Revision Games*

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Abstract

We analyze a situation where players in advance prepare their actions in a game. After the initial preparation, they have some opportunities to revise their actions, which arrive stochastically. Prepared actions are assumed to be mutually observable. We show that players can achieve a certain level of cooperation. The optimal behavior of players can be described by a simple differential equation.

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1 Introduction

In social or economic problems, agents often prepare their actions in advance before they interact. Consider researchers who are competing to win research grants. “Actions” in this context correspond to research proposals to be submitted by a prespecified deadline. Researchers prepare their proposals in advance, and proposals are usually subject to some revisions before submission. Since they have other obligation, such as teaching and committee work, a revision can be made only when an opportunity to work on the proposal arrives. Researchers may also obtain some information about their rivals’ proposals. Based on such information, researchers revise their proposals, and they submit what they have when the deadline comes.

In the present paper, we introduce a stylized model to capture such a situation, which we call a *revision game*. In a revision game, a *component game* is played only once, and players must in advance prepare their actions. They have some opportunities to revise their prepared actions, and the opportunities for revision arrive stochastically. Prepared actions are assumed to be mutually observable, and the final action in the last revision opportunity is played in the component game. We show that, under some regulatory conditions, players can achieve a certain level of cooperation.

Let us contrast our model with the well-known fact that players can cooperate in a repeated game. If players expect a sufficient future reward, they can sustain costly cooperation. It must be the players’ best interest to carry out the future reward, which is guaranteed by reward in the further future, and so forth till indefinitely. In this paper we argue that players can sometimes cooperate *even though the game is played only once.* Cooperation can be sustained by revision process of players’ actions.

The basic mechanism to sustain cooperation in a revision game is similar to that in a repeated game, although the mechanism operates in somewhat disguised way. This is best seen when the revision process is stationary. Suppose players prepare their action in each period, and the prepared actions are played in the component game with a (small) constant probability. Once a component game is played, the game is over and there is no further interaction. In Section 2 we present a simple observation that such a model is actually isomorphic to an infinitely repeated game with a (high) discount factor.

The heart of the paper analyzes a more realistic case, where the component game is played at a *predetermined deadline.* Players obtain revision opportunities according to a Poisson process, and the finally-prepared actions are played at the deadline. In the class of component games that we focus on, we will show that an optimal symmetric trigger-strategy equilibrium exists and it is essentially unique. The equilibrium is characterized by a simple differential equation, which we apply to a variety of economic examples. In particular, the revision game of a Cournot duopoly game can achieve, in expectation, more than 96% of the full collusive payoff.

The key difficulty in sustaining cooperation comes with the fact that the preparation phase ends at a predetermined deadline: As time approaches the deadline, the probability of being rewarded in the future shrinks to zero. This means that the instantaneous cost of cooperation (the gain from deviation) must shrink to zero as well for incentive compatibility to be met at each moment of time.[1] We construct a trigger strategy equilibrium with such a property. On the equilibrium path of play, players prepare action $x(t)$ if they obtain a revision opportunity at time t; upon deviation players revert to the (unique) Nash action. $x(t)$ is a full collusive action when time t is sufficiently far away from the deadline, and it (continuously in t) approaches the Nash action towards the deadline. At the deadline, no more opportunity for reward is expected, so the only sustainable action profile is the static Nash action profile. For a 2-player good exchange game, we depict in Figure 1 the path $x(t)$ of the optimal equilibrium among all the trigger strategy equilibria, and a sample equilibrium path of play given $x(t)$.[2]

As the action approaches the Nash equilibrium, the instantaneous cost of cooperation shrinks to zero. However, it turns out that this is not enough to sustain cooperation. We further need that *the instantaneous cost shrinks sufficiently fast.* To see this point, note that as the action approaches the Nash action, the magnitude of the benefit from the opponent's future cooperation (conditional on there being an opportunity) shrinks to zero as well.[3] Since these benefits realize with a vanishing probability, *the cost must be negligible relative to the benefit* when the action is close to Nash. We show that under an assumption on payoff structure that many economic applications satisfy,[4] the cost indeed shrinks fast enough.

[1] Under certain regularity assumptions.
[2] The formal analysis can be found in Section 5.1.
[3] Under a continuity assumption.
[4] This is expressed in Assumption A4 that we state in Section 4.

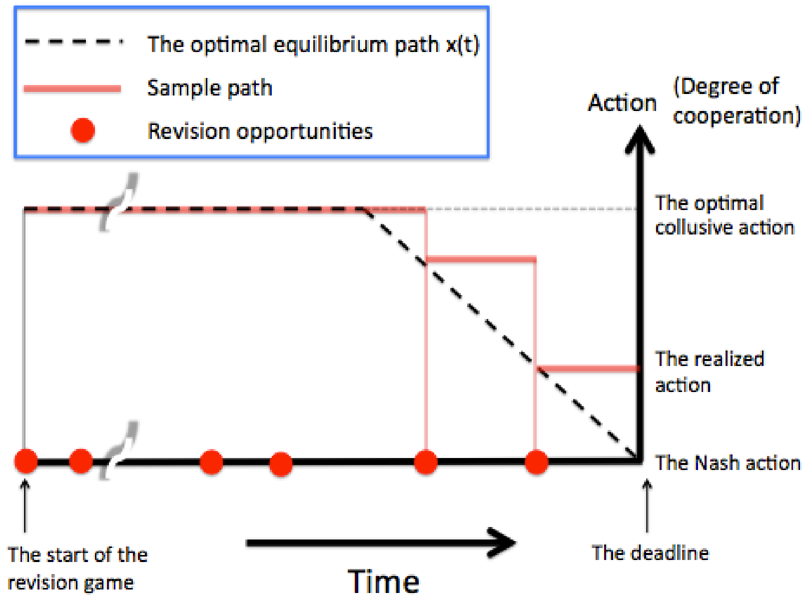

Figure 1: **The optimal path and a sample path for a good exchange game.**

1.1 Related literature

Although in revision games the component game is played only once, features of the model and the dynamic of the equilibrium that they imply are closely related to those in finitely repeated games. A striking fact that the repetition of defections is the only subgame perfect equilibrium in a repeated prisoner's dilemma is overcome by a variety of "twists" in the literature, such as multiple Nash payoffs.[5], bounded rationality[6], reputational effects[7], non-common knowledge of the timing of the deadline[8], social preferences[9], and so forth. Among others, the model of Chou and Geanakoplos (1988) is the most related to ours. They consider finite horizon repeated games in which a player can commit to a (contingent) action at the final period, and show that in "smooth games" a folk theorem obtains. The trigger-strategy equilibrium that we construct is reminiscent of theirs in that the action on the equilibrium path converges to the static Nash action, and the idea is related in the

[5]Harrington (1987) Benoit and Krishna (1985, 1993).
[6]Fudenberg and Levine (1983), Kalai and Neme (1992), Neyman (1985, 1998).
[7]Sobel (1985), Kreps, Milgrom, Roberts, and Wilson (1982), and Fudenberg and Maskin (1986).
[8]Neyman (1999).
[9]Ambrus and Pathak (2011).

sense that in both models a small amount of cooperation at periods close to the deadline builds up a basis for a large cooperation in the entire game. The key difference, besides the fact that the component game is played only once in revision games, is that we do not use commitment to achieve cooperation—our players are fully rational. In our model rational players can cooperate even when the deadline is very close because there is no pre-determined "final period" at which players take actions with a positive probability.

Bernheim and Dasgupta (1995) consider infinite horizon repeated games in which the discount factor falls over time to approximate zero, and show that cooperation can be sustained if the speed at which the discount factor falls is sufficiently slow. They obtain a sufficient condition for the sustainability of cooperation but did not explore characterization of optimal equilibria. Although the mechanism to sustain cooperation in their model is similar to ours, in Section 6.2 we show some crucial differences and demonstrate that our model cannot be mapped into their model.

Pitchford and Snyder (2004) and Kamada and Rao (2009) consider situations in which two parties dynamically transfer a fixed amount of divisible goods that benefit the other party.[10] In the equilibria they construct, a failure to transfer the specified amount of goods at the specified date causes the opponent to stop transferring in the future. The remaining amount of the good in hand converges to zero as the transactions occur a number of times, so the relevant stake of the game gets smaller and smaller over time, reminiscent of our equilibrium form in which $x(t)$ approaches 0 as the deadline comes close. Also, as in our model, there cannot be a final transaction period, since if there can, then the parties do not have an incentive to make a transfer: the transactions need to occur indefinitely. The key difference is that in their models the transaction amount is specified in the way that the game becomes "isomorphic" from one period to the other in an appropriate sense (this is possible since the horizon is infinite), while in our equilibrium the balance of cost and reward for cooperation changes over time, as we have already discussed.

At a technical level, our model is related to that of Ambrus and Lu (2011) who analyze a multilateral bargaining problem in a continuous-time finite-horizon setting where opportunities of proposal arrive via Poisson processes. If an agreement is reached at any time, the game ends then. If no offer is accepted until the deadline, players receive the payoff 0. They show that there is a unique Markov perfect equilibrium in which the first proposal is accepted, so the proposals that different players make converge to the same limit as the

[10]See related papers on gradualism, e.g. Admati and Perry (1991), Marx and Matthews (2000), and Compte and Jehiel (2003) and monotone games, e.g. Gale (1995, 2001).

horizon length becomes large. Although their basic framework is similar to ours, there are two main differences. First, in their model the game can end before the deadline, if an agreement is reached. Second, they focus on Markov perfect equilibrium, which in our model corresponds to the repetition of the component game Nash equilibrium.

The rest of the paper is organized as follows: The next section presents a class of revision games without a deadline to help the readers build up some intuition. The main model with a deadline is presented in Section 3. The results on a general setting with one-dimensional continuous strategies are given in Section 4.. Section 5 provides a number of applications. Section 6 discusses the robustness of our results to fine changes of the specification of the model, identifies the condition for the sustainability of cooperation, and compares our model to an infinite horizon model with a decreasing discount factor. Section 7 concludes.

2 An Example (Two Samurai): Stationary Revision Games

The purpose of this paper is to analyze a class of games where (i) a component game is played only once, (ii) players must prepare their actions in advance, (iii) prepared actions are observable, and (iv) the probability that the prepared actions are actually played is strictly positive but not one. We refer those games as *revision games*. In this section, we start with a simple case, where the problem is stationary in the sense that in each period $t = 0, 1, 2, ...$ there is a fixed, positive probability p with which the component game is played. We refer to this class of revision games as *stationary revision games*. This class will turn out to be isomorphic to a familiar class of games, and it helps to build some intuition on how revision games in general work. The point we make is a simple one, so we just present an example of stationary revision games.

Suppose that a rural village faces an attack of bandits. In each period $t = 0, 1, 2, ...$ the bandits attack the village with probability $p \in (0, 1)$ around midnight. They attack only once. The villagers hired two samurai, $i = 1, 2$, and they must prepare to defend the village (to show up at the village gate around midnight) or not (to hide away and watch the gate from a distance). Hence in each period they observe each other's prepared actions. The acts of preparation themselves (showing up and hiding away) have negligible effects on the samurai's payoffs. When the bandits attack, however, their prepared actions have huge impacts on their payoffs;

	Defend	Hide
Defend	2, 2	-1, 3
Hide	3, -1	0, 0

This is a Prisoner's Dilemma game. Now consider player i's expected payoff. We denote player i's payoff by $\pi_i(t)$, when the bandits' attack occurs at time t. We also assume that players have a common discount factor $\delta \in (0,1)$. Player i's expected payoff is

$$\begin{aligned} & p\pi_i(0) + \delta(1-p)p\pi_i(1) + \delta^2(1-p)^2 p\pi_i(2) + \cdots \\ = \; & p\sum_{t=0}^{\infty} \overline{\delta}^t \pi_i(t), \end{aligned}$$

where $\overline{\delta} := \delta(1-p)$. Hence, *stationary revision games are isomorphic to infinitely repeated games*, and cooperation can be sustained in a subgame perfect equilibrium if p is small. Even though the component game is played only once, when (i) a component game is played only once, (ii) players must prepare for their actions in advance, (iii) prepared actions are observable, and (iv) the probability that the prepared actions are actually played is strictly positive but not one (and the probability with which the game is played is sufficiently low as well as the discount factor is sufficiently high), then players manage to cooperate. The mechanism to sustain cooperation works, for example, as follows. As long as the samurai have been showing up at the gate, they continue to do so (to prepare to defend the village). If anyone hides away, however, they stop preparing to defend.

The next section deals with our main model, where there is a fixed deadline to prepare action in the component game. We will show that some cooperation can be sustained in such games (revision games with a deadline), and the basic mechanism to sustain cooperation is essentially the same as in this bandits story.

3 Revision Games with a Deadline - The Main Model

Consider a normal form game with players $i = 1, ..., N$. Player i's action and payoff are denoted by $a_i \in A_i$ and $\pi_i(a_1, ..., a_N)$, respectively. This game is played at time 0, but players have to prepare their actions in advance, and they also have some stochastic opportunities to revise their prepared actions. Hence, technically the game under consideration is a dynamic game with preparation and revisions of actions, where the normal-form game π is played at the end of the dynamic game (time 0). To distinguish the entire dynamic game and its component π, the former is referred to as a *revision game* and π is referred to as the *component game*.

Specifically, we consider two specifications. In both cases, time is continuous, $-t \in [-T, 0]$ with $T > 0$. At time $-T$, each player i chooses an action from A_i simultaneously. In time interval $(-T, 0]$, revision opportunities arrive stochastically, according to a process defined shortly. There is no cost of revision. At period 0, the payoffs $\pi(a') = (\pi_1(a'_1), \ldots, \pi_N(a'_N))$ materialize, where a'_i is i's finally-revised action.

1. *Synchronous revision game*: There is a Poisson process with arrival rate $\lambda > 0$ defined over the time interval $(-T, 0]$. At each arrival, each player i chooses an action from A_i simultaneously. We analyze this case in the present paper.

2. *Asynchronous revision game*: For each player i, there is a Poisson process with arrival rate $\lambda_i > 0$ defined over the time interval $(-T, 0]$. At each arrival, i chooses an action from A_i. We analyze this case in Kamada and Kandori (2011).

We assume that players observe all the past events in the revision game, and analyze subgame perfect equilibria. In synchronous revision games, if the component game has a unique pure Nash equilibrium, one obvious subgame perfect equilibrium is the strategy profile in which players choose a static Nash action at time $-T$, and they do not revise their actions until time 0. In what follows, we show that, under some regulatory conditions, revision games have other subgame perfect equilibria, where players are better off than in the static Nash equilibrium.

4 Characterization of Optimal Trigger Strategy Equilibrium

In this section, we consider the case of synchronous moves. We restrict ourselves to two players with one-dimensional continuous action space. This case subsumes, for example, good exchange games, Cournot duopolies, Bertrand competitions, and so forth. These applications are discussed in Section 5. We assume two players, but this is just to simplify the exposition: Our results easily extend to the case of N players. The assumption of continuous actions is discussed in a great depth in Section 6.

Consider a general two-person symmetric component game with action $a_i \in A_i$ and payoff function π_i. Two players are denoted $i = 1, 2$, and a player's action space A_i is a convex subset (an interval) in $\mathbb{R}$: Examples include $A_i = [\underline{a}_i, \overline{a}_i]$ or $[0, \infty)$. Symmetry means $A_1 = A_2 =: A$ and $\pi_1(a, a') = \pi_2(a, a')$ for all $a, a' \in A$.[11] We assume that the component game has a unique symmetric pure Nash equilibrium (a^N, a^N), whose payoff is $\pi^N := \pi_i(a^N, a^N)$. Here we confine our attention to symmetric revision game equilibrium $x(t)$ that uses the "trigger strategy." The action path $x(t)$ means that, when a revision opportunity arrives at time $-t$, players are supposed to choose action $x(t)$, given that there has been no deviations in the past. If any player deviates and does not choose the prescribed action $x(t)$, then in the future players prepare the Nash equilibrium action of the component game a^N, whenever a revision opportunity arrives. This is what we mean by the trigger strategy in revision games. Below we identify the optimal symmetric equilibrium in the class of trigger strategy equilibria. By the "optimal equilibrium" in a given class of equilibria, we mean that the strategy profile achieves (ex ante) the highest payoffs in that class. Let the symmetric payoff function be

$$\pi(a) := \pi_1(a, a) = \pi_2(a, a),$$

and define the best symmetric action $a^* := \arg\max_{a \in A} \pi(a)$ and let $\pi^{*:} = \pi(a^*)$ denote the highest symmetric payoff.[12] We assume the following regularity conditions. Unless otherwise noted, *these assumptions are imposed only in this section.*

1. **A1:** A pure symmetric Nash equilibrium (a^N, a^N) exists, and it is different from the best symmetric action profile (a^*, a^*).

2. **A2:** The payoff function π_i for each $i = 1, 2$ is twice continuously differentiable.[13]

3. **A3:** There is a unique best reply $BR(a)$ for all $a \in A$.

4. **A4:** At the best reply, the first and second order conditions are satisfied: For each $i = 1, 2$,
$$\frac{\partial \pi_i(BR(a), a)}{\partial a_i} = 0, \quad \frac{\partial^2 \pi_i(BR(a), a)}{\partial^2 a_i} < 0.$$

[11] When we write $\pi_i(x, x')$, x is player i's action and x' is player $-i$'s action.

[12] Assumption A2 that we state shortly ensures that all these pieces of notation are well-defined.

[13] When A is not an open set, we assume that there exists an open interval $\tilde{A}$ such that $A \subset \tilde{A}$ and π_i can be extended to a function $\tilde{\pi}_i$ over $\tilde{A} \times \tilde{A}$ that is twice continuously differentiable, i.e. $\tilde{\pi}_i(a, a') = \pi_i(a, a')$ if $(a, a') \in A \times A$.

5. **A5:** $\pi(a)$ is strictly increasing if $a < a^*$ and strictly decreasing if $a^* < a$.

6. **A6:** The gain from deviation

$$d(a) := \pi_i(BR(a), a) - \pi_i(a, a) \tag{1}$$

   is strictly decreasing if $a < a^N$ and strictly increasing if $a^N < a$.

A *trigger strategy equilibrium* is characterized by its equilibrium path (revision plan) $x : [0, T] \to A$ (recall that $x(t)$ denotes the equilibrium action to be taken when a revision opportunity arrives at time $-t$). The expected payoff at the beginning of the game (i.e., at time $-T$) associated with x is defined by

$$V(x) := \pi(x(T))e^{-\lambda T} + \int_0^T \pi(x(t))\lambda e^{-\lambda t} dt. \tag{2}$$

We say that a path x is *feasible* if the expected payoff (2) is well-defined. Since (2) represents an expected payoff, the second term in (2) should be regarded as Lebesgue integral. Consequently the *set of feasible paths* is formally defined by

$$X := \{x : [0, T] \to A \mid \pi \circ x \text{ is measurable}\}.$$

Given a feasible path $x \in X$, the incentive constraint at time t is

$$\text{(IC(t)):}\ d(x(t))e^{-\lambda t} \leq \int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds, \tag{3}$$

where $d(x(t))$ represents the gain from deviation (see (1)). The set of trigger strategy equilibrium paths is formally defined as

$$X^* := \{x \in X \mid \text{IC(t) holds for all } t \in [0, T]\}.$$

Thus, by optimal path we mean the path that achieves (ex ante) the highest payoff with in X^*.

First, we show formally that an optimal trigger strategy equilibrium path exists and it is differentiable (Proposition 1). Based on this, we then derive a differential equation to characterize the optimal path (Theorem 1).

Proposition 1 *There exists an optimal trigger strategy equilibrium* $\overline{x}(t)$ *(*$V(\overline{x}) = \max_{x \in X^*} V(x)$*) which is (i) continuous for all* t*, (ii) differentiable in* t *when* $\overline{x}(t) \neq a^N, a^*$*. Furthermore,* $\overline{x}(t)$ *satisfies the following binding incentive constraint if* $\overline{x}(t) \neq a^*$*:*

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{4}$$

(Sketch of the proof): This proposition is proved by a series of propositions in Appendix A. First, we show that the optimal trigger strategy equilibrium exists and it is continuous in t. Then we use the continuity to show that it is differentiable. The proofs rely on the following three elementary technical facts:

1. *For a collection of countably many measurable functions* $\pi^n(t)$, $n = 1, 2, ...,$ $\sup_n \pi^n$ *is measurable.* We use this fact to construct a candidate optimal payoff $\overline{\pi}(t)$ that is measurable (as the supremum of a sequence $\pi(x^n(t))$, where x^n is a sequence in X^* whose payoffs approach $\sup_{x \in X^*} V(x)$: Proposition 12). Then, pretending that this is the optimal payoff, we construct the *candidate* optimal path $\overline{x}(t)$ by the binding "pseudo incentive constraint"

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{5}$$

   Note that we have yet to show that this implies the *true* incentive constraint.

2. *Lebesgue integral* $\int_0^t f(s)ds$ *is continuous in* t *for any measurable function* f. This fact shows that the right-hand side of the above equation (5), whose integrand is measurable by Step 1, is continuous in t, leading to the continuity of $\overline{x}(t)$ (Proposition 12). We also show that $\overline{\pi}(t) \leq \pi(\overline{x}(t))$ so that the pseudo incentive constraint (5) implies the true incentive constraint $d(\overline{x}(t))e^{-\lambda t} \leq \int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right) \lambda e^{-\lambda s} ds$. We go on to show that this weak inequality is actually satisfied with equality (Proposition 13), so that we have the binding incentive constraint (4).

3. *Lebesgue integral* $\int_0^t f(s)ds$ *is differentiable in* t *if* f *is continuous.*[14] This fact shows that the right-hand side of the binding incentive constraint (4), whose integrand is

[14] When f is continuous, $\int_0^t f(s)ds$ is equal to Riemann integral and this is just the well-known fundamental theorem of calculus: $\frac{d}{dt} \int_0^t f(s)ds$ exists and equal to $f(t)$.

continuous by Step 2, is differentiable in t, and this leads to the differentiability of $\overline{x}$ (Proposition 13). Q.E.D.

Next we show that there is *essentially a unique optimal path.* Note first that there are in fact multiple optimal paths which attain the same expected payoff. Let $\overline{x}(t)$ be the optimal trigger strategy equilibrium path identified by the previous proposition. Then,

$$x(t) := \begin{cases} a^N & \text{if } t \text{ is in a measure zero set} \\ \\ \overline{x}(t) & \text{otherwise} \end{cases} .$$

is also a trigger strategy equilibrium path that achieves the same expected payoff as $\overline{x}(t)$ does. However, it is easy to show that the following is true.

Proposition 2 *The optimal path is essentially unique: If $y(t)$ is an optimal trigger strategy equilibrium path, then $y(t) = \overline{x}(t)$ almost everywhere, where $\overline{x}(t)$ is the optimal path that satisfies the binding incentive constraint (4).*

The proof is given in Appendix B. Hereafter, the continuous and differentiable optimal path $\overline{x}(t)$ identified in Proposition 1 is referred to as *the essentially unique optimal path*, or simply as *the optimal path.* Now we are ready to state our main result in this section: The optimal path is characterized by a differential equation.

Theorem 1 *The optimal path $\overline{x}(t)$ is the unique path with the following properties: (i) it is continuous in t and departs a^N at $t = 0$ (i.e., $\overline{x}(t) = a^N$ if and only if $t = 0$), (ii) for $t > 0$, it solves differential equation*

$$\frac{dx}{dt} = \frac{\lambda \left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} =: f(x) \tag{6}$$

until $\overline{x}(t)$ hits the optimal action a^*, and (iii) if $\overline{x}(t)$ hits the optimal action a^*, it satays there ($\bar{x}(t) = a^*$ implies $\bar{x}(t') = a^*$ if $t' > t$). Furthermore, if T is large enough, $\overline{x}(t)$ always hits the optimal action a^* at a fixed finite time,

$$t(a^*) := \lim_{a \to a^N} \int_a^{a^*} \frac{1}{f(x)} dx \tag{7}$$

which is independent of T.

(Sketch of the proof): Technical details can be found in Appendices. Let us now confine our attention to the case $a^N < a^*$ (a symmetric proof applies to the case $a^* < a^N$). Lemma 1 in Appendix A implies that the essentially unique optimal path lies between the Nash and optimal actions ($\overline{x}(t) \in [a^N, a^*]$ for all t). By differentiating the binding incentive constraint (4), we obtain a differential equation (6) when $d'(x) \neq 0$. Under Assumption A6, we have $d'(a^N) = 0$ and $d'(x) > 0$ if $x \neq a^N$ (d' can be shown to exist (Lemma 2)). (Recall that $d'(a^N) = 0$ is the first order condition that the gain from deviation $d(x)$ is minimized at the Nash action $x = a^N$.) Thus we have obtained a differential equation on an open domain $(x, t) \in (a^N, a^* + \varepsilon) \times (-\infty, \infty)$, for some $\varepsilon > 0$.[15] Note well that the domain excludes the Nash action a^N, where $f(a^N)$ is not defined because $d'(a^N) = 0$.

The optimal path $\overline{x}(t)$ satisfies the following conditions:

- (i) it lies in $[a^N, a^*]$ for all t,
- (ii) it is continuous in t,
- (iii) it follows the differential equation when $x \in (a^N, a^*)$, and
- (iv) it starts with Nash action a^N at $t = 0$.

It turns out that there are multiple paths which satisfy conditions (i)-(iv). For example, *trivial constant path* $x(t) \equiv a^N$ satisfies those conditions. In what follows, we identify all paths that satisfy conditions (i)-(iv) and find the optimal one among them.

The crucial step is to show that there is a *non-trivial* path to satisfy (i)-(iv). Is there any solution to $dx/dt = f(x)$ which departs from a^N and reach some action $a^0 \in (a^N, a^*]$ at some finite time? The answer is positive if and only if the following *finite time condition*

$$t(a^0) := \lim_{a \to a^N} \int_a^{a^0} \frac{1}{f(x)} dx < \infty \tag{8}$$

is satisfied. As we will show, $t(a^0)$ represents the time for a solution to the differential equation $\frac{dx}{dt} = f(x)$ to travel from a^N to a^0. The reason is the following. Under Assumptions A5 and A6, $f(x) = \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)} > 0$ when $x \in (a^N, a^*]$. Hence any solution $x(t)$ to $\frac{dx}{dt} = f(x)$ is strictly increasing in t. Therefore, $x(t)$ has inverse function $t(x)$, and its

[15] If a^* is a boundary point of A, extend $f(x)$ to $(a^N, a^* + \varepsilon)$ by any continuously differentiable function and apply the same proof in what follows. This is possible under A2 and footnote 2.

derivative is given by $\frac{dt}{dx} = \frac{1}{f(x)}$. This implies that $t(a^0) = \lim_{a \to a^N} \int_a^{a^0} \frac{dt}{dx} dx$ represents the time for a solution to the differential equation $\frac{dx}{dt} = f(x)$ to travel from a^N to a^0.

It is straightforward to check that this finite time condition (8) is satisfied for any $x^0 \in (a^N, a^*]$, under our assumptions (Lemma 5 in Appendix C). Given those observations, all paths that satisfy (i)-(iv) can be written as follows:

$$x_\tau(t) := \begin{cases} a^N & \text{if } t \in [0, \tau] \\ \\ x^*(t-\tau) & \text{if } t \in (\tau, \tau + t(a^*)) \\ \\ a^* & \text{if } t \in [\tau + t(a^*), \infty) \end{cases},$$

where $x^*(t)$ is the solution to $dx/dt = f(x)$ with boundary condition $x^*(t(a^*)) = a^*$. This path $x_\tau(t)$ departs from a^N at time τ, follows the differential equation, and then hits the optimal action a^* and stays there. (More precisely, we must consider the restriction of $x_\tau(t)$ on $[0, T]$.)

Those paths obviously satisfy (i)-(iv). Next we show the converse: any path satisfying (i)-(iv) is equal to $x_\tau(t)$ for some $\tau \in [0, \infty]$. This comes from the standard result in differential equation: $dx/dt = f(x)$ defined on an open domain $(x, t) \in (a^N, a^* + \varepsilon) \times (-\infty, \infty)$ has a unique solution given any boundary condition, if $f(x)$ is continuously differentiable. Under our assumptions, it is easy to check that $f(x)$ is indeed continuously differentiable on (a^N, a^*) (Lemma 3 in Appendix C). The uniqueness of the solution then implies that any path satisfying (i)-(iv) is equal to $x_\tau(t)$ for some $\tau \in [0, \infty]$.[16]

Among the paths $x_\tau(t)$, $\tau \in [0, \infty]$ the one that departs from a^N immediately (i.e., $x_0(t)$) obviously has the highest payoff. Therefore the optimal path is given by the restriction of $x_0(t)$ on $[0, T]$, which has the stated properties in Theorem 1. Q.E.D.

In the optimal trigger strategy equilibrium identified in the previous theorem, players

[16] Formal proof goes as follows. The trivial path, which satisfy (i)-(iv), is equal to x_τ with $\tau = \infty$. Consider any non-trivial path $x^0(t)$ that satisfy (i)-(iv), where $x^0(t^0) =: a^0 \in (a^N, a^*)$ for some t^0. Define $t' := t^0 - \lim_{a \to a^N} \int_a^{a^0} \frac{1}{f(x)} dx$, so that $x^*(t - t')$ hits a^0 at time t^0. The uniqueness of the solution to the differential equation (for boundary condition $x(t^0) = a^0$) implies $x^0(t) = x^*(t - t')$. If $t' \geq 0$, we obtain the desired result $x^0(t) = x_\tau(t)$ for $\tau = t'$. If $t' < 0$, $x^0(0) = x^*(-t') > a^N$ and $x^0(0)$ cannot satisfy (iv). (We have $x^*(-t') > a^N$ because we are considering the case $a^N < a^*$, where the solution $x^*(t)$ is strictly increasing).

act as follows. Recall that $\overline{x}(t)$ is the action to be taken at time $-t$. If the time horizon is long enough, (i.e., if $T \geq t(a^*)$), players start with the best action a^*, and even if a revision opportunity arrives, they do not revise their actions until time $-t(a^*)$ is reached. After that, if a revision opportunity arrives, they choose an action $\overline{x}(t)$, which is closer to the Nash action. The closer the timing of the revision opportunity is to the end of the game, the closer the revised action $\overline{x}(t)$ is to the Nash equilibrium. At the end of the game, the actions chosen at the last revision opportunity are implemented. Hence the best symmetric trigger strategy equilibrium induces a probability distribution of actions in between the best and Nash actions. The nature of this equilibrium distribution will be examined in the following propositions (Propositions 3 and 4).

One might expect that the outcome of the component game, and hence the payoffs, may depend on the arrival rate λ. The next proposition, which is actually nothing but a simple observation, shows that this is not the case. To state the proposition, we need to introduce the following notation. We denoted the first time to hit the optimal action by $t(a^*)$ in Theorem 1 (see (7)), but to explicitly show its dependence on arrival rate λ, let us now denote it by $t_\lambda(a^*)$.

Proposition 3 ***(Arrival Rate Invariance)*** *Under the best symmetric trigger strategy equilibrium, the probability distribution of action profile at period 0 is independent of the Poisson arrival rate λ, provided that the time horizon T is long enough. Specifically, Let $t_\lambda(a^*)$ be the (first) time to reach the optimal symmetric action, stated in Theorem 1. Then, as long as $t_\lambda(a^*) \leq T$, the probability distribution of the action profile at period 0 is independent of λ.*

Proof. Consider λ such that $t_\lambda(a^*) \leq T$ and call it Model 1. Rewrite this model by changing the time scale in such a way that one unit of time in Model 1 corresponds to λ units in the new model. Under the new time scale, the model is identical to the revision game with arrival rate 1 and time horizon λT. Call it Model 2. The best symmetric trigger strategy equilibrium path in Model 1 should be transformed into the best symmetric trigger strategy equilibrium path in Model 2. In particular, it must be the case that $t_1(a^*)$, the first time the optimal path hits a^* in Model 2, is equal to $\lambda t_\lambda(a^*)$, and this is smaller than the time horizon of Model 2 (λT). Hence in Model 2, there is no revision of action in the best symmetric trigger strategy equilibrium for $t \in [-\lambda T, -t_1(a^*)]$, and therefore the probability distribution of action profile at $t = 0$ is unchanged if the game starts at $-t_1(a^*)$ (instead of $-\lambda T$). Hence, the probability distribution of action profile at $t = 0$ under *any* arrival rate λ such that $t_\lambda(a^*) \leq T$ is equal to the probability distribution under arrival rate 1 and time horizon $t_1(a^*)$. ■

Note that the fact that payoffs realize only at the deadline $t = 0$ played a crucial role in this proposition (otherwise, the expected payoffs would be affected by the arrival rate and the discount factor). Proposition 3 shows the following attractive feature of revision games: we can obtain a unique prediction that does not depend on the fine detail, namely the arrival rate λ of the revision opportunities. In particular, even if λ is sufficiently high (so that there are many chances to revise actions right before the component game), the expected outcome in the component game is the same as in the case of low λ.

The proof also shows how to calculate the cumulative distribution function of symmetric action a, denoted by $F(a)$. Again we consider the case with $a^N < a^*$ (a symmetric argument applies to the other case). Let $x_1(t)$ be the optimal trigger strategy equilibrium path under $\lambda = 1$, and denote the time for $x_1(t)$ to hit $a \in [a^N, a^*]$ by $t_1(a)$. The latter is given by equation (8) for $\lambda = 1$:

$$t_1(a) := \lim_{a' \to a^N} \int_{a'}^{a} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx. \tag{9}$$

For $a \in (a^N, a^*)$, $F(a) = \int_{\{t | x_1(t) \leq a\}} e^{-t} dt = \int_0^{t_1(a)} e^{-t} dt = 1 - e^{-t_1(a)}$. The first equality follows from the fact that the density of action $x_1(t) \leq a$ is the product of

- 1 (the density of revision at time t) and
- e^{-t} (the probability that the revised action at time t, $x(t)$, will never be revised again).

At a^*, the distribution function $F(a)$ jumps by $e^{-t_1(a^*)}$ and $F(a) = 1$ for $a \geq a^*$. This means that the optimal action a^* is played with probability $e^{-t_1(a^*)}$. This is the probability that no revision opportunity arises under $\lambda = 1$. Below we summarize our arguments.

Proposition 4 *Suppose that the time horizon is long enough so that the efficient action a^* is chosen at the beginning of the revision game, under the best symmetric trigger strategy equilibrium. When $a^N < a^*$, the cumulative distribution function of the symmetric action* *realized at $t = 0$ is given by*

$$F(a) = \begin{cases} 0 & \text{if } a < a^N \\ 1 - e^{-t_1(a)} & \text{if } a^N \le a < a^* \\ 1 & \text{if } a^* \le a \end{cases},$$

where $t_1(a)$ is given by (9) and represents the time for the best symmetric trigger strategy action path to reach $a \in [a^N, x^*]$, when the arrival rate is $\lambda = 1$. When $a^* < a^N$, it is given by

$$F(a) = \begin{cases} 0 & \text{if } a < a^* \\ e^{-t_1(a)} & \text{if } a^* \le a \le a^N \\ 1 & \text{if } a^N < a \end{cases}.$$

5 Applications

In this section, we use the general framework given in the previous section to analyze various games of interest. Specifically, we use the differential equation provided in Theorem 1 to analyze good exchange games (prisoner's dilemma), Cournot duopolies, Bertrand competition with product differentiation, and election campaign. Unless otherwise noted, the component games in these examples satisfy Assumptions A1-A6.

We will be considering two measures of the degree of cooperation. Let the expected payoff from the optimal trigger strategy equilibrium when T is sufficiently large be $\tilde{\pi}$. The two measures are:

$$R := \frac{\tilde{\pi}}{\pi(a^*)} \quad \text{and} \quad \tilde{R} := \frac{\tilde{\pi} - \pi^N}{\pi(a^*) - \pi^N}.$$

The first one simply takes the ratio of the equilibrium payoff to the fully collusive payoff. The second is a conservative one, which compares the improvement of the payoff relative to the Nash payoff (the static equilibrium payoff) with the maximum possible payoff improvement.

5.1 Good Exchange Game

For each player $i = 1, 2$, let the payoff function be $\pi_i(a_i, a_{-i}) = a_{-i}^k - c \cdot a_i^2$, where $c > 0$, $k \in (0, 2)$, and the action space is $a_i \in [0, \infty)$. This game represents the following situation. Two players $i = 1, 2$ exchange goods they produce. That is, player 1 produces one unit of good and gives it to player 2 (and *vice versa*). The quality of the good player i produces is equal to a_i^k, and i incurs a convex cost $c \cdot a_i^2$ to provide a good with quality a_i. Alternatively, one can interpret a_i^k as the quantity of goods i provides given effort level a_i and assume that $c \cdot a_i^2$ is the cost to exert the effort level a_i. Note that $a_i = 0$ is the dominant strategy, while the best symmetric action $a^* = \left(\frac{k}{2c}\right)^{\frac{1}{2-k}}$ is strictly positive. Hence this can be regarded as a version of the prisoner's dilemma game with continuous actions. Notice that the larger k is, the smaller is the gain from a very small amount of action (i.e., $a^k < a^{k'}$ if $k > k'$ and a is small).

The differential equation in Theorem 1 for this example is

$$\begin{aligned}\frac{dx}{dt} &= \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} \\ &= \frac{\lambda(cx^2 + (x^k - cx^2) + 0)}{(cx^2)'} = \frac{\lambda x^k}{2cx}.\end{aligned}$$

Note that, since 0 is a dominant action, the Nash payoff π^N is zero, and the best reply to any action is zero: $BR(a_{-i}) = 0$. The latter implies $d(x) = cx^2$. The above differential equation has a solution $x(t) = \left(\frac{2-k}{2c}\lambda t\right)^{\frac{1}{2-k}}$ which departs from 0 (the Nash action) at time $t = 0$. The time at which $x(t)$ reaches the best action, denoted $t(a^*)$, can be calculated by (7), but it is equivalently obtained by solving $a^* = x(t(a^*)) = \left(\frac{2-k}{2c}\lambda t(a^*)\right)^{\frac{1}{2-k}}$. We summarize our findings as follows.

Proposition 5 *In the good exchange game, the optimal trigger strategy equilibrium, $x(t)$, is characterized by*

$$x(t) = \begin{cases} \left(\frac{2-k}{2c}\lambda t\right)^{\frac{1}{2-k}} & \textit{if} \quad t < t(a^*) \\ a^* = \left(\frac{k}{2c}\right)^{\frac{1}{2-k}} & \textit{if} \quad t(a^*) \leq t \end{cases},$$

where $t(a^*) = \frac{k}{\lambda(2-k)}$.

The path characterized in Proposition 5 is depicted in Figure 2.

In Figure 2, as k increases, the time that the path departs from the optimal action ($t(a^*)$) becomes larger, and the path approaches 0 more quickly. These observations suggest that it is more difficult to cooperate when parameter k is large. This is in line with our earlier observation that a larger k implies a smaller gain from cooperation around the Nash equilibrium $(0, 0)$ (hence it is more difficult to sustain cooperation).

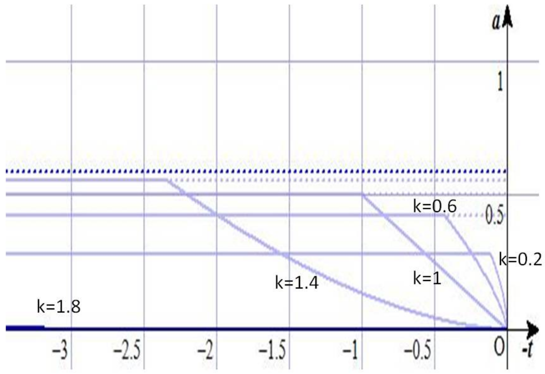

Figure 2: **The optimal path for the good exchange game:** $\lambda = 1$.

Corollary 1 *In the good exchange game, R $(= \tilde{R})$ is decreasing in k. It approaches 1 as $k \searrow 0$, and approaches 0 as $k \nearrow 2$.*

The proof is straightforward calculation and therefore omitted. We can also explicitly calculate the expected payoff when the parameter k is just in the middle of $(0, 2)$. That is, when $k = 1$, the expected payoff is $\frac{1}{2ec^2}$, which implies $R = \tilde{R} = \frac{2}{e} \cong 0.74$ (this is independent of the value of c). The revision game attains 74% of the fully cooperative payoff in this case.

Although there cannot be any cooperation in the Nash equilibrium of the component game, in revision games players can achieve around three fourths of the fully cooperative payoff. The degree of cooperation decreases as the gain from small cooperation decreases. Thus, a higher degree of overall cooperation is more difficult to achieve the less gain there is given a small amount of cooperation.

5.2 Cournot Duopoly: Collusion Through Output Adjustment Achieves 97% of The Monopoly Profit

In this subsection, we consider a Cournot duopoly game with a linear demand curve $P = a - b(q_i + q_j)$ (q_i denotes agent i's quantity) and constant (and identical) marginal cost c. Hence the (component game) payoff function for player i is $\pi_i = (a - b(q_i + q_j) - c)\, q_i$. We suppose $a > c > 0$ and $b > 0$. The differential equation is

$$\begin{aligned}\frac{dq}{dt} &= \frac{\lambda\left(d(q) + \pi(q) - \pi^N\right)}{d'(q)} \\ &= \frac{\lambda}{18}(q - 5\frac{a-c}{3b}).\end{aligned}$$

This comes from $d(q) = \frac{(a-c-3bq)^2}{4b}$, $\pi(q) = (a-c-2bq)q$, and $\pi^N = \frac{(a-c)^2}{9b}$. The differential equation admits a simple solution $q(t) = \frac{a-c}{3b}(5 - 4e^{\frac{\lambda}{18}t})$ which departs from the Cournot Nash output $q^N = \frac{a-c}{3b}$ at $t = 0$, and this path hits the optimal output $q^* = \frac{a-c}{4b}$ at $t(q^*) = \frac{18}{\lambda}\ln\left(\frac{17}{16}\right)$. Therefore, we have obtained the following.

Proposition 6 *In the Cournot duopoly game, the optimal trigger strategy equilibrium, $q(t)$, is characterized by*

$$q(t) = \begin{cases} \frac{a-c}{3b} \cdot r(t) & \textit{if} \quad t < t(q^*) \\ q^* = \frac{a-c}{4b} & \textit{if} \quad t(q^*) \leq t \end{cases},$$

where we let $t(q^*) = \frac{18}{\lambda}\ln\left(\frac{17}{16}\right)$ and $r(t) = 5 - 4e^{\frac{\lambda}{18}t}$.

Note that $r(t)$ is the ratio of the equilibrium quantity at time $-t$ to the static equilibrium quantity, $\frac{a-c}{3b}$.

When the firms collude, they produce less than the Nash quantity, and therefore the optimal trigger equilibrium path that we characterize is *deceasing* in t. That is, the ratio r starts from 1 (due to the initial condition), and decreases monotonically to $\frac{3}{4}$. The path of r with respect to t is depicted in Figure 3.

Next, we consider the welfare implication of the revision game of the Cournot duopoly game. One can compute the equilibrium expected payoff, and it turns out that a surprisingly high degree of collusion can be achieved in this game. The next corollary implies that, when two firms gradually adjust their outputs before the market is open (and if they closely monitor each other's output adjustment processes), then they can achieve almost 97% of the fully collusive profit (this amounts to 72% of the gain relative to the Nash profit). We emphasize that those numbers are independent of the position and the slope of the demand curve (a and b) and the marginal cost c (and also independent of the arrival rate λ of revision opportunities, as Proposition 3 shows).

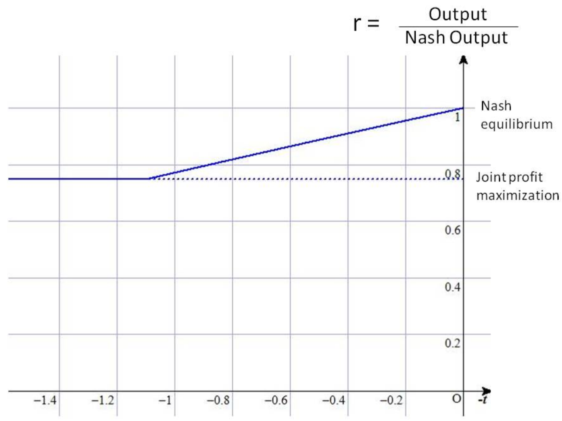

Figure 3: **The optimal path for Cournot duopoly:** $\lambda = 1$.

Corollary 2 *In the Cournot duopoly game,* $R \cong 0.968$ *and* $\tilde{R} \cong 0.714$*, independent of the values of parameters* a*,* b*, and* c*.*

The following story might fit the Cournot revision game. Two fishing boats depart from a harbor early in the morning, and they must return when the fish market at the harbor opens at 6:00 am. They would like to restrict their catch so as to increase the price at the fish market. They first catch a small amount of fish (the collusive quantity). They are operating side by side, closely monitoring each other's behavior. Fish schools occasionally visit them, by Poisson process.[17] The arrival rate is $\lambda = 0.1$ (and the time unit is a minute), so that a fish school comes every ten minutes on average. Since $t(q^*) = \frac{18}{\lambda} \ln\left(\frac{17}{16}\right) = 10.$ 912 minutes, they do not catch any additional fish until eleven minutes before the market opens. In the last eleven minutes, whenever a fish school visit them, they catch additional fish. If no fish school visits, they deliver the collusive amount to the market. If a fish school comes right before 6:00am, they catch Nash amount. If the last visit of a fish school is somewhat before, they catch a smaller amount. On average, they encounter only one revision opportunity in the last eleven minutes (because $\lambda \cdot t(q^*) \simeq 1$), and they can achieve 97% of fully collusive profit.

[17]Pun not intended.

5.3 Bertrand Competition with Product Differentiation

In this subsection we consider a Bertrand competition with product differentiation. We would like to examine how the degree of product differentiation affects the possibility of collusion in the revision game. To this end, we employ the Hotelling model of spatial competition with price setting firms. This model has an advantage of incorporating the case with no product differentiation as a special case.

A continuum of buyers are distributed uniformly over $[0, 1]$. Two firms $i = 1, 2$ are located at positions 0 and 1, respectively. A buyer at x receives payoff $a - d|x - y| - p$ if she buys from the firm at y with price p, where $d \in [0, \frac{2}{3}a)$. Notice that d is the cost of transportation for the buyers, and it measures the degree of product differentiation. In particular, $d = 0$ corresponds to the case in which there is no product differentiation, and a high d means a high degree of product differentiation. If the buyer does not buy anything, the payoff is 0. No buyer would want to buy two or more products. Each firm's marginal cost is normalized at 0, and the firm's payoff is the average revenue from a buyer.[18]

For relatively high product differentiation, namely for $d \in (\frac{2}{7}a, \frac{2}{3}a)$, the differential equation is

$$\frac{dp}{dt} = \lambda\frac{p + 3d}{2}.$$

This comes from $\pi^N = \frac{d}{2}$, $\pi(p) = \frac{p}{2}$, and $d(p) = \frac{(p-d)^2}{8d}$.

When $d \in (0, \frac{2}{7}a]$, however, the degree of differentiation is so small that when the opponent sets a price close to the best collusive price, the best reply is to set a price just enough to take all the buyers, that is, $BR(p) = p - d$ and hence $d(p) = \frac{p}{2} - d$. π^N and $\pi(p)$ are the same as before. Using these formulas, the differential equation in this case can be written as

$$\frac{dp}{dt} = \lambda(2p - 3d).$$

Overall, we obtain the following:

Proposition 7 *In the Bertrand competition game, the optimal trigger strategy equilibrium path, $p(t)$, is characterized as follows:*

[18] Firm i's payoff function has a kink for example when $p_i = p_{-i} - d$, so A2 is violated. However, A2 can be shown to be satisfied at relevant regions ((p, p) and $(BR(p), p)$ for all p weakly between the Nash price and the fully collusive price) when $d \in (\frac{2}{7}a, \frac{2}{3}a)$. When $d \in (0, \frac{2}{7}a)$, A2 is not satisfied at $(BR(p), p)$ for one p on the equilibrium path, but one can show that the optimal path is characterized by solving two differential equations, one for prices below such p and the other for prices above such p.

1. *If* $d \in (\frac{2}{7}a, \frac{2}{3}a)$*,*

$$p(t) = \begin{cases} d\left(4e^{\lambda\frac{t}{2}} - 3\right) & \text{if} \quad t < t(p^*) \\ p^* = a - \frac{d}{2} & \text{if} \quad t(p^*) \le t \end{cases},$$

where $t(p^*) = \frac{2}{\lambda}\ln\left(\frac{a}{4d} + \frac{5}{8}\right)$*.*

2. *If* $d \in (0, \frac{2}{7}a]$*,*

$$p(t) = \begin{cases} d\left(4e^{\lambda\frac{t}{2}} - 3\right) & \text{if} \quad t < t^1 \\ d\left(\frac{8}{27}e^{2\lambda t} + \frac{3}{2}\right) & \text{if} \quad t^1 \le t < t^2 \\ p^* = a - \frac{d}{2} & \text{if} \quad t^2 \le t \end{cases},$$

where $t^1 = \frac{2}{\lambda}\ln\left(\frac{3}{2}\right)$ *and* $t^2 = \frac{2}{\lambda}\ln\left(\frac{a}{d} - 2\right)$*.*

3. *If* $d = 0$*,* $p(t) = 0$ *for all* t*.*

The proposition claims that there is a cooperative path if and only if there is a product differentiation. This highlights the importance of Assumption A4. When $d = 0$, the first order condition does not hold at the static Nash equilibrium, so there cannot be a collusive path. The intuition is as follows: If there is no product differentiation, an infinitesimal price cut can increase the profit discontinuously almost to the double whenever the current price is strictly higher than the marginal cost (which is 0 in this example). This is because all buyers switch to the deviating firm. Hence, if the current price is not equal to the Nash equilibrium, the gain from deviation is not of a smaller order in magnitude than the gain from cooperation. This makes cooperation impossible. If there is a product differentiation, however, only a small fraction of buyers switch to the deviating firm, and this makes the cooperation sustainable.

Note also that p is increasing in d. Hence, the more differentiated the products are, the more collusion there is. This makes sense: When the degree of product differentiation is large, the instantaneous gain from deviation when firms set prices close to the Nash price is small relative to the loss from the punishment because firms need to decrease the price a lot to steal the opponent's share. The path characterized in Proposition 7 when $d > 0$ is depicted in Figure 4. In the figure, we fix $a = 10$ and draw the optimal paths for $d = 1$, $d = 2$, $d = 3$, and $d = 5$. As expected, the collusive path is close to the best collusive price when the degree of product differentiation is high.

The expected payoff under the optimal trigger strategy path can be computed as follows:

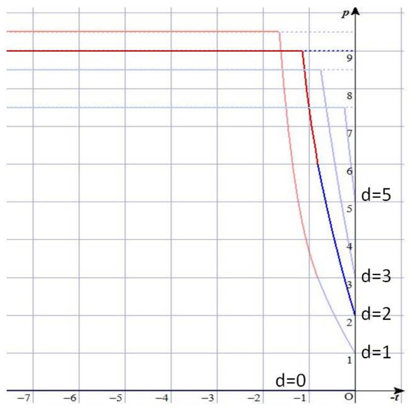

Figure 4: **The optimal path for Bertrand competition:** $\lambda = 1$, $a = 10$.

Corollary 3 *Let $h := \frac{d}{a}$ be the degree of product differentiation. Then, the level of collusion achieved in the revision game, measured by R and $\widetilde{R}$, is expressed as follows.*

1. *If $h \in (\frac{2}{7}, \frac{2}{3})$, $R = \frac{2h(10-7h)}{(2-h)(2+5h)}$ and $\tilde{R} = \frac{8h}{5h+2}$.*
2. *If $h \in (0, \frac{2}{7}]$, $R = \frac{2h(3-7h)}{(2-h)(1-2h)}$ and $\tilde{R} = \frac{2h(2-5h)}{(1-2h)(2-3h)}$.*
3. *If $h = 0$, $R = \tilde{R} = 0$.*
4. *Both R and $\tilde{R}$ are strictly increasing in h.*

The ratios stated in the corollary The ratio of expected payoff relative to the fully collusive payoff is calculated in Table 1 for several values of h. The table shows that in the revision game, firms can achieve quite a bit of cooperation to obtain high expected payoffs. For example, if $h = .5$ then, on average, a buyer's willingness to pay to the worse good is 71.4% of that of the preferred good. In such a circumstance, the table shows that 96% (89% even under the conservative measure of the degree of cooperation) of payoffs can be achieved by the revision game.

5.4 Election Campaign: Policy Platforms Gradually Converge

In this subsection we consider a simple election model with policy-motivated candidates. The policy space is an interval $[0, 1]$. As in the standard model of policy-motivated can-

Degree of product differentiation ($r = \frac{d}{a}$)	0	.1	.2	.3	.5	$.6\dot{6}$
$\frac{\text{Expected payoff}}{\text{Collusive payoff}}$ (R)	0	.30	.59	.80	.96	1
$\frac{\text{Expected payoff - Nash payoff}}{\text{Collusive payoff - Nash payoff}}$ ($\tilde{R}$)	0	.22	.48	.69	.89	1

Table 1: Degrees of product differentiation and cooperation.

didates, the position of the median voter is unknown, but its distribution is known as the uniform distribution over the policy space, $[0, 1]$. There are two candidates, $i = 1, 2$, where candidate 1 chooses policy y_1 and candidate 2 chooses policy y_2.

Given a policy profile (y_1, y_2), let a random variable $w(y_1, y_2)$ represent the "winner" of the election. Let candidate i's realized payoff be

$$g_i(y_i, y_{-i}) = a \cdot \mathbb{I}_{\{i=w(y_1,y_2)\}} + b(|y_{w(y_i,y_{-i})} - \overline{y}_i|)$$

where $a \in (\frac{1}{2}, 1)$ is a positive constant representing the utility of winning *per se*, and $b(\cdot)$ is a "policy preference term," which depends on the distance between the winner's policy (the policy actually implemented) and candidate i's "bliss point," denoted $\overline{y}_i$. We assume that $\overline{y}_1 = 0$ and $\overline{y}_2 = 1$. That is, candidate 1 is "left wing" and candidate 2 is "right wing."

There are two key assumptions that we impose on this standard election model with policy-motivated candidates: *First*, we assume that the payoff function corresponding to the policy preference term is *convex*. As Kamada and Kojima (2009) discuss, such policy preferences are especially relevant for issues that contain religious content (e.g. same-sex marriage, abortion, gun control, and so forth), as in these policy issues it is natural to assume that a player's utility sharply decreases as the implemented policy departs from her bliss point.[19] Convex utility function implies that a profile $(0, 1)$ Pareto-dominates the Nash profile, so there is a potential room for cooperation in a revision game. For simplicity, we assume the following functional form: $b(z) = \max\{\frac{1}{2} - z, 0\}$.[20] *Second*, we assume that candidate 1 chooses policy y_1 from $[0, \frac{1}{2}]$ and candidate 2 chooses policy y_2 from $[\frac{1}{2}, 1]$. The motivation behind this assumption is that candidate 1 (resp. candidate 2) faces a

[19] See Osborne (1995) for a criticism on the use of concave utility function for preferences over electoral policies.

[20] This functional form does not satisfy Assumption A2, but it is straightforward to check that A2 is satisfied over the relevant domain. The assumption that the candidate is exactly indifferent between two policies that are both further away from her bliss point is made only for the purpose of simplicity, and does not play any crucial role in our argument.

reputational concern, so that she never wants to set a policy to the right (resp. left) of the middle (Remember that candidate 1 (resp. candidate 2) is “left wing” (resp. “right wing)). Without this assumption, the best response is always to set a policy as close as possible to the other candidate, and thus there is a huge gain by deviating from the profile close to the Nash equilibrium, which makes cooperation impossible in a revision game (by the violation of A4).

The payoff functions are not symmetric as they are, but by redefining actions by

$$x_1 = y_1 \text{ and } x_2 = 1 - y_2$$

we can retain the symmetry. Notice that the probability of i's winning the election can be calculated as $\frac{1+x_i+x_{-i}}{2}$.

Now, let us explain what the revision game of this election game corresponds to. We interpret the revision phase as the time period for “election campaign.” In the revision phase, candidates obtain opportunities to express their policy positions, for example at an open broadcast on radio or television. At each opportunity, candidates can choose their policy announcement that is possibly different from what they have said before (as is often the case). At “time 0” of the revision game, the election takes place, and candidates are committed to implementing her finally announced policy, given that she is elected.[21]

The differential equation for candidate 1's policy platform $y_1(t) = x(t)$ is

$$\frac{dx}{dt} = \lambda \frac{2x - 2a - 7}{4}.$$

This follows from $\pi(x) = \frac{1}{2}\left(a + \frac{1}{2} - x\right)$, $\pi^N = \frac{1}{2}$, and $d(x) = \frac{1}{8}(a - \frac{1}{2} - x)^2$. This has a solution $x(t) = \frac{7+2a-8\cdot e^{\frac{\lambda}{2}t}}{2}$ which departs from Nash action $x^N = \frac{2a-1}{2}$ at $t = 0$.

Proposition 8 *In the election campaign game, the optimal trigger strategy equilibrium, $(y_1(t), y_2(t))$, is characterized by*

$$y_1(t) = \begin{cases} \frac{7+2a-8\cdot e^{\frac{\lambda}{2}t}}{2} & \text{if} \quad t < t^* \\ 0 & \text{if} \quad t^* \leq t \end{cases},$$

[21] This “policy announcement game” is proposed and analyzed in Kamada and Sugaya (2011), in which they analyze the case where candidates cannot announce inconsistent policies while they have an option to announce an “ambiguous policy.”

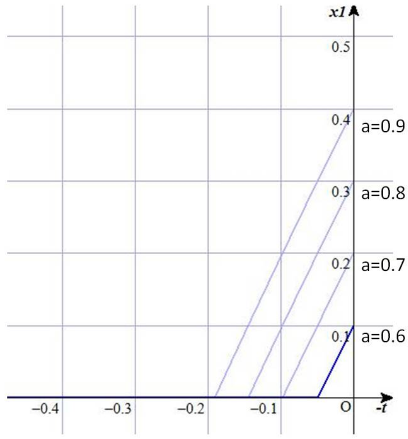

Figure 5: **The optimal path for election campaign (the path of x_1):** $\lambda = 1$.

where $t^* = \frac{2}{3\lambda} \ln\left(\frac{7}{2} + a\right)$ *and* $y_2(t) = 1 - y_1(t)$.

The above proposition shows that in the election campaign game, each candidate starts from announcing their most preferred policies until the time of election becomes close, and then begin catering to the middle in the end. Thus the model captures the well-observed phenomena of candidates changing their policy announcements, moving to the middle when the election is close. The path characterized in Proposition 8 is depicted in Figure 5.

We note that this result does not hold if the policy preference term b is concave, as usually assumed in the political science literature. If candidates' policy preferences are convex, they prefer a diverging policy profile $(0, 1)$ to a converging one $(\frac{1}{2}, \frac{1}{2})$. This is because, for example, candidate 1 does not care about the difference between policies $\frac{1}{2}$ and 1 while she perceives a huge difference between policies 0 and $\frac{1}{2}$. This is why there can be a nontrivial equilibrium path.

6 Discussion

6.1 Robustness of Cooperation

As should be clear at this point, the key to the sustainability of cooperation in revision games is the fact that as the deadline comes close, the gain from defection becomes arbitrarily smaller than the payoff from cooperation. This was made possible because we assume continuous action space and continuous time. First, note that if each player has a dominant action and time is discrete, then by backwards induction it is obvious that the only equilibrium is for each player to play the dominant action at any revision opportunity. Again, drop A1-A6 in this section.

Proposition 9 *Consider a component game with an action set A_i with a strictly dominant action a_i^N for each player i, and consider either of the following two cases:*

1. *A_i is finite.*
2. *There exists $\epsilon > 0$ such that all players are restricted to use strategies that, at any time $-t$, do not condition on what has happened in time $(-t+\epsilon, -t)$.*

Then, whether in synchronous or asynchronous revision games (with homogeneous or heterogeneous arrival rates), there exists a unique subgame perfect equilibrium. In this equilibrium, each player i plays action a_i^N conditional on any history.

For part 1, the proof for the result is straightforward. First observe that if A_i is finite then there exists $\epsilon > 0$ such that given any action of the opponent, a_i^N gives i a payoff at least ϵ greater than any other actions in A_i. This means that, if it is true that each player j prepares an action a_j^N whenever j gets a revision opportunity strictly after time $-t$, then by assumption i's payoff from preparing a_i^N is at least ϵ' greater than preparing any other action for some $\epsilon' > 0$. By continuity of payoffs with respect to probability, this means that there exists $\epsilon'' > 0$ such that i strictly prefers preparing a_i^N to any other action in the time interval $(-t-\epsilon'', -t]$, hence whenever i gets a revision opportunity in $(-t-\epsilon'', -t]$ she prepares a_i^N. This establishes the result.[22] The intuition is that if the time left until the deadline is very little, it is a dominant strategy for players to follow the dominant actions, irrespective of the opponents' strategies.[23] Notice that the above proof is invalid in our main model because there does not exist such $\epsilon > 0$ that we took above.

The restriction on the strategy stated in part 2 describe the situation where there exists a fixed positive "response time," so that any player cannot respond to a defection that has happened in a very close past. The proof is again straightforward. In time $(-\epsilon, 0]$, there

[22]Whether or not ϵ'' depends on t does not matter for the result (In our case, we can actually take ϵ'' independent of t). For a formal proof of this, see Lemma 1 in Calcagno, Kamada, Lovo, and Sugaya (2011).

[23]Calcagno and Lovo (2010) obtained a similar result when the component game is a two-player prisoner's dilemma.

is no reason for any player to play an action other than the strictly dominant action, as the preparation in that time interval does not affect the opponents' future behavior at all. Then, no action prepared in $(-2\epsilon, -\epsilon]$ affects the opponent's future behaviors, so players prepare a_i^N in this time interval as well. Going backwards, we get the desired result.

The proposition shows that *both* the continuity of action space *and* time are needed to obtain cooperation in revision games. In this particular sense, cooperation is not a robust result. However it is not clear why this is the robustness that we should consider with the first-order importance. On the other hand, recent experimental results show that economic agents have altruism motives. In our model, cooperation is retained by a very slight addition of such a behavioral element to the model. To illuminate this issue, we focus on the continuity of time and the situation where players are indifferent between very small cooperation and no cooperation, which is interpreted as an existence of incentives to "give away" a very small amount.[24] Consider a very simple example with the following payoff function:

$$\pi_i(a_i, a_j) = 2a_j - \max\{a_i - \epsilon, 0\}, \qquad \epsilon \geq 0, \quad a_i, a_j \geq 0.$$

This is a version of the good exchange game in Subsection 5.1, where the cost of cooperation takes a different form. We call this game as a *modified good exchange game*.

The cost term is constant at zero near action 0 if $\epsilon > 0$ but it increases linearly otherwise. Notice that when $\epsilon = 0$, there is only one Nash equilibrium in which each player i plays action 0. On the other hand, when $\epsilon > 0$, there are multiple equilibria. In particular, both $(0, 0)$ and (ϵ, ϵ) are Nash equilibria, where the former gives each player the payoff of 0 but the latter gives $2\epsilon > 0$. Also notice that this payoff function does not satisfy Assumption A4 when $\epsilon = 0$.[25]

Now we consider a discrete time version of synchronous revision game (this specification applies only in this subsection). Time is $-t = \ldots, -2, -1, 0$, and at each period, both players have a revision opportunity with probability $p > 0$. The component game is played at time 0 (assume that the revision opportunity may come also at time 0 (with probability p) before the game is played). We construct the optimal symmetric trigger strategy equilibrium path (analogously defined as was done so far) that converges to (ϵ, ϵ) as $t \to 0$ but triggers to $(0, 0)$ upon deviation. A straightforward calculation shows that

[24] An analogous discussion for continuity of actions can be easily done.

[25] Also, this does not satisfy Assumption A2 when $\epsilon > 0$ (as there is a "kink" at $a = \epsilon$).

the path is characterized by[26]:

$$x(t) = \epsilon \cdot \left(\frac{1+p}{1-p}\right)^t.$$

Notice that when $\epsilon = 0$, the path is a trivial one, i.e. $x(t) = 0$ for all t. However, if $\epsilon > 0$, there exists a cooperative path. The nonexistence of cooperative path when $\epsilon = 0$ is straightforward from backwards induction. The existence of cooperative path when $\epsilon > 0$ is that the cost of cooperation does not grow near the Nash action so that players can use the worse equilibria as a threat, and they can use this tiny threat as a foothold for long-run cooperation. This intuition is analogous to the logic of the sustainability of cooperation in Benoit and Krishna (1985), who consider a model of finitely repeated games, and show that an approximate "folk theorem" holds as the horizon becomes infinitely long when each player has multiple Nash payoffs.[27]

Now we turn to our setting with continuous time and smooth payoff function. Specifically, consider a payoff function from the exchange of goods game, $\pi_i(a_i, a_j) = 2a_j - a_i^2$ with $a_i, a_j \geq 0$. There is only one equilibrium at $(0, 0)$, so in the above discrete time setting, there is only one equilibrium in the revision game, by part 2 of Proposition 9. However, recall that there exists a cooperative path when $\epsilon > 0$ in the modified good exchange game, and the sustainability of the path hinges on the fact that the cost of cooperation does not grow near the Nash action so that players can use the worse equilibria as a threat, and they can use this tiny threat as a foothold for long-run cooperation. In the above payoff function, the cost of cooperation, a_i^2, has approximately zero growth near the Nash equilibrium. In the discrete time setting we needed exactly zero growth, but with continuous time, since at no time except at time 0 players are sure that there exists no more revision opportunities, the "growth of approximate zero" (which corresponds to Assumption A4) works as a foothold for long-run cooperation. This is of course not a rigorous proof for why there exists a cooperative path in our model, but this is one of the key parts of the intuition behind our result.

Notice that what is important in the above argument is not the first order condition (A4) per se, but the fact that the gain from defection is smaller than the loss associated

[26] Letting $p = \lambda\Delta\tau$, $\tau = (\Delta\tau) \cdot t$ and taking the limit as $\Delta\tau \to 0$, this converges to the optimal path of $x(\tau) = \epsilon \cdot e^{2\lambda\tau}$ in continuous time, which can also be obtained by a direct computation.

[27] Strictly speaking, Benoit and Krishna (1985) consider the case of flow payoffs (players receive payoffs each period) thus the two settings are slightly different from each other.

with it by a positive order. To make this point clear, consider the following example.

Consider $\pi_i(a_i, a_j) = \sqrt{a_j} - a_i$, with $a_i, a_j \in [0, \infty)$. Note the Nash equilibrium action $a_i = 0$ is a corner solution and the first order condition is not satisfied (the slope is -1). Nevertheless,

$$x(t) = \frac{\lambda^2}{4} t^2$$

constitutes a symmetric trigger strategy path because it satisfies the differential equation (6) in Theorem 1 with $x(0) = 0 = a^N$ (therefore $x(t)$ satisfies the binding incentive constraint $d(x(t))e^{-\lambda t} = \int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds$).[28] This example shows that the first order condition at the Nash equilibrium is not necessary for a nontrivial path to be sustained. What is important in this example is the fact that *the gain from deviation, a, is one order smaller than the value of cooperation,* $\sqrt{a} - a$ *(which can be lost after a deviation) near the Nash action* of $a^N = 0$. In what follows we formulate this observation in a precise way.

First let us generalize the Assumptions A1-A6 imposed in Section 4. In particular we consider a general component game with symmetric action space A and payoff function π.

Proposition 10 *Suppose that there is a symmetric isolated Nash equilibrium* (a^N, a^N) *and that there exists* $\epsilon > 0$ *such that* $[a^N, a^N + \epsilon) \subseteq A \subseteq \mathbb{R}$ *for each player i. Suppose also that there exists* $\epsilon' > 0$, $r > s > 0$ *and* $k, k' > 0$ *such that for all* $a \in (a^N, a^N + \epsilon')$,

$$d(a) \leq k(a - a^N)^r \text{ and} \tag{10}$$

$$k'(a - a^N)^s \leq \pi(a) - \pi^N. \tag{11}$$

Then, in a synchronous revision game, there exists a subgame perfect equilibrium such that non-Nash profiles are prepared at all time $t > 0$ *on the path of play.*

Proof. Take an $\hat{\epsilon} > 0$ such that $[a^N, a^N + \hat{\epsilon}] \subseteq A$, conditions (10) and (11) hold for all $[a^N, a^N, a^N + \hat{\epsilon}]$ with constants k, k', r, and s, and $\frac{k' \lambda e^{-\lambda t}(r-s)}{s+r} t^{\frac{s+r}{r-s}} > k t^{\frac{2r}{r-s}}$. Such $\hat{\epsilon} > 0$ exists if the premise of the proposition holds. We are going to show that a trigger strategy

[28] Note that (unlike in our model in Section 4) the differential equation $\frac{dx}{dt} = f(x) \equiv \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$ is well-defined at $x = a^N$, because $d'(a^N)$ does not vanish ($d' \equiv 1$). In particular, $f(a^N) = 0$. In this case, the differential equation $dx/dt = f$ with boundary condition $x(0) = a^N = 0$ has two solutions. One is $x(t) = \frac{\lambda^2}{4} t^2$, and the other is the constant path $x(0) \equiv a^N = 0$.

path

$$x(t) = \begin{cases} t^{\frac{2}{r-s}} + a^N & \text{if } t < \hat{\epsilon}^{\frac{r-s}{2}} \\ \\ \hat{\epsilon} + a^N & \text{if } t \geq \hat{\epsilon}^{\frac{r-s}{2}} \end{cases} . \tag{12}$$

satisfies the incentive constraint

$$\int_0^t \left(\pi(x(\tau)) - \pi^N\right) \lambda e^{-\lambda\tau} d\tau \geq d(x(t))e^{-\lambda t} \tag{13}$$

for all $t \in [0, T]$. To see this, first consider the case $t < \hat{\epsilon}$. We have

$$\begin{aligned} \int_0^t \left(\pi(x(\tau)) - \pi^N\right) \lambda e^{-\lambda\tau} d\tau &\geq \int_0^t k' \left(x(\tau) - a^N\right)^s \lambda e^{-\lambda\tau} d\tau = \int_0^t k' \tau^{\frac{2s}{r-s}} \lambda e^{-\lambda\tau} d\tau \\ &\geq k'\lambda e^{-\lambda t} \frac{1}{\frac{2s}{r-s}+1} t^{\frac{2s}{r-s}+1} = \frac{k'\lambda e^{-\lambda t}(r-s)}{s+r} t^{\frac{s+r}{r-s}}. \\ &= k t^{\frac{2r}{r-s}} = k(x(t) - a^N)^r \geq d(x(t))e^{-\lambda t}. \end{aligned}$$

Next, consider the case $t \geq \hat{\epsilon}$. We have

$$\begin{aligned} \int_0^t \left(\pi(x(\tau)) - \pi^N\right) \lambda e^{-\lambda\tau} d\tau &\geq e^{-\lambda\hat{\epsilon}} \left(\pi(x(\hat{\epsilon})) - \pi^N\right) + \int_0^{\hat{\epsilon}} \left(\pi(x(\tau)) - \pi^N\right) \lambda e^{-\lambda\tau} d\tau \\ &\geq d(x(\hat{\epsilon}))e^{-\lambda\hat{\epsilon}} = d(x(t))e^{-\lambda\hat{\epsilon}} \geq d(x(t))e^{-\lambda t}. \end{aligned}$$

Hence, the non-trivial path (12) satisfies the incentive constraint (13) for all $t \in [0, T]$. By definition, on the path of play of the subgame perfect equilibrium characterized by this path, non-Nash profiles are prepared for all $t > 0$. This completes the proof. ■

The proposition says that a nontrivial path exists when *the gain from deviation $d(x)$ converges to zero faster than the value of cooperation $\pi(x) - \pi^N$ does, as $x \to a^N$.* If these conditions are met, we can construct a trigger strategy path. Note that those conditions are satisfied in our example (with $d(a) = a$, $a^N = \pi^N = 0$, and $\pi(a) - \pi^N = \sqrt{a} - a$).

A couple of remarks are in order:

- The intuition behind the above proposition can be expressed as follows. As the deadline comes closer and closer, the probability of punishment upon deviation converges to zero. Hence, to maintain the incentive to follow a nontrivial path, the instantaneous gain from deviation need to be infinitesimal relative to the future gain from cooperation, which roughly corresponds to the static loss from reverting to the Nash equilibrium.

- Remember that this condition fails in Bertrand competition without product differentiation and the aforementioned modified good exchange game with $\epsilon = 0$. Thus nonexistence of cooperative path in those examples are consistent with this proposition.

- The above proposition provides a sufficient condition for the existence of a non-trivial path. A necessary and sufficient condition is the finite time condition presented in Section 4 (see the discussion following condition (8)).

Now let us consider a partial converse of this result. Assume that the payoff for each player i has an upper bound $\bar{\pi}$.

Proposition 11 *Suppose that there exists a unique Nash equilibrium a^N and its payoff $\pi(a^N) = \pi^N$. Suppose that* $\inf_{a\in A} \frac{d(a)}{\pi(a)-\pi^N} > 0$*. Then, there exists a unique trigger strategy equilibrium. In this equilibrium, each player prepares a^N given any history.*

Proof. Let $\inf_{a\in A} \frac{d(a)}{\pi(a)-\pi^N} =: m > 0$. We will show that there exists $\epsilon > 0$ such that for any $t \in [0, T]$, if for all time strictly after $-t$ each player prepares a^N given any history then for all time in $(-t-\epsilon, -t]$, each player prepares a^N given any history in any subgame perfect equilibrium. This gives us the desired result.

So take some $t \in [0, T]$ and suppose that for all time strictly after $-t$ each player prepares a^N given any history. Suppose further that at time $-t-\epsilon$ with $\epsilon > 0$, an action profile a is played on the path of play. Then, by the incentive compatibility constraint, it is necessary that

$$d(a)e^{-\lambda(t+\epsilon)} \le e^{-\lambda t}\int_0^\epsilon (\bar{\pi} - \pi^N)\lambda e^{-\lambda\tau} d\tau.$$

This implies

$$\begin{aligned} d(a)e^{-\lambda\epsilon} \le \lambda(\bar{\pi} - \pi^N)\epsilon \quad &\iff \quad d(a) \le \lambda(\bar{\pi} - \pi^N)\epsilon e^{\lambda\epsilon} \\ &\iff \quad \pi(a) - \pi^N \le \frac{\lambda(\bar{\pi} - \pi^N)\epsilon e^{\lambda\epsilon}}{m} \end{aligned}$$

Hence, again by the incentive compatibility constraint it is necessary that

$$d(a)e^{-\lambda(t+\epsilon)} \le e^{-\lambda t}\int_0^\epsilon \frac{\lambda(\bar{\pi} - \pi^N)\epsilon e^{\lambda\epsilon}}{m}\lambda e^{-\lambda\tau} d\tau.$$

This in turn implies that $d(a) \leq \frac{\lambda^2(\bar{\pi}-\pi^N)(\epsilon e^{\lambda\epsilon})^2}{m}$. Iterating, we have that

$$d(a) \leq \frac{\lambda^n}{m^{n-1}}(\bar{\pi} - \pi^N)(\epsilon e^{\lambda\epsilon})^n \quad \text{for all } n = 1, 2, \ldots.$$

Since the right hand side of this inequality goes to zero as n goes to infinity if $\epsilon < \frac{\lambda e^{\lambda\epsilon}}{m}$, $d(a)$ must be zero if $\epsilon < \frac{\lambda e^{\lambda\epsilon}}{m}$. But this means that a must be a Nash equilibrium a^N. Hence in time interval $(-t-\epsilon, -t]$, each player prepares a^N given any history. This completes the proof. ■

The proof is based on the idea that the right hand side of the incentive compatibility condition is at most some constant times the time left to the deadline. That is, if the time left to the deadline is very short, the instantaneous gain from deviation must be very small relative to the payoff from cooperation (See the first remark after Proposition 10). If the ratio of the gain from deviation to the benefit of cooperation has a strictly positive lower bound then this is impossible when the remaining time is sufficiently small.

6.2 Comparison with Infinite Repeated Games with Decreasing Discount Factors

To compare a revision game with a repeated game, let us employ the standard way to measure time: a revision game is played over $[0, T]$ where 0 is the start of the problem and T is the end. The payoff in the revision game at time t is:

$$e^{-\lambda(T-t)}u(a_t) + \int_t^T e^{-\lambda(T-s)}u(a_s)\lambda ds = e^{-\lambda(T-t)}\left[u(a_t) + \int_t^T e^{\lambda(s-t)}u(a_s)\lambda ds\right].$$

Igroring the constant $e^{-\lambda(T-t)}$, we can regard that a player's objective function at time t (i.e., when a revision opportunity arrives at time t) is equal to

$$u(a_t) + \int_t^T e^{\lambda(s-t)}u(a_s)\lambda ds. \tag{14}$$

This highlights the similarity and difference between a revision game and repeated game with shrinking discount factor (Bernheim and Dasgupta, 1995). The objective function in their model at time t is given by

$$u(a_t) + \sum_{s=t+1}^{\infty} u(a_s) \prod_{\tau=t+1}^{s} \delta(\tau),$$

where the time dependent discount factor $\delta(\tau)$ shrinks over time ($\delta(\tau) \to 0$, as $\tau \to \infty$). One obvious (but minor) difference is that their model is in discrete time while ours is in continuous time. Our continuous time formulation enables us to characterize the optimal path by means of a simple differential equation. To compare their model with ours more closely, let us consider a continuous time version of their model, where the stage game is played according to Poisson arrival time. A continuous time version of their objective function would be

$$u(a_t) + \int_t^{\infty} e^{-\int_t^{\tau} \rho(\tau) d\tau} u(a_s) \lambda ds. \tag{15}$$

where instantaneous discount rate diverges ($\rho(\tau) \to \infty$, as $\tau \to \infty$). This is similar to our model in the sense that as time passes by (when t is large), the impact of future payoffs shrinks. However, note the crucial difference that the weight attached to future payoff $u(a_s)$ in our objective function (14), namely $e^{\lambda(s-t)}$, is *increasing* in s. That is, *a larger weight is attached to future payoff in a revision game.* This is an essential feature - as the deadline comes closer, the probability that the prepared action today is implemented becomes larger. One important implication of this fact is that full cooperation cannot be sustained in a revision game. There is always a positive probability that something very close to the Nash equilibrium (an action prepared near the deadline) is played. In contrast, in a repeated game with shrinking discount factor, payoffs in the distant future do not much affect the average payoff, and the full efficiency can be approximately achieved.

The fact that *a larger* weight is attached to future payoff in a revision game implies that there is no natural way to map our objective function to theirs. For example, one may "stretch" the time in our model to map our time domain $[0, T]$ to $[0, \infty)$ by some increasing function $t' = F(t)$, but such a transformation does not alter the property of our model that the weight attached to $u(a_s)$ is increasing in s.

7 Concluding Remarks

We analyzed a new class of games that we call "revision games," a situation where players in advance prepare their actions in a game. After the initial preparation, they have some opportunities to revise their actions, which arrive stochastically. Prepared actions are assumed to be mutually observable. We showed that players can achieve a certain level of cooperation in such a class of games. Specifically, in the class of component games that we focused on, we showed that an optimal symmetric trigger-strategy equilibrium exists and it is essentially unique. We characterized the equilibrium by a simple differential equation and applied it to analyze a variety of economic examples.

While we are circulating the earlier versions of the present paper, several follow-up papers have been written. Calcagno and Lovo (2010) and Kamada and Sugaya (2010a) consider revision games with finite action space and assume that revision opportunities arrive independently across players (asynchronous revision). In contrast to the present paper, they show that the addition of revision phase sometimes narrows down the set of equilibria when the component game has multiple equilibria. They show that when the component game has a strictly Pareto-dominant Nash equilibrium, it is the only profile that can realize in a corresponding revision game when some regulatory conditions are met.[29] They also show that in battle of the sexes games one of the pure Nash equilibrium is played generically. Kamada and Sugaya (2011b) introduce the first model of dynamic election campaigns into the literature on election by using a variant of the revision games framework. In their model, the revision phase corresponds to an election campaign phase where candidates announce their policies, and the component game corresponds to the standard Hotelling-Downs election game.[30] The rich dynamic structure of revision games enables them to endogenize the order of policy announcements, which are exogenously specified in the literature.[31]

We suggest several possible directions for future research. First, we investigate the case of asynchronous revision in a companion paper (Kamada and Kandori, 2011) and show that cooperation is still possible in such a setting. Second, we used trigger strategy equilibrium to sustain cooperation, in which players revert to Nash actions upon deviation. Although this class of strategies is a natural one worth investigation, a severer punishment might be possible. In our continuation work, we consider severer punishment schemes than Nash

[29]Ishii and Kamada (2011) identify the condition under which this result is generalized to the case of a hybrid version of synchronous and asynchronous revisions. Romm (2011) examines the effect of reputation in a variant of revision games proposed by Kamada and Sugaya (2010a).

[30]In their model a policy announcement at each opportunity is restricted by previous announcements in a particular manner, while in our analysis in Section 5.4 no restriction is imposed.

[31]Other recent papers on variants of revision games include Ambrus and Burns (2011) and Kamada and Muto (2011).

reversion.

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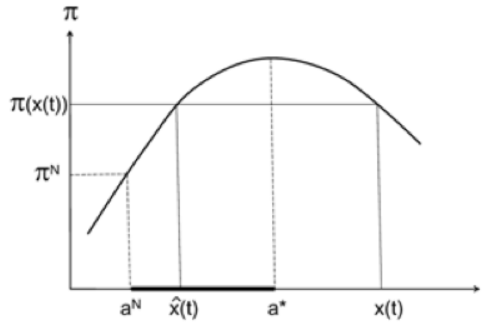

Figure 6: **The graph of $\pi(\cdot)$.**

Appendix A

We provide the proof of Proposition 1 (the existence and differentiability of the optimal path). First, we present a simple but useful lemma. Recall that we are assuming that the optimal action a^* is different form the Nash action a^N (A1). Then consider

$$A^* := \begin{cases} [a^N, a^*] \text{ if } a^N < a^* \\ \\ [a^*, a^N] \text{ if } a^* < a^N \end{cases}$$

The next lemma shows that we can restrict our attention to the trigger strategy equilibria whose action always lies in A^*.

Lemma 1 *For any trigger strategy equilibrium $x \in X^*$, there is a trigger strategy equilibrium $\widehat{x} \in X^*$ such that $\forall t \quad \widehat{x}(t) \in A^*$ and $\pi(\widehat{x}(t)) \geq \pi(x(t))$.*

Proof. We show this for the case of $A^* = [a^N, a^*]$. By assumptions A2 and A5, the graph of π is continuous and "single peaked", and therefore if $\pi(a^N) < \pi(x(t))$ and $x(t) \notin A^*$ then there must be $\widehat{x}(t) \in A^*$ such that $\pi(\widehat{x}(t)) = \pi(x(t))$ and $\widehat{x}(t) < x(t)$ (see Figure 6).

Replace such $x(t)$ by $\widehat{x}(t) \in A^*$ defined above. If $\pi(a^N) \geq \pi(x(t))$, replace $x(t)$ by $\widehat{x}(t) \equiv a^N$. If $x(t) \in A^*$, let $\widehat{x}(t) = x(t)$. Note that $\pi(\widehat{x}(t)) = \max\left\{\pi(x(t)), \pi^N\right\}$ and this is measurable (so that $\widehat{x}$ is feasible). Lastly, we show that $\widehat{x}$ satisfies the incentive constraint (3). Since $\pi(\widehat{x}(t)) \geq \pi(x(t))$, the right hand side of (3) is weakly larger under $\widehat{x}$ for all t. Hence we only need to show $d(\widehat{x}(t)) \leq d(x(t))$ for all t. This is trivially true when $\widehat{x}(t) = a^N$. Otherwise, we have $a^N < \widehat{x}(t) \leq x(t)$. Since $d(a)$ is increasing for $a > a^N$ (by A6), we have $d(\widehat{x}(t)) \leq d(x(t))$. ■

This Lemma shows that the optimal trigger strategy (if any) can be found in the set X^{**} of trigger strategy equilibria whose range is $A^*(= [a^N, a^*]$ or $[a^*, a^N])$:

$$X^{**} := \{x \in X^* | \forall t \; x(t) \in A^*\}.$$

Proposition 12 *There is an optimal trigger strategy equilibrium $\overline{x}(t)$ (i.e., $\overline{x} \in X^*$ and $V(\overline{x}) = \max_{x \in X^*} V(x)$, where V denotes the expected payoff associated with x) which is continuous in t.*

Proof. We show that there is a trigger strategy equilibrium in X^{**} that attains $\max_{x \in X^{**}} V(x)$ (by Lemma 1, it is the true optimal in X^*). We consider the case $a^N < a^*$, so that $x(t) \in A^* = [a^N, a^*]$.

Since $V(x)$ is bounded above by $\pi(a^*) = \max_a \pi(a)$, $\sup_{x \in X^{**}} V(x)$ is a finite number. Hence, by Lemma 1, we can find a sequence x^n, $n = 1, 2, ...$ in X^{**} such that $\lim_{n \to \infty} V(x^n) = \sup_{x \in X^{**}} V(x)$.

Note that $\{\pi(x^n(\cdot))\}_{n=1,2,...}$ is a collection of *countably* many measurable functions. This implies that $\overline{\pi}(t) := \sup_n \pi(x^n(t))(< \infty)$ is also measurable. Now let us define $\overline{x}(t)$ to be the solution to

$$\textbf{Problem P(t)}: \max_{x(t) \in [a^N, a^*]} \pi(x(t))$$

$$\text{s.t. } d(x(t))e^{-\lambda t} \leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{16}$$

Note that the right hand side of the constraint (16) is well-defined, because $\overline{\pi}(\cdot)$ is measurable. Also note that the right hand side is nonnegative by $\overline{\pi}(s) \geq \pi^N$.[32]

Under Assumptions A5 and A6, both $\pi(a)$ and $d(a)$ are increasing on $[a^N, a^*]$. Hence the solution $\overline{x}(t)$ to Problem P(t) is either a^* or the action in $[a^N, a^*)$ with the binding constraint (16) by continuity of d (which follows from A2). Let us write down the solution in the following way. Note first that, by Assumptions A2 and A6, d is continuous and

[32]By A5, $x^n(t) \in [a^N, a^*]$ implies $\pi(x^n(t)) \geq \pi^N$. Hence $\overline{\pi}(t) = \sup_n \pi(x^n(t)) \geq \pi^N$.

strictly increasing on $[a^N, a^*]$, and therefore its continuous inverse d^{-1} exists (if we regard d as a function from $[a^N, a^*]$ to $d([a^N, a^*]) = [0, d(a^*)]$). Then the optimal solution $\overline{x}(t)$ can be expressed as

$$\overline{x}(t) = \begin{cases} a^* & \text{if } d(a^*) < h(t) \\ d^{-1}(h(t)) & \text{otherwise} \end{cases}, \tag{17}$$

where

$$h(t) := e^{\lambda t} \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

A crucial step in the proof is to note that $h(t)$ is continuous in t for any measurable function $\overline{\pi}(\cdot)$.[33] Since d^{-1} is continuous, $\overline{x}(t)$ is continuous whenever $x(t) \in [a^N, a^*)$. Moreover, since $h(t)$ is increasing in t, (17) means that $x(t) = a^*$ implies $x(t') = a^*$ for all $t' > t$. Hence $\bar{x}$ is continuous for all t.

Lastly, we show that $\overline{x}$ is a trigger strategy equilibrium. The continuity of $\overline{x}$ and π implies that $\pi(\overline{x}(\cdot))$ is a measurable function. Therefore, $\overline{x}$ is feasible. We show that $\overline{x}$ also satisfies the (trigger strategy) incentive constraint IC(t) for all t. Recall that x^n is a trigger strategy equilibrium for all $n = 1, 2, ...$ Then we have

$$\begin{aligned} d(x^n(t))e^{-\lambda t} &\leq \int_0^t \left(\pi(x^n(s)) - \pi^N\right) \lambda e^{-\lambda s} ds \quad (x^n \text{ is an equilibrium}) \\ &\leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \qquad (\text{by definition of } \overline{\pi}) \end{aligned}$$

This means that $x^n(t)$ satisfies the constraint of Problem P(t). Since $\overline{x}(t)$ is the solution to Problem P(t), we have

$$\forall n\ \forall t \quad \pi(\overline{x}(t)) \geq \pi(x^n(t)) \tag{18}$$

and therefore

$$\forall t \ \ \pi(\overline{x}(t)) \geq \overline{\pi}(t) = \sup_n \pi(x^n(t)). \tag{19}$$

[33]Note to ourselves (may be omitted): The standard result in measure theory shows that, for any measurable function $f(t)$, the Lebesgue integral $\int_0^t f(s)ds$ is absolutely continuous in t, so it is continuous in t. (See, for example, S. Ito Thm 19.2).

Hence, for all t, $\overline{x}(t)$ satisfies the incentive constraint IC(t):

$$\begin{aligned} d(\overline{x}(t))e^{-\lambda t} &\leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds \quad (\overline{x}(t) \text{ satisfies (16)}) \\ &\leq \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds. \end{aligned}$$

Thus we have shown that $\overline{x}$ is a trigger strategy equilibrium ($\overline{x} \in X^*$), and $V(\overline{x}) \geq V(x^n)$ for all n (by (18)). By definition $\lim_{n\to\infty} V(x^n) = \sup_{x\in X^{**}} V(x)$, and the above inequality implies $V(\overline{x}) \geq \sup_{x\in X^{**}} V(x)$. Since $\overline{x} \in X^{**}$, we must have $V(\overline{x}) = \sup_{x\in X^{**}} V(x) = \max_{x\in X^{**}} V(x) (= \max_{x\in X^{*}} V(x)$ by Lemma 1). Hence we have established that there is an optimal and continuous trigger strategy equilibrium $\overline{x}$. ■

Next, we show that $\overline{x}(t)$ satisfies binding incentive constraint and is differentiable. The continuity of $\overline{x}$ plays a crucial role in the proof.

Proposition 13 *The optimal trigger strategy equilibrium $\overline{x}(t)$ satisfies the binding incentive constraint*

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

if $\overline{x}(t) \neq a^*$, and $\overline{x}(t)$ is differentiable when $\overline{x}(t) \neq a^*, a^N$.

Proof. The proof of Proposition 12 shows that, if $\overline{x}(t) \neq a^*$, then

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds$$

$$\leq \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{20}$$

We now show that the weak inequality above is actually an equality (and therefore we have the binding incentive constraint). If the above inequality were strict for some t, by (19), we would have

$$e^{-\lambda T}\overline{\pi}(T) + \int_0^T \overline{\pi}(s)\lambda e^{-\lambda s} ds < e^{-\lambda T}\pi(\overline{x}(T)) + \int_0^T \pi(\overline{x}(s))\lambda e^{-\lambda s} ds = V(\overline{x}).$$

Since $\overline{\pi}(s) := \sup_n \pi(x^n(t))$, the left hand side is more than or equal to $V(x^n)$ for all n. Since $\lim_{n\to\infty} V(x^n) = \sup_{x\in X^{**}} V(x)$, the above inequality implies $\sup_{x\in X^{**}} V(x) < V(\overline{x})$. This contradicts $\overline{x} \in X^{**}$. Hence (20) should be satisfied with an equality (i.e., $\overline{x}$ satisfies the binding incentive constraint), if $\overline{x}(t) \neq a^*$.

Next we show the differentiability. We continue to consider the case $a^N < a^*$, so that $\overline{x}(t) \in A^* = [a^N, a^*]$. By Assumptions A2 and A6, d is continuous and strictly increasing on $[a^N, a^*]$ and therefore its inverse d^{-1} exists. Hence, if $\overline{x}(t) \neq a^*$, the binding incentive constraint implies

$$\overline{x}(t) = d^{-1}\left(e^{\lambda t}\int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}ds\right).$$

The continuity of $\overline{x}$ implies that $\left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}$ is continuous, and the fundamental theorem of calculus shows that $\int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right)\lambda e^{-\lambda s}ds$ is differentiable with respect to t (with the derivative $\left(\pi(\overline{x}(t)) - \pi^N\right)\lambda e^{-\lambda t}$). Hence the argument of d^{-1} is differentiable with respect to t, and therefore $\overline{x}(t)$ is differentiable whenever $(d^{-1})'$ exists. Note that $(d^{-1})' = 1/d'(\overline{x}(t))$ indeed exists if $\overline{x}(t) \neq a^N$, because d' exists (Lemma 2 in Appendix C) and $d'(\overline{x}(t)) > 0$ (Assumption A6). ■

Appendix B

We provide the proof of Proposition 2 (essential uniqueness of the optimal path):

Proof. Suppose $H := \{t|\pi(y(t)) > \pi(\overline{x}(t))\}$ has a positive measure. Then, define

$$z(t) := \begin{cases} y(t) & \text{if } t \in H \\ \\ \overline{x}(t) & \text{otherwise} \end{cases}.$$

This has a measurable payoff $\pi(z(t)) = \max\{\pi(y(t)), \pi(\overline{x}(t))\}$ and achieves strictly higher expected payoff than $\overline{x}(t)$. Furthermore, z satisfies the incentive constraints

$$\forall t \;\; d(z(t))e^{-\lambda t} \leq \int_0^t \left(\pi(z(s)) - \pi^N\right)\lambda e^{-\lambda s}ds.$$

This follows from the incentive constraints for $\overline{x}$ and y, together with $\pi(z(t)) = \max\{\pi(y(t)), \pi(\overline{x}(t))\}$. Hence, z is a trigger strategy equilibrium path, which achieves a higher payoff than $\overline{x}(t)$ does. This contradicts the optimality of $\overline{x}(t)$, and therefore H must have measure zero. Hence $\pi(y(t)) \leq \pi(\overline{x}(t))$ almost everywhere. If $\{t|\pi(y(t)) < \pi(\overline{x}(t))\}$ has a positive measure, y attains a strictly smaller payoff than $\overline{x}(t)$ does, which contradicts our premise that y is optimal. Therefore we conclude that $\pi(y(t)) = \pi(\overline{x}(t))$ almost everywhere.

Finally we show that $y(t) = \overline{x}(t)$ almost everywhere. Note that π is not monotone and therefore $\pi(y(t)) = \pi(\overline{x}(t))$ may not imply $y(t) = \overline{x}(t)$. Since any trigger strategy equilibrium must play a^N at $t = 0$, suppose $\pi(y(t)) = \pi(\overline{x}(t))$ but $y(t) \neq \overline{x}(t)$, for $t > 0$. This means $y(t) \neq a^*$, so suppose $y(t) \neq a^*$. This will lead to a contradiction.

Consider the case of $a^N \leq a^*$. We must have $a^N < \overline{x}(t) < a^* < y(t)$ (see the graph of π (Figure 6)). Since the incentive constraint is binding when $a^N \leq \overline{x}(t) < a^*$,

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(s)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

This implies that y does not satisfy the incentive constraint, because (i) Assumption A6 and $a^N < \overline{x}(t) < y(t)$ imply $d(\overline{x}(t)) < d(y(t))$, and (ii) $\pi(y(s)) = \pi(\overline{x}(s))$ almost everywhere. This is a contradiction, and therefore $y(t) = \overline{x}(t)$ almost everywhere. ■

Appendix C

We provide auxiliary lemmas to prove Theorem 1. First, we show that d' and d'' exist and are continuous under our assumptions.

Lemma 2 *Under A2-A4, both $d'(x)$ and $d''(x)$ exist and are continuous. In particular,*

$$d'(x) = \frac{\partial \pi_1(BR(x), x)}{\partial x_2} - \frac{\partial \pi_1(x, x)}{\partial x_1} - \frac{\partial \pi_1(x, x)}{\partial x_2}, \tag{21}$$

$$\begin{aligned} d''(x) = &- \left(\frac{\partial^2 \pi_1(BR(x), x)}{\partial x_1 \partial x_2}\right)^2 / \frac{\partial^2 \pi_1(BR(x), x)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(BR(x), x)}{\partial^2 x_2} \\ &- \frac{\partial^2 \pi_1(x, x)}{\partial^2 x_1} - 2\frac{\partial^2 \pi_1(x, x)}{\partial x_1 \partial x_2} - \frac{\partial^2 \pi_1(x, x)}{\partial^2 x_2}, \textit{ and} \end{aligned} \tag{22}$$

$$d''(a^N) = \frac{-\left(\frac{\partial^2 \pi_1(a^N, a^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(a^N, a^N)}{\partial x_1 \partial x_2}\right)^2}{\frac{\partial^2 \pi_1(a^N, a^N)}{\partial^2 x_1}} \tag{23}$$

Proof. We first examine the properties of $BR(x)$. To this end, we apply the implicit function theorem to the first order condition $\frac{\partial \pi_1(BR(x), x)}{\partial x_1} = 0$ (A4). The assumptions of implicit function theorem are satisfied:

- $\frac{\partial^2 \pi_1(BR(x), x)}{\partial^2 x_1} \neq 0$ (by A4) and
- $\frac{\partial \pi_1(x_1, x_2)}{\partial x_1}$ is continuously differentiable (A2).

Hence $BR(x)$ is a continuously differentiable function (and therefore also continuous), with

$$BR'(x) = -\frac{\partial^2 \pi_1(BR(x), x)}{\partial x_1 \partial x_2} / \frac{\partial^2 \pi_1(BR(x), x)}{\partial^2 x_1},$$

and it is finite. Given this, differentiating $d(x) := \pi_1(BR(x), x) - \pi_1(x, x)$ and using the first order condition $\frac{\partial \pi_1(BR(x),x)}{\partial x_1} = 0$ (A4), we obtain (21). Differentiating this once again and using the above formula for $BR'(x)$, we obtain (22). By the twice continuous differentiability of π_1 (A2), $\frac{\partial^2 \pi_1(BR(x),x)}{\partial^2 x_1} \neq 0$ (by A4), and the continuity of $BR(x)$, both d' and d'' are continuous. Lastly, (23) is obtained from (22), by noting that $BR(x) = x$ when x is equal to the Nash action a^N. ■

Next we show $f(x) := \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$, which defies the differential equation $dx/dt = f$, is continuously differentiable.

Lemma 3 *Function* $f(x) := \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$ *is continuously differentiable for* $x \neq a^N$.

Proof. Note that $d'(x) \neq 0$ if $x \neq a^N$ (A6). Then, $f' = \frac{\lambda((d'+\pi')d' - \left(d+\pi-\pi^N\right)d'')}{(d')^2}$ is a continuous function, by Lemma 2. ■

We now examine the behavior of $dx/dt = f(x)$ when x is close to a^N. In particular, we evaluate $f^N := \lim_{x \to a^N} \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$.

Lemma 4

$$\begin{aligned} f^N \quad &: = \lim_{x \to a^N} \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} \\ &= \quad -\lambda\, \pi'(a^N) / \frac{\left(\frac{\partial^2 \pi_1(a^N, a^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(a^N, a^N)}{\partial x_1 \partial x_2}\right)^2}{\frac{\partial^2 \pi_1(a^N, a^N)}{\partial^2 x_1}}. \end{aligned}$$

Under Assumptions A1-A6, f^N *is always non-zero, and* $f^N = \infty$ *or* $-\infty$ *if and only if* $\frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1 \partial x_2} = 0$.

Proof. By de l'Hopital rule,

$$\lim_{x \to a^N} \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} = \frac{\lambda \pi'(a^N)}{d''(a^N)}$$

where we used $d'(a^N) = 0$ (A6). Then the expression of the lemma directly follows from (23) in Lemma 2. The numerator is non-zero, because $\pi' = 0$ only at the optimal action

a^* (A5). By the second order condition at the Nash equilibrium (A4), $\frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1} < 0$. Hence, $f^N \neq 0$ in general, and $f^N = \infty$ or $-\infty$ if and only if $\frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1 \partial x_2} = 0$. ■

Remark 1 *The condition for the finiteness of f^N, $\frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1 \partial x_2} \neq 0$ is equivalent to $BR'(a^N) \neq 1$. This follows from the implicit function theorem $BR' = -\frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} / \frac{\partial^2 \pi_1}{\partial^2 x_1}$.*

Finally we show that the finite time condition (8) is satisfied under our assumptions. Recall that we are looking at the case where $a^N < a^*$.

Lemma 5 *For any $x^0 \in (a^N, a^*]$, $t(x^0) := \lim_{a \to a^N} \int_a^{x^0} \frac{1}{f(x)} dx < \infty$.*

Proof. Recall
$$\frac{1}{f} = \frac{d'(x)}{\lambda \left(d(x) + \pi(x) - \pi^N\right)}$$
and it is finite when $x \in (a^N, a^*)$ because the numerator is finite by Lemma 2 and the denominator is nonzero by A5 and A6. Note that $1/f(x)$ is not defined for $x = a^N$ (both the numerator and denominator of the right hand side is zero at $x = a^N$). By Lemma 4, we have
$$\lim_{a \to a^N} \frac{1}{f(a)} = \frac{1}{f^N} = \frac{-\left(\frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(a^N,a^N)}{\partial x_1 \partial x_2}\right)^2}{\lambda \pi'(a^N) \frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1}},$$
By Assumption A5 (and $a^N < a^*$), we have $\pi'(a^N) > 0$, and also $\frac{\partial^2 \pi_1(a^N,a^N)}{\partial^2 x_1} < 0$ by A4. Also the numerator is finite by A2. Therefore $\lim_{a \to a^N} \frac{1}{f(a)}$ is a finite number. Hence, $t(x^0) := \lim_{a \to a^N} \int_a^{x^0} \frac{1}{f(x)} dx$ is a finite number. ■