavior and Price Discovery," *RAND Journal of Economics*, 32: 221-248.

APPENDIX

A Proof of Theorem 1

We provide the proof of Theorem 1 (the existence and differentiability of the optimal plan). First, we present a simple but useful lemma. Recall that we are focusing on the case $a^N < a^*$.The next lemma shows that we can restrict our attention to the trigger strategy equilibria whose action always lies in $[a^N, a^*]$.

Lemma 2 *For any trigger strategy equilibrium plan $x \in X^*$, there is a trigger strategy equilibrium plan $\widehat{x} \in X^*$ such that $\forall t \;\; \widehat{x}(t) \in [a^N, a^*]$ and $\pi(\widehat{x}(t)) \geq \pi(x(t))$ with a strict inequality if $x(t) \notin [a^N, a^*]$.*

Proof. Construct $\widehat{x}(t)$ from a given $x(t)$ as follows. First, if $x(t) > a^*$, let $\widehat{x}(t) = a^*$. This assures $\pi(\widehat{x}(t)) = \pi(a^*) > \pi(x(t))$ and, by Assumption A4, $d(\widehat{x}(t)) \leq d(x(t))$. Second, if $x(t) < a^N$, let $\widehat{x}(t) = a^N$. This assures $d(\widehat{x}(t)) = 0 < d(x(t))$ and, by Assumption A2, $\pi(\widehat{x}(t)) > \pi(x(t))$. Finally, let $\widehat{x}(t) = x(t)$ if $x(t) \in [a^N, a^*]$. Overall, $\widehat{x}(t)$ provides weakly higher payoffs and weakly smaller gains from deviation, and thus it also satisfies the trigger strategy incentive constraint

$$d(\widehat{x}(t))e^{-\lambda t} \leq \int_0^t \left(\pi(\widehat{x}(s)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

■

This lemma shows that the optimal trigger strategy (if any) can be found in the set X^{**} of trigger strategy equilibria whose range is $[a^N, a^*]$:

$$X^{**} := \left\{x \in X^* | \forall t \; x(t) \in [a^N, a^*]\right\}.$$

Now we are ready to prove Theorem 1.

Proof. We show that there is a trigger strategy equilibrium in X^{**} that attains $\max_{x \in X^{**}} V(x)$ (by Lemma 2, it is the true optimum in X^*).

In the first step, we construct a candidate optimal plan $\overline{x}(t)$ and shows its continuity. In Step 2, we will verify that this plan is feasible and it is indeed the optimal trigger strategy equilibrium plan. In Step 3, we show that the binding incentive constraint holds under the plan $\overline{x}$.

[Step 1] Since $V(x)$ is bounded above by $\pi(a^*) = \max_a \pi(a)$, $\sup_{x\in X^{**}} V(x)$ is a finite number. Hence, we can find a sequence $x^n \in X^{**}$ such that $\lim_{n\to\infty} V(x^n) = \sup_{x\in X^{**}} V(x)$.

Note that $\{\pi(x^n(\cdot))\}_{n=1,2,\dots}$ is a collection of *countably* many measurable functions. This implies that $\overline{\pi}(t) := \sup_n \pi(x^n(t))(< \infty)$ is also measurable. Now let us define $\overline{x}(t)$ to be the solution to

$$\textbf{Problem P(t)}: \quad \max_{x(t)\in[a^N,a^*]} \pi(x(t))$$

$$\text{s.t. } d(x(t))e^{-\lambda t} \leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{19}$$

Note that the right hand side of the constraint (19) is well-defined, because $\overline{\pi}(\cdot)$ is measurable. Also note that the right hand side is nonnegative by $\overline{\pi}(s) \geq \pi^N$.[13]

Under Assumptions A2 and A4, both $\pi(a)$ and $d(a)$ are strictly increasing on $[a^N, a^*]$. Furthermore, by Assumption A3, $d(a)$ is continuous by Berge's Theorem of Maximum. Hence the solution $\overline{x}(t)$ to Problem P(t) is either a^* or the action in $[a^N, a^*)$ with the binding constraint (19). Let us write down the solution $\overline{x}(t)$ to the above problem P(t) in the following way. Since d is continuous and strictly increasing on $[a^N, a^*]$, on this interval its continuous inverse d^{-1} exists. Then the optimal solution $\overline{x}(t)$ can be expressed as

$$\overline{x}(t) = \begin{cases} a^* & \text{if } d(a^*) < h(t) \\ d^{-1}\left(h(t)\right) & \text{otherwise} \end{cases}, \tag{20}$$

where

$$h(t) := e^{\lambda t} \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

A crucial step in the proof, that shows the continuity of the optimal plan, is to note that the integral $\int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds$ in the definition of $h(t)$ is continuous in t for any measurable function $\overline{\pi}(\cdot)$.[14] Since d^{-1} is continuous, this observation implies that $\overline{x}(t)$ is continuous whenever $x(t) \in [a^N, a^*)$. Moreover, since $h(t)$ is increasing in t, (20) means that $x(t) = a^*$ implies $x(t') = a^*$ for all $t' > t$. Hence $\bar{x}$ is continuous for all t.

[13] By A2, $x^n(t) \in [a^N, a^*]$ implies $\pi(x^n(t)) \geq \pi^N$. Hence $\overline{\pi}(t) = \sup_n \pi(x^n(t)) \geq \pi^N$.

[14] A standard result in measure theory shows that, for any measurable function $f(t)$, the Lebesgue integral $\int_0^t f(s)ds$ is absolutely continuous in t. Therefore, it is continuous in t.

[**Step 2**] We show that $\overline{x}$ is actually feasible and a trigger strategy equilibrium. The continuity of $\overline{x}$ and π implies that $\pi(\overline{x}(\cdot))$ is a measurable function. Therefore, $\overline{x}$ is feasible. We show that $\overline{x}$ also satisfies the (trigger strategy) incentive constraint IC(t) for all t. Recall that x^n is a trigger strategy equilibrium for all $n = 1, 2, ...$ Then we have, for all $n = 1, 2, \ldots,$

$$\begin{aligned} d(x^n(t))e^{-\lambda t} &\leq \int_0^t \left(\pi(x^n(s)) - \pi^N\right) \lambda e^{-\lambda s} ds \quad (x^n \text{ is an equilibrium}) \\ &\leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \qquad (\text{by the definition of } \overline{\pi}) \end{aligned}$$

This means that $x^n(t)$ satisfies the constraint of Problem P(t). Since $\overline{x}(t)$ is the solution to Problem P(t), we have

$$\forall n \ \forall t \quad \pi(\overline{x}(t)) \geq \pi(x^n(t)) \tag{21}$$

and therefore

$$\forall t \ \ \pi(\overline{x}(t)) \geq \overline{\pi}(t) = \sup_n \pi(x^n(t)). \tag{22}$$

This implies that, for all t, $\overline{x}(t)$ satisfies the incentive constraint IC(t):

$$\begin{aligned} d(\overline{x}(t))e^{-\lambda t} &\leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds \quad (\overline{x}(t) \text{ satisfies (19)}) \\ &\leq \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds. \end{aligned}$$

Thus we have shown that $\overline{x}$ is a trigger strategy equilibrium ($\overline{x} \in X^*$), and $V(\overline{x}) \geq V(x^n)$ for all n (by (21)). By definition $\lim_{n\to\infty} V(x^n) = \sup_{x \in X^{**}} V(x)$, and the above inequality implies $V(\overline{x}) \geq \sup_{x \in X^{**}} V(x)$. Since $\overline{x} \in X^{**}$, we must have $V(\overline{x}) = \sup_{x \in X^{**}} V(x) = \max_{x \in X^{**}} V(x)(= \max_{x \in X^*} V(x)$ by Lemma 2). Hence we have established that there is an optimal and continuous trigger strategy equilibrium $\overline{x}$.

[**Step 3**] Lastly we prove that the optimal plan $\overline{x}$ satisfies the binding incentive constraint. Step 1 shows that, if $\overline{x}(t) \neq a^*$, then the following "pseudo" binding incentive constraint is satisfied:

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{23}$$

Our remaining task is to show the "true" binding incentive constraint

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

Since $\pi(\overline{x}(t)) \geq \overline{\pi}(t)$ for all t (inequality (22)), the pseudo binding incentive constraint (23) implies

$$d(\overline{x}(t))e^{-\lambda t} \leq \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{24}$$

We show that this is satisfied with an equality. If the above inequality were strict for some t, we would have $\int_0^t \overline{\pi}(s)\lambda e^{-\lambda s} ds < \int_0^t \pi(\overline{x}(s))\lambda e^{-\lambda s} ds$. Given $\pi(\overline{x}(s)) \geq \overline{\pi}(s)$ for all $s \in (t, T]$ (inequality (22)), we would have

$$e^{-\lambda T}\overline{\pi}(T) + \int_0^T \overline{\pi}(s)\lambda e^{-\lambda s} ds < e^{-\lambda T}\pi(\overline{x}(T)) + \int_0^T \pi(\overline{x}(s))\lambda e^{-\lambda s} ds = V(\overline{x}).$$

Since $\overline{\pi}(s) := \sup_n \pi(x^n(t))$, the left hand side is more than or equal to $V(x^n)$ for all n. Since $\lim_{n\to\infty} V(x^n) = \sup_{x\in X^{**}} V(x)$, the above inequality implies $\sup_{x\in X^{**}} V(x) < V(\overline{x})$. In contrast $\overline{x} \in X^{**}$ implies $\sup_{x\in X^{**}} V(x) \geq V(\overline{x})$, and this is a contradiction. Hence (24) should be satisfied with an equality (i.e., $\overline{x}$ satisfies the binding incentive constraint), if $\overline{x}(t) \neq a^*$. ■

B Proof of Proposition 1

We provide the proof of Proposition 1 (essential uniqueness of the optimal plan):

Proof. Suppose $H := \{t | \pi(y(t)) > \pi(\overline{x}(t))\}$ has a positive measure.[15] Then, define

$$z(t) := \begin{cases} y(t) & \text{if } t \in H \\ \\ \overline{x}(t) & \text{otherwise} \end{cases}.$$

This has a measurable payoff $\pi(z(t)) = \max\{\pi(y(t)), \pi(\overline{x}(t))\}$ and achieves a strictly

[15]Since y is a feasible plan, H is a measurable set.

higher expected payoff than $\overline{x}(t)$. Furthermore, z satisfies the incentive constraints

$$\forall t \quad d(z(t))e^{-\lambda t} \leq \int_0^t \left(\pi(z(s)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

This follows from the incentive constraints for $\overline{x}$ and y, together with $\pi(z(t)) = \max\{\pi(y(t)), \pi(\overline{x}(t))\}$. Hence, z is a trigger strategy equilibrium plan, which achieves a strictly higher payoff than $\overline{x}(t)$ does. This contradicts the optimality of $\overline{x}(t)$, and therefore H must have measure zero. Hence $\pi(y(t)) \leq \pi(\overline{x}(t))$ almost everywhere. If $\{t|\pi(y(t)) < \pi(\overline{x}(t))\}$ has a positive measure, y attains a strictly smaller payoff than $\overline{x}(t)$ does, which contradicts our premise that y is optimal. Therefore, we conclude that $\pi(y(t)) = \pi(\overline{x}(t))$ almost everywhere.

Now, note that Lemma 2 shows that if $\{t|y(t) \notin [a^N, a^*]\}$ has a positive measure, then $y(t)$ cannot be optimal in X^*. This implies that $y(t) \in [a^N, a^*]$ almost everywhere. This and $\pi(y(t)) = \pi(\overline{x}(t))$ almost everywhere imply that $y(t) = \overline{x}(t)$ almost everywhere because π is strictly increasing on $[a^N, a^*]$. ■

C Proof of Theorem 2

We prove Theorem 2.

Proof. By Theorem 1 and Lemma 1, the optimal plan $\overline{x}(t)$ satisfies the following conditions:

(i) it lies in $[a^N, a^*]$ for all t,

(ii) it is continuous in t,

(iii) it follows the differential equation $dx/dt = f(x)$ if $x \in (a^N, a^*)$, and

(iv) it starts with Nash action a^N at $t = 0$.

We first show that the existence of a non-trivial optimal plan implies the Finite Time Condition. Properties (i), (ii) and (iv) imply that, if $\overline{x}(t)$ is non-trivial (i.e., not equal to the Nash action a^N for all t), then $\overline{x}(t^0) = a^0 \in (a^N, a^*)$ for some $t^0 > 0$ and some a^0. At this point the optimal plan satisfies the differential equation $dx/dt = f(x)$ by (iii). By A2 and A5, $f(x) = \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)} > 0$ (recall that $d(a^N) = 0$ and $\pi(a^N) = \pi^N$), when $x \in (a^N, a^*)$. Hence, once the optimal path departs from the Nash action a^N, it is strictly increasing and never goes back to a^N. Given that the optimal path starts with a^N, this implies the following. First, the plan stays at the Nash action for some time interval $[0, t^N]$ (this interval may be degenerate: t^N may be equal to 0). Second, after this time interval, the plan is continuous and strictly increasing with $\overline{x}(t^N) = a^N < \overline{x}(t^0) = a^0$. Therefore, on $[t^N, t^0]$, function $\overline{x}(t)$ has a continuous inverse that we denote by $t(x)$, and its derivative is defined on $(t^N, t^0]$ and equal to $\frac{dt}{dx} = \frac{1}{f(x)}$. This implies that $\lim_{a \downarrow a^N} \int_a^{a^0} \frac{dt}{dx} dx = \lim_{a \downarrow a^N} (t(a^0) - t(a)) = t(a^0) - t(a^N)$, where the last equality follows from the continuity of the inverse function $t(\cdot)$. By definition, $t(a^0) = t^0$ and $t(a^N) = t^N$, and therefore $\lim_{a \downarrow a^N} \int_a^{a^0} \frac{dt}{dx} dx < \infty$ holds. In addition, since $f(x)$ is Lipchitz continuous over $[a^0, a^*]$, the differential equation $\frac{dx}{dt} = f(x)$ with an initial condition $x(t^0) = a^0$ has a unique solution, and $\overline{x}$ is equal to such a solution. Hence, letting $\hat{t}$ be $\overline{x}(\hat{t}) = a^*$, we obtain $\hat{t} < \infty$. Hence, $\int_{a^0}^{a^*} \frac{1}{f(x)} dx = \hat{t} - t^0 < \infty$. Overall, we conclude that

$$\lim_{a \downarrow a^N} \int_a^{a^*} \frac{dt}{dx} dx = \left(\lim_{a \downarrow a^N} \int_a^{a^0} \frac{dt}{dx} dx \right) + \int_{a^0}^{a^*} \frac{dt}{dx} dx < \infty,$$

so the Finite Time Condition (11) holds.

Next, we show that the Finite Time Condition implies that the optimal plan is non-trivial. Choose any $a^0 \in (a^N, a^*)$. By Assumption A6 (the Lipschitz continuity), the differential equation $dx/dt = f(x)$ with boundary condition $x(0) = a^0$ has a unique solution, denoted by $x^\varepsilon(t)$, on $(a^N + \varepsilon, a^*)$ for any small enough $\varepsilon > 0$. By the same argument as above, our assumptions ensure $dx/dt = f(x) > 0$ for $x \in (a^N, a^*)$. Define

$$t^* := \lim_{\varepsilon \to 0} \int_{a^N + \varepsilon}^{a^0} \frac{1}{f(x)} dx < \infty, \tag{25}$$

where the finiteness follows from the Finite Time Condition (11). The above argument shows that there is a solution to the differential equation $x(t)$ such that $x(0) = a^0$ and $x(t) \downarrow a^N$ as $t \to -t^*$. Shift the origin of time and construct a new plan $y(t) := x(t - t^*)$. The new plan is also a solution to the differential equation, and it satisfies $y(t^*) = a^0$ and $y(t) \to a^N$ as $t \to 0$. Now, construct another plan $z(t)$ by suitably extending $y(t)$:

$$z(t) = \begin{cases} a^N & \text{if } t = 0 \\ \\ y(t) & \text{if } t \in (0, t^*] \\ \\ a^0 & \text{if } t > t^* \end{cases}$$

This plan satisfies the trigger strategy incentive constraint (6): the incentive constraint is binding on $[0, t^*]$ (because it satisfies the differential equation), and for $t > t^*$ the incentive constraint is satisfied with strict inequality. Hence $z(t)$ is a non-trivial trigger strategy equilibrium. This implies that the optimal trigger strategy equilibrium is non-trivial. ■

D Proof of Corollary 1

We prove Corollary 1.

Proof. Recall that the optimal plan $\overline{x}(t)$ satisfies conditions (i)-(iv) in Appendix C. It turns out that there are multiple plans which satisfy those conditions. For example, *trivial constant plan* $x(t) \equiv a^N$ satisfies those conditions. In what follows, we identify all plans that satisfy conditions (i)-(iv) and find the optimal one among them.

The Finite Time Condition is

$$t(a^*) := \lim_{a \downarrow a^N} \int_a^{a^*} \frac{1}{f(x)} dx < \infty.$$

The proof of Theorem 2 shows that there is a solution to the differential equation $x^*(t)$ that satisfies $x^*(t(a^*)) = a^*$ and $x^*(t) \to a^N$ as $t \to 0$. From $x^*(t)$, construct the following plan

$$x_\tau(t) := \begin{cases} a^N & \text{if } t \in [0, \tau] \\ \\ x^*(t - \tau) & \text{if } t \in (\tau, \tau + t(a^*)) \\ \\ a^* & \text{if } t \in [\tau + t(a^*), \infty) \end{cases} .$$

This plan $x_\tau(t)$ departs from a^N at time τ, follows the differential equation, and then hits the optimal action a^* and stays there (In a revision game with time horizon T, we must consider the restriction of $x_\tau(t)$ on $[0,T]$).

Those plans $x_\tau(t), \tau \geq 0$ obviously satisfy (i)-(iv). Next we show the converse: any plan satisfying (i)-(iv) is equal to $x_\tau(t)$ for some $\tau \in [0,\infty]$. This comes from the standard result in differential equation: $dx/dt = f(x)$ defined on an open domain $(x,t) \in (a^N, a^*) \times (-\infty, \infty)$ has a unique solution given any boundary condition, if $f(x)$ is Lipschitz continuous. The uniqueness of the solution then implies that any plan satisfying (i)-(iv) is equal to $x_\tau(t)$ for some $\tau \in [0,\infty) \cup \{\infty\}$.[16]

Among the plans $x_\tau(t)$, $\tau \in [0,\infty]$ the one that departs from a^N immediately (i.e., $x_0(t)$) obviously has the highest payoff. Therefore the optimal plan is given by the restriction of $x_0(t)$ on $[0,T]$, which has the stated properties in Corollary 1. ■

E Proof of Proposition 2

We prove Proposition 2. First, we show the following technical lemma.

Lemma 3 *Under A1, A2, A4, A5 and S1-S3, both $d'(x)$ and $d''(x)$ exist and are continuous. In particular,*

$$d'(x) = \frac{\partial \pi_1(BR(x),x)}{\partial x_2} - \frac{\partial \pi_1(x,x)}{\partial x_1} - \frac{\partial \pi_1(x,x)}{\partial x_2}, \tag{26}$$

$$\begin{aligned} d''(x) = &-\left(\frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1 \partial x_2}\right)^2 \Big/ \frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1^2} + \frac{\partial^2 \pi_1(BR(x),x)}{\partial x_2^2} \\ &- \frac{\partial^2 \pi_1(x,x)}{\partial x_1^2} - 2\frac{\partial^2 \pi_1(x,x)}{\partial x_1 \partial x_2} - \frac{\partial^2 \pi_1(x,x)}{\partial x_2^2}, \text{ and} \end{aligned} \tag{27}$$

$$d''(a^N) = \frac{-\left(\frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1^2} + \frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1 \partial x_2}\right)^2}{\frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1^2}}. \tag{28}$$

[16] A formal proof goes as follows. The trivial path, which satisfies (i)-(iv), is equal to x_τ with $\tau = \infty$. Consider any non-trivial path $x^0(t)$ that satisfies (i)-(iv), where $x^0(t^0) =: a^0 \in (a^N, a^*)$ for some t^0. Define $t' := t^0 - \lim_{a \downarrow a^N} \int_a^{a^0} \frac{1}{f(x)} dx$ (which is finite by the Finite Time Condition), so that $x^*(t-t')$ hits a^0 at $t = t^0$. The uniqueness of the solution to the differential equation (for boundary condition $x(t^0) = a^0$) implies $x^0(t) = x^*(t-t')$ for $t \geq t'$. If $t' \geq 0$, we obtain the desired result $x^0(t) = x_\tau(t)$ for $\tau = t'$. If $t' < 0$, $x^0(0) = x^*(-t') > a^N$ and $x^0(0)$ cannot satisfy (iv) ($x^*(-t') > a^N$ leads to a contradiction because we are considering the case $a^N < a^*$).

Proof. We first examine the properties of $BR(x)$. To this end, we apply the implicit function theorem to the first order condition $\frac{\partial \pi_1(BR(x),x)}{\partial x_1} = 0$ (S2). The assumptions of the implicit function theorem are satisfied:

- $\frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1^2} \neq 0$ (by S2) and
- $\frac{\partial \pi_1(x_1,x_2)}{\partial x_1}$ is continuously differentiable (S1).

Hence $BR(x)$ is a continuously differentiable function (and therefore also continuous), with

$$BR'(x) = -\frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1 \partial x_2} / \frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1^2},$$

which is finite. Given this, differentiating $d(x) := \pi_1(BR(x),x) - \pi_1(x,x)$ and using the first-order condition $\frac{\partial \pi_1(BR(x),x)}{\partial x_1} = 0$ (S2), we obtain (26). Differentiating this once again and using the above formula for $BR'(x)$, we obtain (27). By the twice continuous differentiability of π_1 (S1), $\frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1^2} \neq 0$ (by S2), and the continuity of $BR(x)$, both d' and d'' are continuous. Lastly, (28) is obtained from (27), by noting that $BR(x) = x$ when x is equal to the Nash action a^N. ■

Now we are ready to prove Proposition 2.

Proof. Since A3 is directly implied by S1 and S2, we need to derive the Lipschitz condition A6 and the Finite Time Condition (11) (then the Proposition follows from Corollary 1).

Let us show that $f(x) = \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$ is Lipschitz continuous on $[a^N + \varepsilon, a^*]$ for any $\varepsilon \in (0, a^*]$. A sufficient condition for the Lipschitz continuity is that $f(x)$ is continuously differentiable on $[a^N + \varepsilon, a^*]$.[17] Sufficient conditions for this is that $d(a)$, $\pi(a)$, and $d'(a)$ are continuously differentiable, and $d'(a) > 0$ on $[a^N + \varepsilon, a^*]$. Continuous differentiability of π is directly assumed in S1. Lemma 3 above shows the continuous differentiability of d and d' (d' and d'' exist and are continuous). Finally, $d'(a) > 0$ is directly assumed in A5.

It remains to show the Finite Time Condition

$$\lim_{a \downarrow a^N} \int_a^{a^*} \frac{d'(x)}{\lambda\left(d(x) + \pi(x) - \pi^N\right)} dx < \infty.$$

[17] If $f(x)$ is continuously differentiable, the function $|f'(x)|$ is continuous and thus has the maximum value on the compact interval $[a^N + \varepsilon, a^*]$. This maximum value serves as the (Lipschitz) constant K in the definition of Lipschitz continuity: $\left|\frac{f(x)-f(y)}{x-y}\right| \leq K$.

Note that the integral is over a bounded interval, and the integrand is a finite number for all $x \in (a^N, a^*]$ under our assumptions. Hence the Finite Time Condition is satisfied if the integrand $\frac{d'(x)}{\lambda(d(x)+\pi(x)-\pi^N)}$ tends to a finite number as $x \downarrow a^N$. The denominator tends to $d(a^N) + \pi(a^N) - \pi^N = 0 + \pi^N - \pi^N = 0$. The numerator $d'(x)$ also tends to zero, under our smoothness conditions:

$$d'(a^N) = 0. \tag{29}$$

This follows from Lemma 3, which states

$$d'(x) = \frac{\partial \pi_1(BR(x), x)}{\partial x_2} - \frac{\partial \pi_1(x, x)}{\partial x_1} - \frac{\partial \pi_1(x, x)}{\partial x_2}.$$

At the Nash action $x = a^N$, the second term is zero by the first-order condition. The first and third terms cancel out, because $BR(x) = x$ at the Nash action $x = a^N$. Hence, we obtain $d'(a^N) = 0$.

Given that both the numerator and denominator of $\frac{d'(x)}{\lambda(d(x)+\pi(x)-\pi^N)}$ tend to zero as $x \downarrow a^N$, we can apply l'Hopital's rule:

$$\lim_{x \downarrow a^N} \frac{d'(x)}{\lambda\left(d(x) + \pi(x) - \pi^N\right)} = \frac{d''(a^N)}{\lambda\left(d'(a^N) + \pi'(a^N)\right)}$$

By (29), the denominator of the right hand side is equal to $\lambda \pi'(a^N)$, and it is non-zero (strictly positive) by S3. Lemma 3 shows that the numerator $d''(a^N)$ is a finite number. Hence, the right hand side is a finite number, and therefore the Finite Time Condition is satisfied. ■

F Calculation of Expected Payoffs for Cournot Duopoly

The expected payoff can be calculated as follows:

$$
\begin{aligned}
&\int_0^{t(x^*)} (a-2bx(t)-c)\,x(t)\lambda e^{-\lambda t}dt + e^{-\lambda t(x^*)}(a-2bx^*-c)\,x^* \\
&= \int_0^{t(x^*)} \left(a-2b\frac{a-c}{3b}\left(5-4e^{\frac{\lambda}{18}t}\right)-c\right)\frac{a-c}{3b}\left(5-4e^{\frac{\lambda}{18}t}\right)\lambda e^{-\lambda t}dt + e^{-\lambda t(x^*)}\left(a-2b\frac{a-c}{4b}-c\right)\frac{a-c}{4b} \\
&= \frac{(a-c)^2}{9b}\int_0^{t(x^*)}\left(-7+8e^{\frac{\lambda}{18}t}\right)\left(5-4e^{\frac{\lambda}{18}t}\right)\lambda e^{-\lambda t}dt + e^{-\lambda t(x^*)}\frac{(a-c)^2}{8b} \\
&= \frac{(a-c)^2}{9b}\left[35e^{-\lambda t}-72e^{-\frac{17}{18}\lambda t}+36e^{-\frac{8}{9}\lambda t}\right]_0^{t(x^*)} + e^{-\lambda t(x^*)}\frac{(a-c)^2}{8b} \\
&= \frac{(a-c)^2}{9b}\left(35\left(\frac{17}{16}\right)^{-18}-72\left(\frac{17}{16}\right)^{-17}+36\left(\frac{17}{16}\right)^{-16}+1\right)+\left(\frac{17}{16}\right)^{-18}\frac{(a-c)^2}{8b} \\
&= \frac{(a-c)^2}{8b}\left(\left(\frac{17}{16}\right)^{-18}\left(35\cdot\frac{8}{9}-64\cdot\frac{17}{16}+32\left(\frac{17}{16}\right)^2+1\right)+\frac{8}{9}\right) \\
&= \frac{(a-c)^2}{8b}\left(\left(\frac{17}{16}\right)^{-18}\left(-\frac{323}{9}+32\left(\frac{17}{16}\right)^2\right)+\frac{8}{9}\right)
\end{aligned}
$$

On the other hand, the collusive payoff is:

$$\left(a-2b\frac{a-c}{4b}-c\right)\frac{a-c}{4b} = \frac{(a-c)^2}{8b}.$$

Thus, the ratio between these two values is:

$$\left(\frac{17}{16}\right)^{-18}\left(-\frac{323}{9}+32\left(\frac{17}{16}\right)^2\right)+\frac{8}{9} \simeq 0.96817\ldots$$

G Proof of Theorem 3

We first prove Part [1] of Theorem 3. Condition $\lim_{x\downarrow a^N}\inf\frac{d(x)}{\pi(x)-\pi^N}>0$ implies that there is a constant $h>0$ such that

$$\frac{d(x)}{\pi(x)-\pi^N}\geq h$$

for x sufficiently close to a^N. Hence, for $a \in (a^N, a^*]$,

$$\begin{aligned} t(a) &= \lim_{\underline{a}\downarrow a^N} \int_{\underline{a}}^{a} \frac{d'(x)}{\lambda\left(d(x)+\pi(x)-\pi^N\right)}dx \geq \lim_{\underline{a}\downarrow a^N} \int_{\underline{a}}^{a} \frac{d'(x)}{\lambda(1+\frac{1}{h})d(x)}dx \\ &= \frac{1}{\lambda(1+\frac{1}{h})} \lim_{\underline{a}\downarrow a^N} \int_{\underline{a}}^{a} \frac{d'(x)}{d(x)}dx \\ &= \frac{1}{\lambda(1+\frac{1}{h})} \log d(a) - \lim_{\underline{a}\downarrow a^N} \log d(\underline{a}). \end{aligned}$$

Since $d(a^N) = 0$, $-\lim_{\underline{a}\downarrow a^N} \log d(\underline{a}) = \infty$ and the Finite Time Condition $t(a) < \infty$ fails.

Now we show Part [2] of Theorem 3. First, $\lim_{x\downarrow a^N} \frac{d(x)^k}{\pi(x)-\pi^N} = 0$ for some $k \in (0,1)$ implies

$$t(a) = \lim_{\underline{a}\downarrow a^N} \int_{\underline{a}}^{a} \frac{d'(x)}{\lambda\left(d(x)+\pi(x)-\pi^N\right)}dx \leq \lim_{\underline{a}\downarrow a^N} \int_{\underline{a}}^{a} \frac{d'(x)}{\lambda\left(d(x)+(d(x))^k\right)}dx.$$

if $a > a^N$ is close enough to a^N. Now calculate the integral with respect to d instead of x. For the sake of exposition, let us now denote $d(x)$ by $D(x)$ and recall that it is strictly increasing for $x > a^N$. Then, by noting $D(a^N) = 0$, we obtain

$$\begin{aligned} t(a) &\leq \lim_{\underline{a}\downarrow a^N} \int_{\underline{a}}^{a} \frac{\frac{dD}{dx}}{\lambda\left(D+D^k\right)}dx \\ &= \lim_{\underline{D}\downarrow 0} \int_{\underline{D}}^{D(a)} \frac{1}{\lambda\left(D+D^k\right)}dD \\ &\leq \lim_{\underline{D}\downarrow 0} \int_{\underline{D}}^{D(a)} \frac{1}{\lambda D^k}dD \\ &= \lim_{\underline{D}\downarrow 0} \left[\frac{1}{\lambda(1-k)}D^{1-k}\right]_{\underline{D}}^{D(a)} < \infty, \end{aligned}$$

where the last inequality follows from $\lim_{\underline{D}\downarrow 0} \underline{D}^{1-k} = 0$ because $k \in (0,1)$. Hence the finite time condition is indeed satisfied.

H Impossibility of Mapping between the Revision-Games Payoffs and BD's Payoffs

Here we show that it is impossible to rewrite (18) in such a way that properties 1 and 2 hold with a continuous arrival-rate function and the continuous instantaneous discount rate diverges as time goes to infinity.

Such a rewriting is possible if the expression (18) is equal to

$$\pi(a_{f(s')}) + \int_{s'}^{\infty} \pi(a_{f(s)}) e^{\left(\int_{s'}^{s} -[\widetilde{\rho}(\tau)] d\tau\right)} \widetilde{\lambda}(s) ds \tag{30}$$

for some continuous functions $\tilde{\lambda}$ and $\tilde{\rho}$ such that $\tilde{\rho}(s) \to \infty$ as $s \to \infty$. The equality of two payoffs (18) and (30) requires that

$$\forall s \forall s' \quad e^{\left(\int_{s'}^{s} -[\widetilde{\rho}(\tau)] d\tau\right)} \widetilde{\lambda}(s) = e^{\left(\int_{s'}^{s} -[-\lambda f'(\tau)] d\tau\right)} \lambda f'(s) \tag{31}$$

holds.[18] This condition (31) for $s = s'$ implies

$$\forall s' \quad \widetilde{\lambda}(s') = \lambda f'(s').$$

[18] To see this, suppose that there exist s and s' such that the left hand side in (31) is strictly larger than the right hand side. By the continuity of $\tilde{\rho}$, $\tilde{\lambda}$ and f', both sides of this equality are continuous, so there exists $\varepsilon > 0$ such that

$$e^{\left(\int_{s'}^{s''} -[\widetilde{\rho}(\tau)] d\tau\right)} \widetilde{\lambda}(s'') > e^{\left(\int_{s'}^{s''} -[-\lambda f'(\tau)] d\tau\right)} \lambda f'(s'') \text{ for all } s'' \in [s, s+\varepsilon]. \tag{32}$$

Now normalize the component game payoff such that $\pi(a') = 1$ and $\pi(a'') = 0$ for some actions a' and a''. Let

$$a_{f(s'')} = \begin{cases} a' & \text{for } s'' \in [s, s+\varepsilon] \\ a'' & \text{otherwise} \end{cases}.$$

Then, (32) implies

$$\int_{s'}^{\infty} \pi(a_{f(s)}) e^{\left(\int_{s'}^{s} -[-\lambda f'(\tau)] d\tau\right)} [\lambda f'(s)] \, ds > \int_{s'}^{\infty} \pi(a_{f(s)}) e^{\left(\int_{s'}^{s} -[\widetilde{\rho}(\tau)] d\tau\right)} \widetilde{\lambda}(s) ds,$$

and therefore (18) and (30) do not represent the same payoff function. Hence, for (18) and (30) represent the same payoff, condition (31) must be satisfied. A symmetric proof shows that there cannot exist s and s' such that the left hand side in (31) is strictly smaller than the right hand side, either.

Plugging this into (31), we obtain

$$\forall s \forall s' \qquad \int_{s'}^{s} -[\widetilde{\rho}(\tau)]\, d\tau = \int_{s'}^{s} -[-\lambda f'(\tau)]\, d\tau$$

This implies $\widetilde{\rho}(\tau) = -\lambda f'(\tau)$ for each τ.[19] Hence, $\tilde{\rho}(s)$ does not diverge to ∞ as $s \to \infty$ (note that $f(s) < T$ for all T).

[19]To see this, suppose that there exists $\bar{\tau}$ such that $\widetilde{\rho}(\bar{\tau}) > -\lambda f'(\bar{\tau})$. By the continuity of $\widetilde{\rho}$ and f', there exist s and s' with $s' < \bar{\tau} < s$ such that $\widetilde{\rho}(\tau) > -\lambda f'(\tau)$ for all $\tau \in (s', s)$. This implies $\int_{s'}^{s} -[\widetilde{\rho}(\tau)]\, d\tau < \int_{s'}^{s} -[-\lambda f'(\tau)]\, d\tau$, leading to a contradiction. A symmetric proof shows that it is not possible to have $\widetilde{\rho}(\tau) < -\lambda f'(\tau)$, either.