Revision Games[*]

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Abstract

This paper proposes a class of games called *revision games*. In a revision game, players start with initially prepared actions, followed by a sequence of random revision opportunities until a predetermined deadline. In the course of revisions, players monitor each other's behavior. It is shown that players can cooperate and that their behavior under the optimal equilibrium is described by a simple differential equation. We present the necessary and sufficient conditions for cooperation to be sustained in revision games. We also present applications to the preopening activities in the stock exchange and to an electoral campaign.

1   Introduction

We show that cooperation can be sustained when players prepare and revise their strategies before playing a game. Specifically, we propose a class of stylized models, which we call *revision games*, to formulate and analyze such a situation. A revision game starts at time $-T$ and ends at time 0 (time is continuous). Players prepare actions at the beginning, and then they obtain revision opportunities according to a Poisson process. When a revision opportunity arises, players simultaneously revise their actions.[1] The initial and revised actions are mutually observable, and the actions chosen at the last revision opportunity are played at time 0. We show that, under a certain set of conditions, players can cooperate by using a version of the trigger strategy defined over the revision process. The optimal revision plan has a tractable characterization: it is given by a simple differential equation.

The revision game represents a situation where common random revision opportunities arrive before the "deadline" to play a game. For example, in the stock exchange markets, traders prepare and revise their orders before the opening of the market. The prepared orders are observable on the public screen, which is refreshed frequently. Each refreshment of the screen can be regarded as a random revision opportunity (see Section 4.1).[2] In an electoral campaign, an effective revision opportunity of candidates' policies is likely to be tied to the arrival of a random event that triggers (i) debates within each party, (ii) voters' attention, and (iii) voters' willingness to accept policy changes. In the 2017 Korean presidential election campaign, the two candidates announced and revised their national defense policies after the arrival of important political news (see Section 4.2 for the detail).

Let us illustrate how the revision game works when applied to the familiar game of Cournot duopoly.[3] One can imagine that two fishing boats, operating side by side, dynamically adjust their catch until the fish market opens. Random revision opportunities correspond to the arrivals of fish schools. Players' behavior in the revision game is represented by a *revision plan* $q(t)$. When a revision opportunity arrives at time $-t$, they are supposed to adjust their quantity to $q(t)$. The *trigger strategy* (in revision games) stipulates that, if anyone fails to follow the revision plan, players

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[1]In Section 6, we analyze the case where revision opportunities are player-specific.

[2]Calcagno and Lovo (2010) were the first to point out that the "preopening phase" of stock exchange is a good example of the situation where players revise their strategies before the deadline.

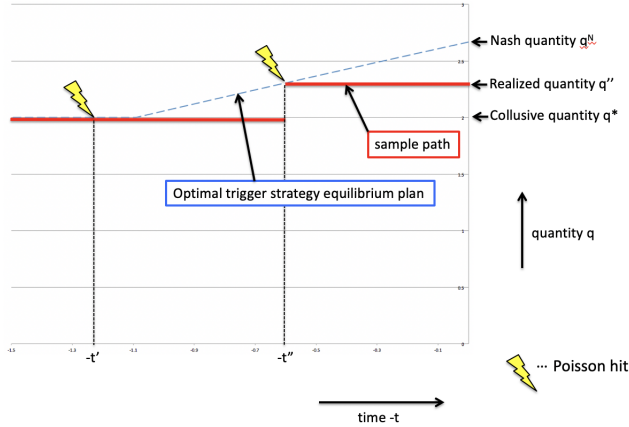[3]See Example 2 in Section 3 for the detail.

Figure 1: The optimal trigger strategy equilibrium plan and a sample path for the revision game of Cournot duopoly. Arrival rate $\lambda = 1$.

choose the static Nash equilibrium quantity at all *future* revision opportunities. We will show that the optimal revision plan supported by the trigger strategy is given by a simple differential equation

$$\frac{dq}{dt} = \frac{\lambda \left(d(q) + \pi(q) - \pi^N \right)}{d'(q)},$$

where $d(q)$ and $\pi(q)$ are the gain from deviation and the payoff when each player chooses quantity q, π^N is the Nash equilibrium payoff, and λ is the Poisson arrival rate. A closed-form solution $q(t)$ is obtained and it is depicted as a dashed curve of Figure 1.[4]

This optimal trigger strategy equilibrium plan starts at the collusive quantity and then follows the differential equation. Finally, it reaches the Nash equilibrium quantity at the end of the revision game. In contrast, the bold segments depict a typical *realized path* of quantity. Whenever a revision opportunity arrives, the prepared quantity is adjusted according to the revision plan. At time $-t'$, the first revision opportunity arrives, but players do not revise their quantity. Closer to the deadline at time $-t''$, a revision opportunity comes again, and the players increase their quantity to q''. Then, they encounter no more revision opportunities, and the amount shipped into the market is q'' (thus, the final outcome is random and determined by when the last revision opportunity arrives).

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[4]See equation (8) in Section 3.

Why can we sustain cooperation in revision games? When a revision opportunity arrives near the end of a revision game, there is still a small but positive probability that another revision is possible in the remaining time. This means that, if a player cheats at the current revision opportunity, she has some (small) probability of being punished in the remaining time. Hence, players can cooperate a little near the end of the game. Using this as a foothold, players can cooperate more when a revision opportunity arrives earlier. By a repeated application of this mechanism, players can cooperate substantially when they are far away from the end of the revision game. This logic does not always hold, however, and in some games no cooperation is possible. We provide necessary and sufficient conditions for cooperation to be sustained in revision games.

1.1   Related literature

Several papers examine revision processes that are different from ours. Kalai (1981) and Bhaskar (1989) examine models where players *can always react* to their opponents' revisions, and show that a unique and efficient outcome is obtained. In contrast, players in our revision games may not obtain a chance to react, and thus full cooperation cannot be obtained. In Caruana and Einav (2008a,b), a revision is possible at *any moment of time* with some *switching costs*[5] Thus, players have an incentive to act like the Stackelberg leader, using the switching cost as a commitment device. Their second paper (2008b) considers the Cournot duopoly and shows that, in contrast to our model, the firms end up producing *more* than the Nash quantity in their equilibrium. Hence, unlike in our model, the outcome is *less* cooperative than the static Nash equilibrium. Nishihara (1997) considers a model in which uncertainty about the order of moves induces the possibility of cooperation. The domain to which his cooperation mechanism applies is different from ours because his mechanism relies on a specific signal structure in which only defection is revealed to the next mover, while we assume perfect information.

Some existing work examine random revision opportunities. Vives (1995) and Medrano and Vives (2001) present infinite-horizon discrete-time models, where in each period n, with probability γ_n, the stage game payoff in that period is realized and the game ends immediately. The probability γ_n is nondecreasing in n. Those

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[5]See also Iijima and Kasahara (2016).

papers consider a continuum of agents (and a single large agent in Medrano and Vives (2001)), and therefore there does not exist the kind of strategic interaction that is the main focus of our paper. Ambrus and Lu (2015) analyze a multilateral bargaining problem in a continuous-time finite-horizon setting where opportunities of proposals arrive via Poisson processes. Although their model is similar to ours, they focus on (the unique) Markov perfect equilibrium, which corresponds, in our model, to the trivial equilibrium where the Nash equilibrium action is always played. Ambrus, Burns and Ishii (2014) analyze a model of eBay-style auctions in which bidders have chances to submit their bids at Poisson opportunities before the deadline.[6] They show that there are equilibria in which bidders gradually increase their bids. Their equilibria are built on the existence of multiple best replies in the eBay-style auction, and the mechanism behind the revision behavior in their model is different from ours.

Revision games are similar to repeated games, in the sense that the possibility of future punishment provides incentives to cooperate.[7] We elaborate on this point in Section 7.2. In this respect, the paper by Chou and Geanakoplos (1988) is the most closely related to ours. They consider T-period repeated games in which a player can commit to a ε-optimal action in the final period and prove that, if the stage game has a "smooth" payoff function, the folk theorem obtains as $T \to \infty$. That is, a little bit of commitment at the last period provides a basis for substantial cooperation in the entire game. Our paper shows that a similar mechanism can operate in revision processes, without the need for suboptimal behavior at the deadline.

Finally, various follow-up papers have been written while we were circulating earlier versions of the present paper. Given the growing volume of such research, here we provide a brief summary of recent work. First, a group of papers are concerned with uniqueness of equilibrium (and comparative statics on such a unique equilibrium) in revision games where (i) revision opportunities are player-specific and (ii) there are finitely many actions. Those papers include: Calcagno, Kamada, Lovo and Sugaya (2014), Ishii and Kamada (2011), Romm (2014), Kamada and Sugaya (2019), and Gensbittel, Lovo, Renault and Tomala (2018). Second, some other papers attempt to obtain general properties of revision games. For example, Moroni (2015) and Lovo and

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[6]See also Hopenhayn and Saeedi (2016).

[7]Such punishment for deviations can also be carried out in the form of commitment in a simultaneous-move game, for example using price-matching guarantees (e.g., Edlin (1997)) or the use of computer programs (e.g., Tennenholtz 2004). In our dynamic setting, an action at a given time cannot condition on the future actions, and this feature is absent in these papers.

Tomala (2015) show the existence of equilibria in their respective generalizations of revision games. Finally, Roy (2017) conducts laboratory experiments that are related to our quantity revision game.[8] The paper shows that the experimental results exhibit some important features of the trigger-strategy equilibrium that we identify in the present paper.

2 Revision Games

We consider a normal-form game with two players $i = 1, 2$.[9] Player i's action and payoff are denoted by $a_i \in A_i$ and $\pi_i(a_1, a_2)$. Players prepare their actions in advance, and they also have stochastic revision opportunities. Formally, time is continuous, $-t \in [-T, 0]$ with $T > 0$. At the beginning (at time $-T$), players simultaneously choose their actions. Then, revision opportunities arrive according to Poisson process(es). Players can revise their actions costlessly, and they observe each other's initial and revised actions. At time 0, which we call the *deadline*, player i obtains payoff $\pi_i(a'_1, a'_2)$, where a'_j is j's action chosen in the last revision opportunity. There are two specifications of the revision process:

1. *Synchronous case*: There is a single Poisson process with arrival rate $\lambda > 0$. At each arrival, players simultaneously choose their revised actions.

2. *Asynchronous case*: For each player, there is a Poisson process with arrival rate $\lambda_i > 0$ (independent across players). At each arrival for player i, she/he chooses a revised action.

We mainly focus on the synchronous case. In Section 6, however, we show that our main results for the synchronous case continue to hold for the asynchronous case when the payoff function satisfies a certain condition. The entire dynamic game of revision process is referred to as the *revision game*, while the game $\pi_i(a_1, a_2)$ is referred to as its *component game*.[10]

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[8]Avoyan and Ramos (2018) also conduct experiments on revision games.

[9]Our analysis can be extended to the N-player case.

[10]The assumption that revisions can be made even at times very close to the deadline may not always be realistic. Section 7.1 discusses a modification of the revision game by introducing willingness to punish a deviator and/or a soft deadline. For the main part of the paper, however, we focus on our simple setup presented in the current section.

3　The Optimal Trigger Strategy Equilibrium

We analyze a symmetric revision-game equilibrium that uses a trigger strategy defined below. Consider a symmetric component game, where $A_1 = A_2 =: A$ and $\pi_1(a, a') = \pi_2(a', a)$ for all $a, a' \in A$. The action space A is a convex subset (interval) in $\mathbb{R}$. In what follows, we present the main logic to characterize the optimal trigger strategy equilibrium, while all technical propositions are relegated to Appendix A.

We assume the following properties, where $\pi(a) := \pi_1(a, a) = \pi_2(a, a)$.

- **A1:** There exist a unique pure symmetric Nash equilibrium action a^N with payoff $\pi^N := \pi(a^N)$ and a unique best symmetric action a^* that maximizes $\pi(\cdot)$. The Nash and the best actions are distinct.

- **A2:** If $a^N < a^*$, $\pi(a)$ is strictly increasing for $a < a^*$ (a symmetric condition holds if $a^* < a^N$).

- **A3:** $\pi_1(a_1, a_2)$ is continuous, and $\max_{a_1} \pi_1(a_1, a_2)$ exists for all a_2 so that we can define the (maximum) *gain from deviation* at a symmetric profile (a, a) by

$$d(a) := \max_{a_1} \pi_1(a_1, a) - \pi_1(a, a). \tag{1}$$

- **A4:** If $a^N < a^*$, $d(a)$ is strictly increasing on $[a^N, a^*]$ and non-decreasing for $a^* < a$ (symmetric conditions hold if $a^* < a^N$).

Assumptions A1 and A3 are mostly innocuous technical assumptions (the uniqueness of Nash equilibrium makes it harder to sustain cooperation in our setting). The requirement that actions are continuous variables, however, is crucial. We discuss this issue in Section 5.2. A2 and A4 are monotonicity conditions that simplify our analysis. Assumption A2 requires that the symmetric payoff $\pi(a)$ monotonically decreases as we move away from the optimal action a^* (in the relevant region for our analysis). Assumption A4 says that the gain from deviation monotonically increases as we move away from the Nash equilibrium (again in the relevant region for our analysis).

At this moment we define the term "cooperation" (or "collusion"). We say that action a is cooperative (collusive) or achieves (some degree of) cooperation (collusion) if it provides a higher payoff than the Nash equilibrium: $\pi(a) = \pi_i(a, a) > \pi^N$.

A symmetric **trigger strategy** is characterized by its **revision plan** $x : [0, T] \to A$. Players start with the initial action $x(T)$, and when a revision opportunity arrives

at time $-t$, they choose action $x(t)$. Note that t, the argument of revision plan $x(t)$, refers to the *remaining time* in the revision game which can be regarded as the payoff-relevant state variable. If any player fails to choose $x(t)$ when a revision opportunity arrives at $-t$, both players choose the Nash action a^N in all future revision opportunities.

We would like to remind the reader that the plan $x(\cdot)$ should not be confused with the path of revised actions. As Figure 1 in the Introduction shows, even when the graph of plan $x(\cdot)$ is a smooth curve, the realized path of revised actions is discontinuous and piecewise constant.

We will characterize the optimal trigger strategy equilibrium, which we formally define as follows: First, in order to guarantee that the expected payoffs are well defined, we restrict attention to trigger strategies with plan $x(\cdot)$ in the following *set of feasible plans*:

$$X := \{x : [0, T] \to A \mid \pi \circ x \text{ is measurable}\}.$$

Then, the expected payoff (to each player) associated with plan $x(\cdot) \in X$ can be written as:

$$V(x) := \pi(x(T))e^{-\lambda T} + \int_0^T \pi(x(t))\lambda e^{-\lambda t}dt. \tag{2}$$

The coefficient of $\pi(x(T))$, $e^{-\lambda T}$, is the Poisson probability that no revision opportunity arises in the future. In that case, the initial action $x(T)$ is implemented. To interpret the integral in (2), note that λ is a density of a Poisson arrival at any moment of time and $e^{-\lambda t}$ is the probability of no Poisson arrival in the remaining time t. Therefore, $\lambda e^{-\lambda t}$ is *the density of the last revision opportunity*. Overall, the integral represents the expected payoff when the last revision opportunity arrives in $(-T, 0]$.

The *incentive constraint* at time t for the trigger strategy with plan $x(t)$ to be a subgame-perfect equilibrium is

$$(\text{IC}(t)): \underbrace{d(x(t))e^{-\lambda t}}_{\text{deviation gain}} \leq \underbrace{\int_0^t \left(\pi(x(s)) - \pi^N\right)\lambda e^{-\lambda s}ds}_{\text{punishment}}. \tag{3}$$

The left hand side is interpreted as follows: With probability $e^{-\lambda t}$, there is no revision opportunity in the remaining time t, and with this probability the deviation gain $d(x(t))$ (defined by (1)) materializes. The right hand side shows that the payoff

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is decreased under the trigger strategy, when a revision opportunity arrives in the remaining time. In particular, when s is the last revision opportunity in the future, the realized payoff is decreased from $\pi(x(s))$ to the Nash payoff π^N. As we explained above, the density of the last revision opportunity is $\lambda e^{-\lambda s}$, and therefore the integral in (3) represents the expected loss. In summary, the incentive constraint (3) is the condition for subgame-perfect equilibrium: it shows that, at any time $-t$, a player cannot increase her payoff by deviating now and possibly obtaining the equilibrium payoff in the subgames after the deviation (= the Nash payoff).[11] Note that (3) implies an obvious requirement $x(0) = a^N$ (if a revision opportunity arrives at the deadline, players should choose the Nash action).

The set of *trigger strategy equilibrium plans* is formally defined as

$$X^* := \{x \in X \mid \text{IC(t) holds for all } t \in [0,T]\}.$$

A plan that achieves the highest ex ante expected payoff within X^* is referred to as an *optimal trigger strategy equilibrium plan*.[12]

Proposition 5 in Appendix A shows that under assumptions A1-A4, (i) the maximum trigger strategy equilibrium payoff exists and (ii) it is achieved by plan $x(\cdot)$ that is continuous and satisfies the incentive constraint (3) with equality as long as $x(t) \neq a^*$.[13] This plan, which we call the *optimal plan*, therefore satisfies the following binding incentive constraint if $x(t) \neq a^*$.

$$\underbrace{d(x(t))e^{-\lambda t}}_{\text{deviation gain}} = \underbrace{\int_0^t \left(\pi(x(s)) - \pi^N\right) \lambda e^{-\lambda s} ds}_{\text{punishment}}. \tag{4}$$

The binding incentive constraint (4) reveals the following central mechanism to sustain cooperation in the revision game. Near the deadline, the punishment in the future revision opportunities happens with a very small probability. Therefore, not

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[11]Under the trigger strategies, it is obviously a mutual best reply to play the stage game Nash action after a deviation. Given this, checking one-time deviation on the equilibrium path of play is sufficient to verify subgame perfection.

[12]An optimal trigger strategy equilibrium plan achieves the highest ex ante payoff across all subgame-perfect equilibria if $\pi(a^N)$ is the minimax payoff of game π. This includes the case when π is additively separable across two players' actions.

[13]To be more precise, there are multiple plans that achieve the maximum trigger strategy equilibrium payoff, but they are essentially the same in the sense that they differ only on a measure zero set in $(0,T)$ (Proposition 6 in Appendix A).

much cooperation is sustained. This is captured by the condition (4) for t close to 0: the right hand side is close to zero and so is the deviation gain $d(x(t))$. Hence, when t is small, $x(t)$ must be close to the Nash action, where the deviation gain is small. As the time to the deadline t increases, players have more opportunities to punish (the right hand side of (4) increases) and therefore more cooperation can be sustained. This effect can be shown by differentiating the both sides of (4).

Actually, with the following additional assumption, we can formally show that the optimal plan is differentiable (Proposition 7 in Appendix A).

- **A5:** On $(a^N, a^*]$, the gain from deviation d is differentiable and $d' > 0$ if $a^N < a^*$ (a symmetric condition holds if $a^* < a^N$).

By differentiating both sides of the binding incentive constraint (4), we obtain

$$\left(d'(x(t)) \frac{dx}{dt} - \lambda d(x) \right) e^{-\lambda t} = \left(\pi(x(t)) - \pi^N \right) \lambda e^{-\lambda t}. \tag{5}$$

This is interpreted as follows. If we marginally increase the remaining time from t to $t + dt$, the magnitude of future punishment (the right hand side of the binding incentive constraint (4)) increases, where the increment is equal to the right hand side of (5) times dt. This stronger punishment sustains a more cooperative action at time $-(t + dt)$, which is captured by $\frac{dx}{dt}$ on the left hand side of (5).[14] The other term on the left hand side, $-\lambda d(x)e^{-\lambda t}$, reflects the fact that a deviation becomes less profitable as the time to the deadline t increases. This is because the probability that the current deviation gain materializes (= the probability of no revision in the remaining time) decreases. This effect, which reduces the temptation to deviate, also helps to sustain a more cooperative action at $t + dt$.

Equation (5) can be rearranged to obtain the differential equation

$$\frac{dx}{dt} = \frac{\lambda \left(d(x) + \pi(x) - \pi^N \right)}{d'(x)}, \tag{6}$$

as long as $d'(x) \neq 0$. The optimal plan is a solution to this differential equation that

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[14]When increasing the action (from the Nash action) increases the total payoff, the optimal plan satisfies $x(t) > a^N$ and therefore $d'(x(t)) > 0$ for $t > 0$ (by Assumption A5). This and condition (5) imply $dx/dt > 0$ (players can increase their action as the remaining time increases). A symmetric argument applies when decreasing the action (from the Nash action) increases the total playoff $(d'(x(t)) < 0$ and $dx/dt < 0)$.

satisfies $x(0) = a^N$. However, there is an important caveat to solve the differential equation with $x(0) = a^N$ as the boundary condition:

Remark 1. The differential equation may not be defined at the Nash action $x = a^N$. This happens, for example, when the deviation gain d is a smooth function and a^N is in an interior point in the action space A. Since the deviation gain is minimized at a^N, in such a case $d'(a^N) = 0$ (the first order condition of the minimization) and therefore the denominator of the right hand side of (6) would be zero. $\quad\square$

Because of this fact, our formal statements are a bit convoluted. Let us now formally summarize our findings by the following theorem. In this section and Section 3.1, we focus on the case where $a^N < a^*$. Symmetric statements apply to the case where $a^* < a^N$.

Theorem 1. *Under Assumptions A1 - A5, there is a plan $\bar{x}(\cdot)$ (the optimal plan) that achieves the maximum trigger strategy equilibrium payoff. It is continuous in t with $\bar{x}(0) = a^N$ and satisfies the differential equation $\frac{d\bar{x}}{dt} = f(\bar{x}(t))$ if $\bar{x}(t) \neq a^*$ and $d'(\bar{x}(t)) \neq 0$, where*

$$f(x) := \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)}.$$

Moreover, $\bar{x}(t) \in [a^N, a^*]$ for all $t \in [0, T]$ and $f(x) > 0$ on (a^N, a^*).

Proof. This follows from Propositions 5 and 7 in Appendix A. $\quad\square$

Next, we derive a necessary and sufficient condition for the optimal plan to sustain (some degree of) cooperation (i.e., $\bar{x}(t) \in (a^N, a^*]$ for some t).[15] The following example shows that sometimes the optimal plan is identically equal to the Nash action ($\bar{x}(t) = a^N$ for all t) so that no cooperation is sustained by trigger strategies in the revision game.

Example 1 (Linear Exchange Game[16]). Two players $i = 1, 2$ exchange goods. Player i chooses a quantity (or quality) $a_i \in [0, 1]$ of the goods she provides to the other player. The cost of effort of player i is equal to $a_i/2$. In total, player i's payoff is equal to $\pi_i = a_{-i} - a_i/2$.

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[15]In fact, under our necessary and sufficient condition, a stronger property $\bar{x}(t) \in (a^N, a^*]$ for all $t > 0$ holds.

[16]This is also known as the linear public good provision game.

The Nash action is no effort $a^N = 0$, the optimal action is $a^* = 1$, the symmetric payoff is $\pi(a) = a - a/2 = a/2$, and the gain from deviation is $d(a) = a/2$ (saving the cost of effort $a/2$). This example satisfies A1 - A5, but the optimal plan is the trivial one that always plays the Nash action. The differential equation for the optimal plan is

$$dx/dt = \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)} = 2x,$$

which has a *unique* solution $x(t) \equiv 0(= a^N)$ under the boundary condition $x(0) = a^N$.[17] □

Note that the differential equation was derived from the binding incentive constraint (4), and it always has a trivial solution $x(t) \equiv a^N$. In the above example, this was the only solution. Our remaining task is to find out the condition under which the binding incentive constraint (and the associated differential equation) admits another solution that achieves cooperative actions. To this end, we introduce an innocuous technical assumption.

- **A6:** $f(x) := \frac{\lambda\left(d(x)+\pi(x)-\pi^N\right)}{d'(x)}$ is Lipschitz continuous on $[a^N + \varepsilon, a^*]$ for any $\varepsilon \in (0, a^* - a^N]$, if $a^N < a^*$ (a symmetric requirement holds if $a^* < a^N$).

Lipschitz continuity of f, which says that the slope of f is bounded, guarantees the uniqueness of a solution to the differential equation. Recall, however, that we may not use the Nash action as the boundary condition to solve the differential equation (Remark 1). A6 implies a unique solution when the boundary condition is given by a *non*-Nash action in $(a^N, a^*]$.

When the optimal plan sustains a cooperative action ($\bar{x}(t') = a' \in (a^N, a^*]$ for some $t' > 0$), Assumption A6 guarantees a unique solution to the differential equation with boundary condition $\bar{x}(t') = a'$, and the optimal plan must satisfy such a differential equation by Theorem 1. The unique solution travels from this action a' to the optimal action a^* in finite time because $dx/dt = f(x)$ is positive and can be shown to be bounded away from 0 for $x \in [a', a^*)$ under our assumptions. From action a', the solution (the optimal plan) must approach the Nash action a^N in the remaining time t' because $\bar{x}(0) = a^N$. Hence, when the optimal plan sustains cooperation, the following condition must be satisfied:

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[17]Note that the Nash action 0 is a boundary point in the action space and $d'(0) = 1/2 \neq 0$, so that the differential equation is defined at the Nash action (cf. Remark 1). Hence the optimal plan must be a solution to the differential equation with the boundary condition $x(0) = a^N = 0$.

- Finite Time Condition (FTC): The solution to $dx/dt = f(x)$ travels from optimal action a^* to Nash action a^N (as t decreases) in a finite amount of time.[18]

Conversely, if the above condition is satisfied, the solution that travels from a^* to a^N in a finite time corresponds to an equilibrium plan because it satisfies the binding incentive constraint. Therefore, the Finite Time Condition is the necessary and sufficient condition for the optimal trigger strategy to sustain cooperation. Lastly, note that $\int_a^{a^*} \frac{1}{f(x)} dx = \int_a^{a^*} \frac{dt}{dx} dx$ represents the time to reach a^* from a (as t increases), following the solution to the differential equation. Hence, FTC is represented as follows:

$$\text{(FTC: Finite Time Condition)} \quad \lim_{a \downarrow a^N} \int_a^{a^*} \frac{1}{f(x)} dx < \infty. \tag{7}$$

Our argument is formally summarized as follows:

Theorem 2. *Under Assumptions A1-A6, the optimal trigger strategy equilibrium plan $\overline{x}(\cdot)$ sustains cooperation (i.e., $\overline{x}(t) \in (a^N, a^*]$ for some t) if and only if the Finite Time Condition (7) is satisfied.*

Proof. See Appendix A. □

The preceding theorems and some minor additional analysis completely characterize the optimal plan as follows. Recall that we are focusing on the case $a^N < a^*$.

Corollary 1 (**Summary of the Main Results**). *Under Assumptions A1 - A6, cooperation is sustained by the trigger strategy, if and only if the Finite Time Condition (7) is satisfied. Under this condition, the optimal plan $\overline{x}(\cdot)$ exists, and it is a unique plan that has the following properties: (i) it is continuous in t and departs a^N at $t = 0$, (ii) for $t > 0$, it solves the differential equation $\frac{d\overline{x}}{dt} = f(\overline{x}(t))$, where*

$$f(x) := \frac{\lambda \left(d(x) + \pi(x) - \pi^N \right)}{d'(x)} > 0$$

until $\overline{x}(t)$ hits the optimal action a^*, and (iii) if $\overline{x}(t)$ hits the optimal action a^* it stays there.[19] Furthermore, if the time horizon T is large enough, the optimal plan always

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[18]Formally, $\exists t^0 > 0$ such that the solution $x(\cdot)$ to $dx/dt = f(x)$ with terminal condition $x(t^0) = a^*$ satisfies $\lim_{t \downarrow 0} x(t) = a^N$ (the solution travels from a^* to a^N in finite time t^0). The limit is taken because the differential equation may not be defined at the Nash action a^N, as Remark 1 shows.

[19]That is, $\overline{x}(t') = a^*$ for some $t' \leq T$ implies $\overline{x}(t'') = a^*$ for all $t'' \in [t', T]$.

hits the optimal action a^* when the remaining time is

$$t(a^*) := \lim_{a \downarrow a^N} \int_a^{a^*} \frac{1}{f(x)} dx.$$

Proof. See Appendix A. □

The unique optimal plan has the following features: If the time to deadline is long enough, players start with the best action a^*, and they do not revise their actions until the remaining time reaches $t(a^*)$. After that, if a revision opportunity arrives, they choose an action $\bar{x}(t)$, which is closer to the Nash action. The closer the revision opportunity $-t$ is to the deadline 0, the closer the revised action $\bar{x}(t)$ is to the Nash equilibrium. We illustrate the corollary for the familiar game of Cournot duopoly.

Example 2 (**Cournot duopoly**). Consider firms $i = 1, 2$ with constant and identical marginal cost $c > 0$, and a linear demand curve $P = a - bQ$ with $a > c$ and $b > 0$. The component game payoff function for firm i is $\pi_i = (a - b(q_i + q_{-i}) - c) q_i$, where q_i is i's quantity. This game satisfies assumptions A1-A6. Hence, Corollary 1 implies that the optimal trigger strategy equilibrium plan $x(\cdot)$ departs from the Cournot Nash quantity $\frac{a-c}{3b}$, and solves the following differential equation:[20]

$$\frac{dx}{dt} = \frac{\lambda \left(d(x) + \pi(x) - \pi^N \right)}{d'(x)} = \frac{\lambda}{18} \left(x - 5\frac{a-c}{3b} \right)$$

for t such that $x(t) \in \left(\frac{a-c}{4b}, \frac{a-c}{3b} \right)$ where $\frac{a-c}{4b}$ is the optimal quantity. This differential equation has a simple solution:

$$\bar{x}(t) = \frac{a-c}{3b}(5 - 4e^{\frac{\lambda}{18}t}). \tag{8}$$

For all $t \geq t(\frac{a-c}{4b}) = \frac{18}{\lambda} \ln \left(\frac{17}{16} \right)$, $x(t)$ stays at the optimal quantity, $\frac{a-c}{4b}$. Thus, the optimal plan is non-decreasing over time, starting with a small collusive quantity and gradually increasing towards the Nash quantity (recall that t refers to the remaining time in the revision game, so it refers to time $-t$). Figure 1 in the Introduction shows the shape of the optimal plan and a realized path. The Supplementary Information on the authors' websites offers a possible story for the revision game of Cournot quantity

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[20]This follows from $d(q) = \frac{(a-c-3bq)^2}{4b}$, $\pi(q) = (a - c - 2bq)q$, and $\pi^N = \frac{(a-c)^2}{9b}$.

competition as well as the calculation of the expected payoff from the optimal trigger strategy equilibrium. □

One can also show that suboptimal cooperation can also be sustained. Specifically, for any expected payoff $\pi \in [\pi^N, \check{\pi}]$ where $\check{\pi}$ is the expected payoff from the optimal plan, there exists a plan that achieves π.[21]

The necessary and sufficient condition for cooperation, the Finite Time Condition (7), is easy to check, but it is not clear what it requires intuitively. In Section 5, we show that the Finite Time Condition is satisfied when *the cost of cooperation ($d(a)$) tends to zero faster than the benefit of cooperation ($\pi(a) - \pi^N$) does, as action a tends to the Nash equilibrium.*

3.1 Action Distribution and Arrival Rate Invariance

Since the action at the deadline in the revision game depends on random revision opportunities, the revision game induces a probability distribution over the set of action profiles. In this subsection, we determine this distribution, which enables us to calculate the expected payoff. First, we show that the outcome of the revision game does not depend on the Poisson arrival rate λ in the following sense.

Remark 2. (Arrival Rate Invariance) Under the optimal plan, the probability distribution of the final action is independent of the Poisson arrival rate λ, as long as the optimal action a^* is chosen at time $-T$.

The simple reason is as follows. Compare two revision games (T, λ) and (T, λ') where $\lambda < \lambda'$, assuming that the optimal actions are chosen initially in those games. In general, when two revision games with (T, λ) and (T^0, λ^0) share the same outcome distribution under the optimal plan, let us write $(T, \lambda) \approx (T^0, \lambda^0)$. We will show

$$(T, \lambda) \underset{(A)}{\approx} (\frac{\lambda}{\lambda'}T, \lambda') \underset{(B)}{\approx} (T, \lambda'),$$

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[21]To see this, we first note that any plan $x(\cdot)$ such that there exist $t', t'' \in \mathbb{R}_+$ with $t' \leq t''$ satisfying the following three properties is a trigger strategy equilibrium plan: (i) $x(t) = a^N$ for $t \in [0, t']$, (ii) $x(t)$ follows the differential equation presented in Theorem 1 in $t \in [t', t'']$ and $x(t)$ travels from a^N to a^*, and (iii) $x(t) = a^*$ for $t \in [t'', T]$. Now, if $t' = T$ (i.e., $x(t) = a^N$ for all $t \in [0, T]$), the expected payoff is the Nash payoff π^N. As t' changes from T to 0, the expected payoff continuously increases from π^N to the expected payoff from the optimal plan $\check{\pi}$. This shows the desired property.

The equivalence (A) holds because the two games can be mapped to each other by changing the unit to measure time.[22] Since the optimal action a^* is played in game (T, λ), the same is true in the equivalent game $(\frac{\lambda}{\lambda'}T, \lambda')$. This implies that starting the game $(\frac{\lambda}{\lambda'}T, \lambda')$ earlier at $-T$ $(< -\frac{\lambda}{\lambda'}T)$ does not change the distribution of the final outcome under the optimal plan because players just keep playing a^* until time $-\frac{\lambda}{\lambda'}T$ and then they follow the optimal plan in game $(\frac{\lambda}{\lambda'}T, \lambda')$ (Corollary 1). This shows the equivalence (B), which completes the proof. $\qquad\square$

The above remark shows that the probability distribution of the realized action profile can be obtained by focusing on the case $\lambda = 1$. Let $x_1(\cdot)$ be the optimal trigger strategy equilibrium plan under $\lambda = 1$. Assume that Assumptions A1-6 and the Finite Time Condition are satisfied so that $x_1(\cdot)$ departs the Nash action and eventually hits the optimal action (if the horizon is long). To calculate the distribution, the time for $x_1(\cdot)$ to hit $a \in [a^N, x_1(T)]$, denoted by $t_1(a)$, turns out to be useful. Formally, $t_1(a)$ is given by

$$t_1(a) := \lim_{a' \downarrow a^N} \int_{a'}^{a} \frac{dt}{dx} dx = \lim_{a' \downarrow a^N} \int_{a'}^{a} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx.$$

Now consider the density of realized action $x_1(t) \leq a$. The density is $\lambda e^{-\lambda t} = e^{-t}$, which is the product of

- $\lambda = 1$ (the density of revision opportunity at time t) and

- $e^{-\lambda t} = e^{-t}$ (the probability that the revised action at time t, $x_1(t)$, will never be revised again).

Therefore, the cumulative distribution function of realized action a, denoted by $F(\cdot)$, is given by

$$F(a) = \int_{\{t|x_1(t) \leq a\}} e^{-t} dt = \int_0^{t_1(a)} e^{-t} dt = 1 - e^{-t_1(a)},$$

for $a \in [a^N, x_1(T))$. For $a \geq x_1(T)$, $F(a) = 1$ holds because the realized action cannot be more than $x_1(T)$. This implies that, at $x_1(T)$, the distribution function

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[22]Formally, introduce a new time variable s in game (T, λ) by $s = \frac{\lambda}{\lambda'}t$. The resulting alternative representation is the game $(\frac{\lambda}{\lambda'}T, \lambda')$. Note that the probability of a Poisson arrival in an infinitesimal time interval in the game $(\frac{\lambda}{\lambda'}T, \lambda')$ is $\lambda' ds = \lambda' \frac{\lambda}{\lambda'} dt = \lambda dt$, and the two games share the same expected number of revision opportunities, λT.

$F(\cdot)$ jumps by $e^{-t_1(x_1(T))}$. The jump means that a probability mass of $e^{-t_1(x_1(T))}$ is attached to action $x_1(T)$. This is the probability that no revision opportunity arises after time $-t_1(x_1(T))$ under the Poisson arrival rate $\lambda = 1$. Below we summarize our arguments.

Proposition 1. *Suppose that Assumptions A1 - A6 and the Finite Time Condition (7) are satisfied. When $a^N < a^*$, the cumulative distribution function of the symmetric action realized at $t = 0$ is given by*

$$F(a) = \begin{cases} 0 & \text{if } a < a^N \\ 1 - e^{-t_1(a)} & \text{if } a^N \le a < x_1(T) \\ 1 & \text{if } x_1(T) \le a \end{cases} , \quad \text{where} \quad t_1(a) := \lim_{a' \downarrow a^N} \int_{a'}^{a} \frac{d'(x)}{d(x) + \pi(x) - \pi^N} dx.^{23}$$

4   Applications

In this section, we present two applications of the revision-game model: preopening in the stock exchange and electoral campaigns. All proofs are found in the Online Appendix.

4.1   Preopening in the Stock Exchange

Stock exchanges across the world such as Nasdaq or Tokyo Stock Exchange have the "preopening" phase before the opening time. As Calcagno and Lovo (2010) first observed, revision games can be regarded as a stylized model of this situation. Traders can submit and revise orders in the preopening phase, and the final set of orders at the opening time determines the opening price and quantities. The aggregate orders in the preopening phase are displayed on a public screen. This public screen is refreshed at (frequent but) stochastic times, which can be regarded as the revision opportunities in our model. We aim to point out that participants may have an incentive to change their orders over time in the preopening phase. To illustrate this point, we consider the simplest setting with two sellers and a fixed demand.

The challenge in this application is to deal with the rich strategy space, which is the set of all supply schedules. We will show, however, that we can effectively reduce the strategy space to a one-dimensional interval and apply our technique.

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[23] When $a^* < a^N$, it is given by $F(a) = 0$ if $a < x_1(T)$, $F(a) = e^{-t_1(a)}$ if $x_1(T) \le a \le a^N$, and $F(a) = 1$ if $a^N < a$, where $t_1(a)$ is defined symmetrically to the case with $a^N < a^*$.
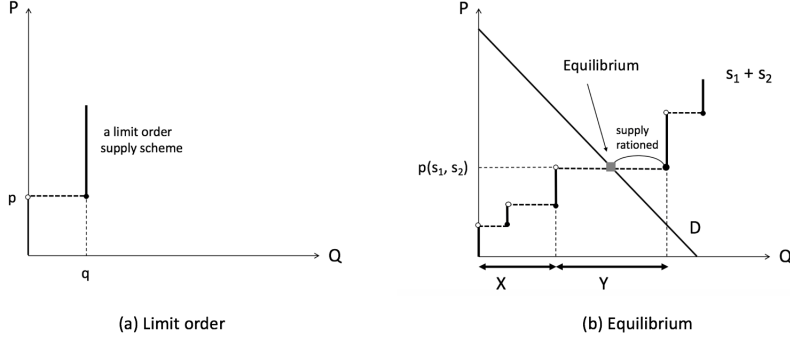
P

a limit order
supply scheme

p

q

Q

(a) Limit order

P

Equilibrium

$s_1 + s_2$

supply
rationed

$p(s_1, s_2)$

D

X   Y

Q

(b) Equilibrium

Figure 2: Limit orders and a supply scheme.

Component game (supply-schedule game)

There are two sellers $i = 1, 2$, who value the stock at a constant (and identical) value $c > 0$. Demand curve is linear $p = a - bq$ ($a > c$, $b > 0$) and the associated demand function is denoted by $D(p)$.[24]

In the stock exchange, sellers can submit a "limit order" to sell quantity q at any price no less than p (panel (a) of Figure 2). Sellers can submit any number of limit orders, and therefore they can approximately submit any non-decreasing supply schedules. Accordingly, we assume that seller i can choose any $s_i \in S = \{s : \mathbb{R}_+ \to \mathbb{R}_+ \mid s$ is non-decreasing$\}$. Panel (b) of Figure 2 illustrates a stock market equilibrium, and it shows that, given supply schedules $(s_1, s_2) \in S^2$, price $p(s_1, s_2)$ is given by

$$p(s_1, s_2) = \sup\{p \in \mathbb{R}_+ \mid s_1(p) + s_2(p) \leq D(p)\},$$

when $s_1(a) + s_2(a) > 0$.[25] The "sup" in this formula deals with the case where rationing is required to clear the market, as we explain in what follows.

If no supply rationing happens, the quantity sold by seller i, denoted $q_i(s_1, s_2)$, is equal to her supply $s_i(p(s_1, s_2))$. When supply rationing happens as in Figure 2 (b), "orders below the market price" (X in Figure 2 (b)) are accepted but "orders at the market price" (Y in Figure 2 (b)) are rationed.[26] For each seller i, define $X_i := \lim_{p \uparrow p(s_1, s_2)} s_i(p)$ (the "orders below the market price") and $Y_i := s_i(p(s_1, s_2)) - X_i$

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[24]Specifically $D(p) = \max\{0, \frac{a-p}{b}\}$.

[25]If $s_1(a) + s_2(a) = 0$, we set $p(s_1, s_2) = a$ and $q_i(s_1, s_2) = 0$.

[26]This corresponds to the "price first" rationing rule in stock exchanges in reality.

(the "order at the market price"). Formally, when supply is rationed,

$$q_i(s_1, s_2) = X_i + f_i(s_1, s_2)Y_i.$$

where $f_i(s_1, s_2) \in [0, 1]$ is any rationing rule that satisfies $q_1(s_1, s_2) + q_2(s_1, s_2) = D(p(s_1, s_2))$.[27] The requirements that supply schedules be non-decreasing and the orders below the market price be accepted play crucial roles in the analysis. Finally, seller i's payoff is

$$\pi_i(s_1, s_2) = (p(s_1, s_2) - c) \cdot q_i(s_1, s_2).$$

This component game has a large number of Nash equilibria. It turns out that submitting a "vertical supply" at the Cournot-Nash quantity q^N ($\forall p\ s_i(p) = q^N$) constitutes a Nash equilibrium. The "Bertrand outcome" with $p(s_1, s_2) = c$ and zero profit is also supported by a Nash equilibrium.[28]

Despite the large strategy space (the set of all supply schedules), we are able to identify the set of all (pure strategy) Nash equilibria. The set of corresponding quantity profiles is the triangular area bounded by the reaction curves R_1, R_2 and the zero-profit line $q_1 + q_2 = \bar{q}$, where $\bar{q} = \frac{a-c}{2}$ satisfies $D(\bar{q}) = c$.[29]

Lemma 1 (Nash equilibria). $(s_1, s_2) \in S^2$ is a Nash equilibrium of the supply-schedule game if and only if the associated quantity profile $(q_1(s_1, s_2), q_2(s_1, s_2))$ is in

$$Q^N := \left\{ (q_1, q_2) \in \mathbb{R}_+^2 \mid q_i \geq R_i(q_{-i}) \text{ for each } i = 1, 2, \text{ and } q_1 + q_2 \leq \bar{q} \right\},$$

[27]Tokyo Stock Exchange uses a rationing rule that does not depend on the times of submitting orders. At some other markets, the "time priority" rule is used (after using the "price first" rule) where the excess demand is rationed so that the first seller submitting the order gets priority in supplying stocks. An analysis similar to the one given here can be made for those markets as well. We note that Tokyo Stock Exchange allows the sellers to observe all the supply schedules submitted in the past if they subscribe for the "full quotes" functionality (similar functionality is present at other places as well), being consistent with our model.

[28]A symmetric equilibrium strategy to achieve zero-profit is to supply $D(c)/2$ if $p < c$ and to supply large quantity $D(0)$ if $p \geq c$. Note that the opponent is unable to raise price above c under this supply scheme.

[29]Klemperer and Meyer (1989) analyzed supply function equilibria for general demand and marginal costs. They showed that any quantity profile that satisfies a certain condition can be sustained by a supply function equilibrium. In the case of linear demand and constant identical marginal costs, one can check that their condition is satisfied in the strictly positive-profit region of the Nash quantity profile set Q^N in Lemma 1. The zero-profit points of Q^N (the line segment $q_1 + q_2 = \bar{q}$, $q_1, q_2 \geq 0$) cannot be sustained in their model because they require that supply functions be twice continuously differentiable. The zero-profit points, however, are crucial to construct the optimal trigger strategy equilibrium, and we show they can be sustained in our model by discontinuous supply functions, which correspond to combinations of limit orders.

where $R_i(q_{-i}) = \frac{a-c-bq_{-i}}{2b}$ is the reaction function of the Cournot game and $\overline{q} = \frac{a-c}{b}$ is the sum of the quantities that induces zero-profit.

The best symmetric Nash equilibrium of the component game corresponds to the Cournot-Nash equilibrium, and the best trigger strategy equilibrium plan we will construct approaches that quantity as $t \to 0$. If a deviation happens, however, the best trigger strategy equilibrium will implement the worst Nash equilibrium of the component game that induces zero profit.

Reduction of the problem

It turns out that we can effectively reduce the supply-schedule component game into a much simpler one that we call a *semi-Cournot competition*. This game is *obtained from the Cournot game by incorporating an additional action $\emptyset$ that induces a zero-profit Nash equilibrium*. More specifically, the semi-Cournot competition has a payoff function $(\bar{\pi}_i)_{i=1,2}$ such that the following hold:

1. Each seller $i = 1, 2$ chooses $q_i \in \mathbb{R}_+ \cup \{\emptyset\}$.

2. $\bar{\pi}_i(q_1, q_2)$ is specified as the Cournot profit for any $(q_1, q_2) \in \mathbb{R}_+^2$.[30]

3. $\bar{\pi}_i(q_1, q_2) = 0$ if $q_i = \emptyset$ for some $i = 1, 2$.

It has two symmetric pure strategy Nash equilibria, the Cournot-Nash equilibrium (q^N, q^N) and $(\emptyset, \emptyset)$, the latter of which induces zero-profit to each player. Note that they correspond to the best and worst symmetric Nash equilibria in the supply schedule game.

In the Online Appendix, we prove that the reduction works in the sense that the optimal trigger strategy equilibria for the supply-schedule game and for the semi-Cournot game share the same outcome (the same quantity plan at almost all time $-t$). The key idea is the following: For any supply schedule profile, we can consider an alternative profile of vertical supply schedules that achieves the same price and quantities. The gains from deviation under the vertical profile turn out to be no greater than those under the original profile as long as the associated outcome Pareto

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[30]Specifically, $\bar{\pi}_i(q_1, q_2) = (\max\{a - b(q_1 + q_2), 0\} - c) \cdot q_i$ for any $(q_1, q_2) \in \mathbb{R}_+^2$.

dominates the Cournot-Nash equilibrium.[31] As a result, one can confine attention to such vertical profiles, which correspond to the quantities in the semi-Cournot competition game. The zero-profit equilibrium of this game $(\emptyset, \emptyset)$, in contrast, corresponds to the worst Nash equilibrium of the supply-schedule game that is implemented by the optimal trigger strategy equilibrium after a deviation.

Optimal plan

The reduction argument enables us to use the characterization of the optimal trigger strategy equilibrium plan in Section 3. For a plan $q(\cdot)$ to be the optimal trigger strategy equilibrium plan in the semi-Cournot competition, it needs to be the solution for the following differential equation for $q(t) \in (\frac{a-c}{4b}, \frac{a-c}{3b}]$:[32]

$$\frac{dq}{dt} = \frac{\lambda\left(d(q) + \pi(q) - 0\right)}{d'(q)} = \lambda\frac{(a - c - bq)^2}{6b(3bq - (a - c))}.$$

One can solve this differential equation with the initial condition $\lim_{t \downarrow 0} q(t) = q^N$. In the Online Appendix, we formally define trigger strategies and the optimal plan in an analogous way as in Section 3. We also define essential uniqueness of the optimal plan, taking care of zero-measure events (see footnote 13). With such terminology, we obtain the following:

Proposition 2 (Optimal plan). *In the revision game of the supply-schedule game, there exists an essentially unique optimal plan of quantities $q(t)$. For $t \in [0, t^*]$, where $t^* = \frac{1}{\lambda}\left(36 \ln(3) - 52 \ln(2) - 2\right)$, $q(t)$ is the unique solution to*

$$\ln\left(\frac{3 - \frac{q(t)}{q^N}}{2}\right) + \left(\frac{2}{3 - \frac{q(t)}{q^N}}\right) = \frac{\lambda}{18}t + 1$$

that satisfies $q(t) \leq q^N$ for all $t \in [0, T]$, and $q(t)$ is equal to the optimal quantity $\frac{a-c}{4b}$

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[31]The reason is sketched out as follows: Suppose supply schedule profile (s_1, s_2) induces price p^* and quantities (q_1, q_2). Given that the supply schedules are non-decreasing, we have $s_i(p) \leq q_i$ for $p < p^*$ for each i. Hence, the residual demand for the opponent, $D(p) - s_i(p)$, is reduced if i changes her supply schedule from s_i to the vertical supply at q_i, for $p < p^*$. When $(q_1.q_2)$ Pareto dominates the Cournot-Nash profile, players have an incentive to reduce the price and steal the customers from the opponent. Switching to the vertical supply makes such a deviation less profitable, by making the residual demand smaller. Note that the requirement that the supply schedule be non-decreasing plays a crucial role.

[32]Note that $\frac{a-c}{4b}$ is the most collusive quantity and $\frac{a-c}{3b} = q^N$ is the Nash quantity in the standard Cournot competition. The differential equation follows from $d(q) = \frac{(a-c-3bq)^2}{4b}$ and $\pi(q) = (a - c - 2bq)$.
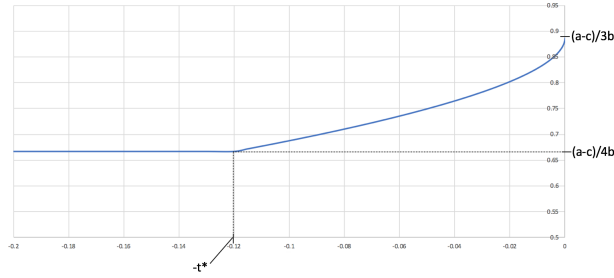
Figure 3: The optimal trigger strategy equilibrium plan for the supply-schedule game.

for $t \in [t^*, \infty)$.

Note that the formula in the proposition above shows $q(0) = q^N$. In the Appendix, we compute the expected profit from the optimal equilibrium and show that it achieves a significant percentage of the most collusive payoff. The plan of quantities is depicted in Figure 3, where the quantities increase over time towards the Cournot-Nash quantity. As a result, the price decreases over time in the preopening phase, and the opening price is higher than any Nash equilibrium prices of the component game with probability one. This conclusion, of course, should be taken with some reservation because we have not modeled the dynamics on the demand side. Our results should be interpreted not as a perfect description of the preopening phase, but rather a demonstration of the possibility of implicit collusion among the participants.

Remark 3 (Contribution to the literature)**.** A body of empirical literature investigated how accurately the prices in the preopening period reflect the fundamental stock values (Barclay and Hendershott (2003), Barclay, Hendershott and Jones (2008), Biais, Hillion and Spatt (2009)). Medrano and Vives (2001) presented a stylized model of preopening and analyzed how a large informed trader, who knows the fundamental stock value, can manipulate the behavior of other traders. To the best of our knowledge, the possibility of collusion in preopening has not been systematically analyzed. □

4.2 Election Campaign: Policy Platforms Gradually Approach the Median

In this application, we show that candidates may revise their policies during the election campaign from their bliss points towards that of the median voter. As an

example, consider the Korean presidential election in 2017, in which Moon Jae-in and Ahn Cheol-soo fought and the deployment of THAAD (Terminal High Altitude Area Defense) was considered to be one of the biggest issues.[33] The election had originally been planned for December 2017, but due to the unexpected impeachment of the former president, voted in December 2016, the election date was shifted to May 2017.

When THAAD was first introduced into South Korea in July 2016, both candidates announced a position against THAAD.[34] After the news of impeachment, however, they shifted their positions towards supporting THAAD.[35] From the beginning of 2016 until the election day, the approval of the deployment of THAAD was always the most popular poll result, and thus it would be difficult to explain the two candidates' move by a change of the voters' preferences.

Our model presumes that revision opportunities are synchronized for players and arrive exogenously. The Korean presidential election case admittedly does not perfectly match those assumptions, but the model captures some important aspects of reality. Although the presidential candidates can announce policy changes at any time, the matters should be discussed and approved within their parties, and also the messages should be effectively transmitted to the voters. Hence, an effective revision opportunity is likely to be tied to the arrival of important news or event, which triggers (i) debates within each party, (ii) voters' attention, and (iii) voters' willingness to accept policy changes. In the Korean presidential election case, there were two such events: the introduction of THAAD by the former president in July 2016, and the unexpected shift of the election date in December 2016. Note that these two events are both exogenous. Moreover, since two candidates respond to the same event each time, their announcements can be considered effectively synchronous.

Model

The policy space is $[0, 1]$. The position 0 represents perfect opposition to THAAD,

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[33]THAAD is an American defense system designed to shoot down ballistic missiles in their terminal phase by intercepting with a hit-to-kill approach. The warhead of nuclear-tipped ballistic missiles will not detonate on a kinetic-energy hit by THAAD.

[34]Moon said "[THAAD] will do more harm than good to our national interest," and Ahn also released a statement expressing his firm opposition to THAAD.

[35]In an interview in January 2017, Moon said "I am not saying that [the issue] should be handed over to the next government with the policy of canceling the decision of the THAAD deployment." Ahn said "South Korea and the United States have already concluded an agreement on the deployment" in February 2017.

and position 1 represents perfect support. Positions in $(0, 1)$ represent various degrees of middle grounds. There are two candidates, A and B. To formulate the situation where the majority favors THAAD, we assume that, given the policy profile (x_A, x_B), the probability of A winning is

$$P_A(x_A, x_B) = \frac{1 + \delta(x_A - x_B)}{2},\tag{9}$$

where $\delta \in (0, 1]$. This functional form has a feature that, if $x_A = x_B$, then the winning probability is $1/2$, and it increases if candidate A is more favorable to THAAD $(x_A > x_B)$.[36]

We define candidate A's expected payoff as follows (candidate B's payoff is symmetrically defined):

$$\pi_A(x_A, x_B) = P_A(x_A, x_B)((1 - x_A) + w) + P_B(x_A, x_B)\gamma(1 - x_B),\tag{10}$$

where $w \geq 0$ and $\gamma \in [0, 1)$. This payoff function assumes that, in the event that candidate A wins, A's payoff is $(1 - x_A) + w$, while her payoff is $\gamma(1 - x_B)$ if she loses. This specification is motivated as follows: The defense policy of candidate i is a combination of the deployment of THAAD and the *candidate's own alternative defense policy*, where the weights of the former and the latter are x_i and $(1 - x_i)$, respectively. Both candidates dislike THAAD, and its value is normalized at 0 for both A and B. We assume that a candidate values his own alternative defense policy more than the opponent's, so that the value of his own alternative defense policy is 1 while the value of the opponent's is $\gamma < 1$. Additionally, candidates receive utility w from being in the office. Call this component game the *election game*.

This game has a unique pure strategy Nash equilibrium and it is symmetric. We denote this Nash equilibrium by (x^N, x^N). It depends on the strength of office motivation w (the utility of being elected *per se*) as follows:

$$x^N = \begin{cases} 0 & \text{if } w \leq \frac{(1-\delta)+\delta\gamma}{\delta} \\ \frac{w - \gamma - (\frac{1}{\delta} - 1)}{1 - \gamma} & \text{if } \frac{(1-\delta)+\delta\gamma}{\delta} < w \leq \frac{1}{\delta} \\ 1 & \text{if } \frac{1}{\delta} < w \end{cases}.\tag{11}$$

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[36]The winning probability (9) has a microfoundation based on voters' utility from each candidate and their voting behavior, whose detail we present in the Supplementary Information on the authors' webpages.
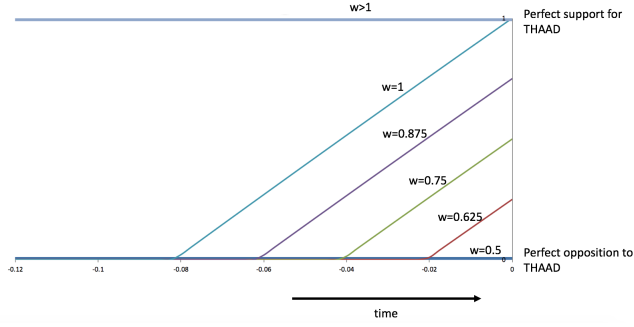
Figure 4: The optimal trigger strategy equilibrium plan $\bar{x}(t)$ for various values of w for the election game: $\lambda = 1$, $\gamma = \frac{1}{2}$, and $\delta = 1$.

Optimal trigger strategy equilibrium plan

The differential equation for the optimal plan is

$$\frac{dx}{dt} = -\lambda \frac{(1-\gamma)^2(1-x^1) + (1-\gamma)\delta w + 3 + 5\gamma}{2\delta(1-\gamma)^2}.$$

This admits the following closed-form solution.

Proposition 3. *In the revision game of the election game, the optimal trigger strategy equilibrium plan, $\bar{x}(t)$, is characterized by the following:*

1. *If $w \leq \frac{(1-\delta)+\delta\gamma}{\delta}$, then $\bar{x}(t) = 0$ for all t.*

2. *If $\frac{(1-\delta)+\delta\gamma}{\delta} < w \leq \frac{1}{\delta}$, then*

$$\bar{x}(t) = \begin{cases} x^N - \frac{(e^{\frac{\lambda}{2}t}-1)(4+4\gamma)}{\delta(1-\gamma)^2} & \text{if } t \leq t^* \\ 0 & \text{if } t^* < t \end{cases}, \quad \text{where } t^* = \frac{2}{\lambda}\ln\left(\frac{\delta(1-\gamma)(w+1-\gamma)+3+5\gamma}{4+4\gamma}\right).^{37}$$

3. *If $\frac{1}{\delta} < w$, then $\bar{x}(t) = 1$ for all t.*

The above proposition shows that, when the office motivation is not too large or too small (Case 2), each candidate starts from announcing their most preferred policy, which is the far-left. They stick to their original announcements until a certain time (time $-t^*$) before the election day, and then begin catering to the right towards the end of the campaign period. Thus, the model captures the dynamics of policy

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[37] This case exists because $\frac{(1-\delta)+\delta\gamma}{\delta} < \frac{1}{\delta}$ holds for any $\gamma < 1$ and $\delta \in (0,1]$.

announcements in the 2017 Korean Presidential election. The plan characterized in Proposition 3 is depicted in Figure 4 for various values of parameter w, supposing $\gamma = \frac{1}{2}$ and $\delta = 1$. Notice that there is a discontinuity at $w = 1/\delta = 1$, i.e., the limit of the optimal plan as $w \downarrow 1$ (which is a trivial plan such that $\bar{x}(t) = 1$ for all t) does not converge to the optimal plan at $w = 1$ (which is a nontrivial plan).

Remark 4. A simple modification of the above model can capture the situation where two candidates have opposing bliss points (-1 and 1 in the policy space $[-1, 1]$) and their policies gradually approach the middle (0). See the Supplementary Information on the authors' webpages for the detail. $\qquad\square$

Remark 5 (Contribution to the election literature)**.** Previous literature on the dynamics of election campaigns mainly focused on the case in which elections are repeated or there are primary and general elections (see e.g., Alesina (1988) and Meirowitz (2005)). Kamada and Sugaya (2019) was the first to apply our revision-games framework to analyze the dynamics within a single election campaign. Unlike in our model, they considered the case in which policy announcements are irreversible and there are candidate-specific revision opportunities. They showed that the candidates remain silent until shortly before the deadline. Their analysis is based on an analogue of backward induction, and thus different from ours. $\qquad\square$

5 What Determines the Possibility of Cooperation?

Our main result (Theorem 2) shows that the Finite Time Condition is necessary and sufficient for cooperation to be sustained *by a trigger strategy equilibrium.* In this section, we consider all revision game strategies and ask a more general question: When is cooperation sustained in revision games *by any equilibrium*? We find a simple answer that the following is necessary and almost sufficient for cooperation to be sustained in revision games.

- [**Convergence Condition**] As the action profile converges to the Nash equilibrium, the gain from deviation $d(a)$ tends to zero faster than the benefit of cooperation $\pi(a) - \pi^N$ does.

More precisely, if a slightly stronger version of this condition is satisfied, cooperation is sustained by an equilibrium (actually, by a trigger strategy equilibrium: see Section 5.1). If this condition is not satisfied, cooperation cannot be sustained by *any equilibrium* (Section 5.2). We are able to show the latter result for a very general class of component games. Note that those results also clarify when the Finite Time Condition (FTC) is satisfied: the Convergence Condition above is necessary for the FTC, and a slightly stronger version of this condition ((13) in the next section) is sufficient for the FTC.

Intuitively, why is the Convergence Condition equivalent to the possibility of cooperation in revision games? Near the deadline, say at time $-\varepsilon$, the probability that a current deviation is retaliated in the remaining time is very small. Hence, we may only sustain a near-Nash action a with a small gain from deviation $d(a)$. Note that the continuity of the action space plays an important role here. The associated benefit of cooperation $\pi(a) - \pi^N$ contributes to the magnitude of punishment to sustain cooperation slightly before time $-\varepsilon$. If $\pi(a) - \pi^N$ is large, much more cooperation is sustained if we further move away from the deadline. Therefore, when small $d(a)$ can provide large $\pi(a) - \pi^N$, as players move away from the deadline, they can quickly achieve more and more cooperative actions. This "snowball effect" enables players to depart from the Nash equilibrium. The formal analysis in the following sections makes this intuition precise.

5.1 Sufficient Condition for Cooperation

We derive a sufficient condition for cooperation under a set of assumptions weaker than A1-A6. We consider the case where larger actions are more efficient than the Nash action a^N.[38] We assume that the component game has a symmetric action space $A \subseteq \mathbb{R}$ and a measurable symmetric payoff function π with a unique symmetric Nash equilibrium a^N. Let $d(a) := \sup_{a' \in A} \pi(a', a) - \pi(a)$.

Assumption (*): There exists $\epsilon > 0$ such that:

1. $[a^N, a^N + \epsilon] \subseteq A$,

2. d is continuous in $[a^N, a^N + \epsilon]$, and

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[38]Symmetric statements apply to the opposite case.

3. $\pi(a) \geq \pi^N$ for all $a \in [a^N, a^N + \epsilon]$.

Call the component games satisfying the above assumptions the *symmetric uni-dimensional component games*.

Under Assumption (*), a straightforward formulation of the Convergence Condition would be

$$\lim_{x \downarrow a^N} \frac{d(x)}{\pi(x) - \pi^N} = 0. \tag{12}$$

Since condition (12) itself is not quite strong enough for the sustainability of cooperation, we will slightly strengthen condition (12).[39] Note that, when $d(x)$ is small (less than 1), for any $k \in (0, 1)$, $d(x)^k > d(x)$. Our condition for cooperation is that this larger value $d(x)^k$ tends to zero faster than the benefit of cooperation $\pi(x) - \pi^N$ does:

$$\lim_{x \downarrow a^N} \frac{d(x)^k}{\pi(x) - \pi^N} = 0 \text{ for some } k \in (0, 1). \tag{13}$$

Note that this is only a slightly weakened version of (12) because the constant k can be arbitrarily close to 1. The following theorem shows that cooperation is sustained under this condition.

Theorem 3. *Consider the revision game with a symmetric uni-dimensional component game. If condition (13) holds, cooperation can be sustained by a trigger strategy equilibrium.*[40]

We prove this theorem by explicitly constructing a trigger strategy plan, which is presented in the Online Appendix.[41] Here we illustrate the construction in the following example. Consider the component game with

$$\pi_i(a_1, a_2) = ((a_{-i})^l + (a_{-i})^m) - (a_i)^m \text{ with } 0 < l < m.$$

In this case,

$$\frac{d(a)^k}{\pi(a) - \pi^N} = \frac{a^{mk}}{a^l},$$

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[39] To be more precise, we have been unable to show that (12) implies the Finite Time Condition.

[40] "Cooperation can be sustained" means that cooperative action profiles that provide higher payoffs than the Nash equilibrium can be realized with a positive probability.

[41] We do not show that condition (13) implies FTC but construct an equilibrium plan. The reason is that Theorem 3 considers a symmetric uni-dimensional component game, which imposes a weaker restriction than A1-A6 (under which Theorem 2 shows that FTC is equivalent to sustainability of cooperation).

and therefore condition (13) holds for any $k \in (\frac{l}{m}, 1)$.

The incentive constraint at time $-t$ for a trigger strategy equilibrium plan $x(\cdot)$ can be written as:

$$e^{-\lambda t} x(t)^m \leq \int_0^t x(t)^l \lambda e^{-\lambda \tau} d\tau.$$

Since $e^{-\lambda \tau} \geq e^{-\lambda t}$ for all $\tau \in [0, t]$, the following condition is sufficient for this incentive constraint to hold.

$$x(t)^m \leq \int_0^t x(\tau)^l \lambda d\tau.$$

For sufficiently small t, we consider $x(t) = t^{\frac{2}{m-l}}$. Then, this sufficient condition is equivalent to:

$$t^{\frac{4m}{m-l}} \leq \int_0^t \tau^{\frac{2l}{m-l}} \lambda d\tau = \frac{1}{\frac{2l}{(m-l)} + 1} t^{\frac{2l}{(m-l)}+1} \lambda = \frac{\lambda}{\frac{2l}{(m-l)} + 1} t^{\frac{l+m}{m-l}},$$

which holds for sufficiently small $t \geq 0$ because $l < m$.

5.2 Sufficient Condition for No Cooperation

We consider a very general component game with an arbitrary action space A_i, $i = 1, 2$. Let $A = A_1 \times A_2$. Suppose that there exists a unique pure Nash action profile $a^N = (a_1^N, a_2^N)$. For any action profile $a \in A$, let $\pi_i(a)$ and $d_i(a) := \sup_{a_i' \in A_i} \pi_i(a_i', a_{-i}) - \pi_i(a)$ be the payoff and the "supremum gain" from deviation for player i. Let $\pi_i^N = \pi_i(a^N)$, and assume that $\bar{\pi}_i = \sup_{a \in A} \pi_i(a)$ and $\underline{\pi}_i = \inf_{a \in A} \pi_i(a)$ exist. Call this component game the *general component game*.

When the gain from deviation $d_i(a)$ is bounded away from zero outside a neighborhood of the Nash profile a^N, the negation of the Convergence Condition (12) (when the limit in (12) exists) is

$$\inf_{a \in A \setminus \{a^N\}} \frac{d_i(a)}{|\pi_i(a) - \pi_i^N|} > 0.^{42} \tag{14}$$

For example, this condition is always satisfied if the action space A_i is finite for each $i = 1, 2$. The next theorem shows that condition (14) implies no cooperation in revision games under any subgame-perfect equilibrium.

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[42]In the environment under A1-A6, condition (14) is stronger than the negation of (12) because A1-A6 allows for a possibility that $\lim_{a \downarrow \inf A} d(a) = 0$ when $(\inf A) \notin A$.

Theorem 4. *Consider the revision game with a general component game defined above. Under condition (14), there exists a unique pure strategy subgame-perfect equilibrium. It specifies the Nash action profile a^N at any revision opportunity after any history.*

The proof is based on a continuous-time backward induction: Fix any time $-t$ and suppose that only the Nash action can be played at every time strictly after time $-t$ in any pure strategy subgame perfect equilibrium (SPE). Note that this is vacuously satisfied at time $-t = 0$. Then we show that there exists $\epsilon > 0$ such that for all time in $[-t - \epsilon, -t]$, only the Nash action can be played in any SPE. Proceeding in this way, we can show that there is no time $-t$ such that a non-Nash action is taken. A formal proof is given in the Online Appendix.

6   Asynchronous Revisions

In some real-life cases, players' revision opportunities are not necessarily synchronized. In this section, we show that our analysis carries over to the asynchronous case if the component-game payoff function satisfies a certain property.

Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be player 1 and 2's arrival rates, respectively. We assume that players observe all the past events in the revision game, including when revision opportunities have arrived to the opponent (so i can see if j has actually followed the equilibrium action plan), and analyze the optimal symmetric trigger strategy equilibrium. Assume that *the payoff function is additively separable with respect to each player's action*: For each $i = 1, 2$,

$$\pi_i(a_i, a_{-i}) = b(a_{-i}) - c(a_i). \tag{15}$$

We assume that there is a unique minimizer of $c(\cdot)$, a^N, and normalize the payoff so that $b(a^N) = c(a^N) = 0$. Notice that there is a unique Nash equilibrium, $(a_1, a_2) = (a^N, a^N)$ that induces zero payoffs. The linear exchange game in Section 3 and the games we will present in Section 7.1 with $\epsilon = 0$ fit this framework.

In general, player i's incentive in the asynchronous case depends not only on how much time is left but also on the opponent's action that is fixed at the time of the revision. Consequently, player i's equilibrium revision plan at $-t$ would depend on the payoff-relevant state variable (t, a_{-i}), where a_{-i} is the fixed action of the

opponent at revision time $-t$. For this reason, the analysis of the asynchronous case is substantially more complicated. If the payoff is separable across players' actions, however, a player's current gain from deviation is not affected by the fixed action of the opponent, and the size of the opponent's future punishment is not affected by the current deviation action of the player. This is the main reason why we can apply the same analysis as in the synchronous case.

Specifically, for each $i = 1, 2$, consider revision plan $x_i : [0, T] \to A$ for each player $i = 1, 2$ that depends only on the remaining time. For any opponent's current action a_j, player i's continuation payoff under this revision plan at time $-t$ is

$$e^{-\lambda_j t} b(a_j) + \int_0^t b(x_j(\tau)) \lambda_j e^{-\lambda_j \tau} d\tau - \left(e^{-\lambda_i t} c(x_i(t)) + \int_0^t c(x_i(\tau)) \lambda_i e^{-\lambda_i \tau} d\tau \right).$$

Under the Nash reversion, i's continuation payoff under defection at time $-t$ is

$$e^{-\lambda_j t} b(a_j).$$

Hence, the incentive compatibility condition for player i at time $-t$ is:

$$e^{-\lambda_i t} c(x_i(t)) \leq \int_0^t \left(b(x_j(\tau)) \lambda_j e^{-\lambda_j \tau} - c(x_i(\tau)) \lambda_i e^{-\lambda_i \tau} \right) d\tau. \tag{16}$$

Notice that this condition does not depend on a_j, the fixed action of the opponent.

In the case of homogeneous arrival rates $\lambda_1 = \lambda_2 = \lambda$, the incentive compatibility condition (16) is identical to the incentive compatibility condition (3) of the synchronous case. This gives us the following proposition, which implies that our main results apply to the case of asynchronous revisions if the arrival rates are homogeneous.

Proposition 4. *When the component-game payoff is separable as in (15) and the arrival rates are equal $\lambda = \lambda_1 = \lambda_2$ in the asynchronous case, symmetric trigger strategy plan $x_1(t) = x_2(t) = x(t)$ constitutes an equilibrium if and only if $x(t)$ is a symmetric trigger strategy equilibrium plan in the synchronous case.*

We end this subsection with two remarks. First, although the equilibrium plans are the same in the synchronous and asynchronous cases, the probability distributions of action profiles at the deadline are different from each other. This is because two

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players' actions are perfectly correlated under synchronous revisions, while they are independent under asynchronous revisions. However, by additive separability, the expected payoffs stay the same in the two cases.

Second, when payoffs are additively separable and arrival rates are heterogeneous $\lambda_1 \neq \lambda_2$, we need to work with two distinct incentive constraints ((16) for the two players) simultaneously. As a consequence, the optimal trigger strategy equilibrium has a property $x_1(t) \neq x_2(t)$. Despite the difference, some key features of optimal equilibria we found in this paper are robust. For example, the incentive constraints are binding when the time is close to the deadline, $x_i(t)$ is increasing in t when t is sufficiently small, and $x_i(t) = a^*$ when t is sufficiently large. Those results for the asynchronous case under additively separable payoffs and heterogeneous arrival rates are the subject of Kamada and Kandori (2012). A full-fledged analysis of the asynchronous case under general payoff functions is an important open problem.

7 Further Analysis

In this section, we briefly sketch further analysis on the robustness of our main results and the relationship between our model and related dynamic games (stochastic games). The full details can be found in the Supplementary Information on the authors' webpages.

7.1 Robustness

For cooperation to be sustained in the revision game, it is essential that there is always a chance that another revision can happen before the deadline is reached. In real-life applications, however, if a revision opportunity arrives near the deadline, no more revisions are practically possible. We can show, however, that the main message of the revision game survives in such a situation if we allow that either (i) the deadline is soft or (ii) players have a small incentive to punish a deviator. We consider the effects of introducing (i) and (ii) in a discrete time baseline model with $t = -T, ..., -2, -1, 0$, where a revision opportunity arrives with probability $\gamma > 0$ in each period. Note that, since this baseline model has the last revision opportunity, the backward induction implies that players cannot sustain cooperation.[43]

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[43]We continue to assume the uniqueness of the Nash equilibrium of the component game.

First, we introduce a soft deadline by adding periods $1, 2, ...$ The game goes on until time 0 (the deadline) with probability 1, while it ends with probability $1-\epsilon \in (0, 1]$ at the end of each period $0, 1, 2, \ldots$, independently. Parameter ϵ measures how "soft" the deadline is. If $\epsilon = 0$, then the deadline is firm. It is shown that our main results survive in the following sense:

- The sufficient condition (13) for cooperation in the revision game implies that cooperation can be sustained under a trigger-strategy equilibrium for any degree of non-zero softness of the deadline ($\epsilon > 0$).

- Under the sufficient condition (14) for no cooperation in the revision game, no cooperation can be sustained in any trigger-strategy equilibrium if the deadline is sufficiently firm.[44]

Secondly, to examine the effects of a small incentive to punish, we consider modifications of the linear exchange game in Section 3, with payoff function $\pi_i = a_{-i} - c_\epsilon(a_i)$, $a_i \in [0, 1]$.[45] Parameter ϵ, which determines the cost of cooperation $c_\epsilon(\cdot)$, is interpreted as the willingness to punish a deviator. We consider the following models.

- Model 1 : $c_\epsilon(a_i) = \max\{a_i^2 - \epsilon, 0\}$.

- Model 2 : $c_\epsilon(a_i) = \max\{a_i - \epsilon, 0\}$.

When $\epsilon = 0$, the component game has the unique Nash action $a_i = 0$, and one can check that cooperation in the revision game is possible in Model 1 but not in Model 2.

When $\epsilon > 0$, in contrast, every $a_i \in [0, \epsilon]$ constitutes a Nash equilibrium of the component game. This means that players can switch from the "good Nash action" $a_i = \epsilon$ to the "bad one" $a_i = 0$ in the last period in our discrete time model, when a deviation has happened. By solving the model backwards, we find that the main results of the revision game survive: substantial cooperation is possible if and only if cooperation is sustained in the revision game. In particular, substantial cooperation

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[44]When the unique Nash equilibrium is the minimax point in the component game (e.g., the linear exchange game in Example 1 or its modification with $\epsilon = 0$ that we will shortly discuss), the trigger-strategy entails the severest punishment; hence the result that no cooperation can be sustained under any trigger-strategy equilibrium implies that the same is true for any equilibrium.

[45]A well-documented fact shows that real people have small incentive to punish a deviator (for example, see Fehr and Gachter (2002)).

is possible in Model 1 even for small ϵ, while substantial cooperation in Model 2 is not possible for small ϵ.

7.2 Comparison to Related Dynamic Games

Although payoffs accrue only at time 0 in the revision game, we can show that the revision game is strategically equivalent[46] to a discrete-time model where a flow payoff accrues to each player in periods $n = 0, 1, 2, ...$ and the game terminates randomly in a way similar to Shapley (1953). This game has a state variable $s(n) \in [0, T]$, which corresponds to the remaining time in the revision game, and the nth period corresponds to the nth revision opportunity in the revision game. In each period n, player i receives payoff $\pi_i(a_i, a_{-i})e^{-\lambda s(n)}$ where (a_i, a_{-i}) is the action profile taken at period n.

By using this equivalence, we can directly compare the revision game to other related dynamic games with flow payoffs. In particular, we can show that (i) the revision game does not belong to the class of dynamic games (stochastic games) where the folk theorems have been proved (Dutta (1995) and Hörner, Sugaya, Takahashi and Vieille (2011)) and (ii) the revision game is closely related but not equivalent to the model with a decreasing discount factor (Bernheim and Dasgupta, 1995).

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[46]"Strategically equivalent" means that this game has the same strategy spaces and expected payoff functions as in the revision game.

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A Appendix: Technical Results on the Optimal Trigger Strategy Equilibrium

Assumptions A1-A4 guarantee the existence of the optimal trigger strategy equilibrium plan that is continuous and satisfies the binding incentive constraint. To show this, we first present a simple but useful lemma. Recall that we are focusing on the case $a^N < a^*$. The next lemma shows that we can restrict our attention to the trigger strategy equilibria whose actions always lie in $[a^N, a^*]$.

Lemma 2. *For any trigger strategy equilibrium plan $x \in X^*$, there is a trigger strategy equilibrium plan $\widehat{x} \in X^*$ such that $\forall t$ $\widehat{x}(t) \in [a^N, a^*]$ and $\pi(\widehat{x}(t)) \geq \pi(x(t))$ with a strict inequality if $x(t) \notin [a^N, a^*]$.*

Proof. Construct $\widehat{x}(t)$ from a given $x(t)$ as follows. First, if $x(t) > a^*$, let $\widehat{x}(t) = a^*$. This assures $\pi(\widehat{x}(t)) = \pi(a^*) > \pi(x(t))$ and, by Assumption A4, $d(\widehat{x}(t)) \leq d(x(t))$.

Second, if $x(t) < a^N$, let $\widehat{x}(t) = a^N$. This assures $d(\widehat{x}(t)) = 0 < d(x(t))$ and, by Assumption A2, $\pi(\widehat{x}(t)) > \pi(x(t))$. Finally, let $\widehat{x}(t) = x(t)$ if $x(t) \in [a^N, a^*]$. Overall, $\widehat{x}(t)$ provides weakly higher payoffs and weakly smaller gains from deviation, and thus it also satisfies the trigger strategy incentive constraint

$$d(\widehat{x}(t))e^{-\lambda t} \leq \int_0^t \left(\pi(\widehat{x}(s)) - \pi^N \right) \lambda e^{-\lambda s} ds.$$

□

This lemma shows that the optimal trigger strategy (if it exists) can be found in the set X^{**} of trigger strategy equilibria whose range is $[a^N, a^*]$:

$$X^{**} := \left\{ x \in X^* | \forall t \; x(t) \in [a^N, a^*] \right\}.$$

Now we are ready to prove the following.

Proposition 5. *Under Assumptions A1-A4, there is an optimal trigger strategy equilibrium plan $\overline{x}(t)$ ($V(\overline{x}) = \max_{x \in X^*} V(x)$) that is continuous for all t and satisfies the binding incentive constraint when $\overline{x}(t) \neq a^*$:*

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(s)) - \pi^N \right) \lambda e^{-\lambda s} ds. \tag{17}$$

Furthermore, $\overline{x}(t) \in [a^N, a^*]$ for all t if $a^N < a^*$ (and a symmetric condition holds if $a^* < a^N$).

Proof. We show that there is a trigger strategy equilibrium in X^{**} that attains $\max_{x \in X^{**}} V(x)$ (by Lemma 2, it is the true optimum in X^*).

In the first step, we construct a candidate optimal plan $\overline{x}(t)$ and show its continuity. In Step 2, we will verify that this plan is feasible and it is indeed the optimal trigger strategy equilibrium plan. In Step 3, we show that the binding incentive constraint holds under the plan $\overline{x}$.

[**Step 1**] Since $V(x)$ is bounded above by $\pi(a^*) = \max_a \pi(a)$, $\sup_{x \in X^{**}} V(x)$ is a finite number. Hence, we can find a sequence $x^n \in X^{**}$ such that $\lim_{n \to \infty} V(x^n) = \sup_{x \in X^{**}} V(x)$.

Note that $\{\pi(x^n(\cdot))\}_{n=1,2,\dots}$ is a collection of *countably* many measurable functions. This implies that $\overline{\pi}(t) := \sup_n \pi(x^n(t))(< \infty)$ is also measurable. Now let us define

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$\overline{x}(t)$ to be the solution to

$$\textbf{Problem P(t)}: \max_{x(t) \in [a^N, a^*]} \pi(x(t))$$

$$\text{s.t. } d(x(t))e^{-\lambda t} \leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{18}$$

Note that the right hand side of the constraint (18) is well-defined because $\overline{\pi}(\cdot)$ is measurable. Also note that the right hand side is nonnegative by $\overline{\pi}(s) \geq \pi^N$.[47]

Under Assumptions A2 and A4, both $\pi(a)$ and $d(a)$ are strictly increasing on $[a^N, a^*]$. Furthermore, by Assumption A3, $d(a)$ is continuous by Berge's Theorem of Maximum. Hence, the solution $\overline{x}(t)$ to Problem P(t) is either a^* or the action in $[a^N, a^*)$ with the binding constraint (18). Let us write down the solution $\overline{x}(t)$ to the above problem P(t) in the following way. Since d is continuous and strictly increasing on $[a^N, a^*]$, on this interval its continuous inverse d^{-1} exists. Then, the optimal solution $\overline{x}(t)$ can be expressed as

$$\overline{x}(t) = \begin{cases} a^* & \text{if } d(a^*) < h(t) \\ d^{-1}\left(h(t)\right) & \text{otherwise} \end{cases}, \tag{19}$$

where

$$h(t) := e^{\lambda t} \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

A crucial step in the proof, that shows the continuity of the optimal plan, is to note that the integral $\int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds$ in the definition of $h(t)$ is continuous in t for any measurable function $\overline{\pi}(\cdot)$.[48] Since d^{-1} is continuous, this observation implies that $\overline{x}(t)$ is continuous whenever $x(t) \in [a^N, a^*)$. Moreover, since $h(t)$ is increasing in t, (19) means that $x(t) = a^*$ implies $x(t') = a^*$ for all $t' > t$. Hence $\overline{x}$ is continuous for all t.

[**Step 2**] We show that $\overline{x}$ is actually feasible and a trigger strategy equilibrium. The continuity of $\overline{x}$ and π implies that $\pi(\overline{x}(\cdot))$ is a measurable function. Therefore, $\overline{x}$ is feasible. We show that $\overline{x}$ also satisfies the (trigger strategy) incentive constraint IC(t) for all t. Recall that x^n is a trigger strategy equilibrium for all $n = 1, 2, ...$

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[47]By A2, $x^n(t) \in [a^N, a^*]$ implies $\pi(x^n(t)) \geq \pi^N$. Hence $\overline{\pi}(t) = \sup_n \pi(x^n(t)) \geq \pi^N$.

[48]A standard result in measure theory shows that, for any measurable function $f(t)$, the Lebesgue integral $\int_0^t f(s)ds$ is absolutely continuous in t. Therefore, it is continuous in t.

Then we have, for all $n = 1, 2, \ldots$,

$$d(x^n(t))e^{-\lambda t} \leq \int_0^t \left(\pi(x^n(s)) - \pi^N\right) \lambda e^{-\lambda s} ds \quad (x^n \text{ is an equilibrium})$$

$$\leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \qquad (\text{by the definition of } \overline{\pi})$$

This means that $x^n(t)$ satisfies the constraint of Problem P(t). Since $\overline{x}(t)$ is the solution to Problem P(t), we have

$$\forall n \ \forall t \quad \pi(\overline{x}(t)) \geq \pi(x^n(t)) \tag{20}$$

and therefore

$$\forall t \ \ \pi(\overline{x}(t)) \geq \overline{\pi}(t) = \sup_n \pi(x^n(t)). \tag{21}$$

This implies that, for all t, $\overline{x}(t)$ satisfies the incentive constraint IC(t):

$$d(\overline{x}(t))e^{-\lambda t} \leq \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds \quad (\overline{x}(t) \text{ satisfies } (18))$$

$$\leq \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

Thus, we have shown that $\overline{x}$ is a trigger strategy equilibrium ($\overline{x} \in X^*$), and $V(\overline{x}) \geq V(x^n)$ for all n (by (20)). By definition $\lim_{n \to \infty} V(x^n) = \sup_{x \in X^{**}} V(x)$, and the above inequality implies $V(\overline{x}) \geq \sup_{x \in X^{**}} V(x)$. Since $\overline{x} \in X^{**}$, we must have $V(\overline{x}) = \sup_{x \in X^{**}} V(x) = \max_{x \in X^{**}} V(x) (= \max_{x \in X^*} V(x)$ by Lemma 2). Hence, we have established that there is an optimal and continuous trigger strategy equilibrium $\overline{x}$.

[**Step 3**] Lastly we prove that the optimal plan $\overline{x}$ satisfies the binding incentive constraint. Step 1 shows that, if $\overline{x}(t) \neq a^*$, then the following "pseudo" binding incentive constraint is satisfied:

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\overline{\pi}(s) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{22}$$

Our remaining task is to show the "true" binding incentive constraint

$$d(\overline{x}(t))e^{-\lambda t} = \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

Since $\pi(\overline{x}(t)) \geq \overline{\pi}(t)$ for all t (inequality (21)), the pseudo binding incentive constraint (22) implies

$$d(\overline{x}(t))e^{-\lambda t} \leq \int_0^t \left(\pi(\overline{x}(t)) - \pi^N\right) \lambda e^{-\lambda s} ds. \tag{23}$$

We show that this is satisfied with an equality. If the above inequality were strict for some t, we would have $\int_0^t \overline{\pi}(s)\lambda e^{-\lambda s} ds < \int_0^t \pi(\overline{x}(s))\lambda e^{-\lambda s} ds$. Given $\pi(\overline{x}(s)) \geq \overline{\pi}(s)$ for all $s \in (t, T]$ (inequality (21)), we would have

$$e^{-\lambda T}\overline{\pi}(T) + \int_0^T \overline{\pi}(s)\lambda e^{-\lambda s} ds < e^{-\lambda T}\pi(\overline{x}(T)) + \int_0^T \pi(\overline{x}(s))\lambda e^{-\lambda s} ds = V(\overline{x}).$$

Since $\overline{\pi}(s) := \sup_n \pi(x^n(t))$, the left hand side is more than or equal to $V(x^n)$ for all n. Since $\lim_{n\to\infty} V(x^n) = \sup_{x \in X^{**}} V(x)$, the above inequality implies $\sup_{x \in X^{**}} V(x) < V(\overline{x})$. In contrast, $\overline{x} \in X^{**}$ implies $\sup_{x \in X^{**}} V(x) \geq V(\overline{x})$, and this is a contradiction. Hence, (23) should be satisfied with an equality (i.e., $\overline{x}$ satisfies the binding incentive constraint), if $\overline{x}(t) \neq a^*$. □

Let us make a technical remark about the multiplicity of the optimal plans. Recall that $\overline{x}$ is a particular optimal trigger strategy equilibrium plan with the binding incentive constraint (the one that is described in Theorem 5). This plan $\overline{x}$ is referred to as *the optimal plan*. There are, however, other optimal plans if $\overline{x}$ is not trivial. For example,

$$x(t) := \begin{cases} a^N & \text{if } t \text{ is in a measure zero set in } (0, T) \\ \\ \overline{x}(t) & \text{otherwise} \end{cases}.$$

is also a trigger strategy equilibrium plan that satisfies the incentive constraint (3) and achieves the same expected payoff as $\overline{x}$ does. This is because the probability that revision opportunities happen in the measure-zero set in $(0, T)$ is zero. Hence, the above plan is also optimal. Formally, there is *essentially a unique optimal plan* in the following sense.

Proposition 6. *The optimal plan is essentially unique: If $y(\cdot)$ is an optimal trigger strategy equilibrium plan, then $y(t) \neq \overline{x}(t)$ only on a measure zero set in $(0, T)$, where $\overline{x}$ is the optimal plan that satisfies the binding incentive constraint (17).*

Proof. Suppose $H := \{t \in (0, T)|\pi(y(t)) > \pi(\overline{x}(t))\}$ has a positive measure.[49] Then,

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[49]Since y is a feasible plan, H is a measurable set.

define

$$z(t) := \begin{cases} y(t) & \text{if } t \in H \\ \\ \overline{x}(t) & \text{otherwise} \end{cases}.$$

This has a measurable payoff $\pi(z(t)) = \max\{\pi(y(t)), \pi(\overline{x}(t))\}$ and achieves a strictly higher expected payoff than $\overline{x}(t)$. Furthermore, z satisfies the incentive constraints

$$\forall t \quad d(z(t))e^{-\lambda t} \leq \int_0^t \left(\pi(z(s)) - \pi^N\right) \lambda e^{-\lambda s} ds.$$

This follows from the incentive constraints for $\overline{x}$ and y, together with $\pi(z(t)) = \max\{\pi(y(t)), \pi(\overline{x}(t))\}$. Hence, z is a trigger strategy equilibrium plan, which achieves a strictly higher payoff than $\overline{x}(t)$ does. This contradicts the optimality of $\overline{x}$, and therefore H must have measure zero. Hence, $\pi(y(t)) \leq \pi(\overline{x}(t))$ almost everywhere in $(0, T)$. If $\{t \in (0, T) | \pi(y(t)) < \pi(\overline{x}(t))\}$ has a positive measure, y attains a strictly smaller payoff than $\overline{x}(t)$ does, which contradicts our premise that y is optimal. Therefore, we conclude that $\pi(y(t)) = \pi(\overline{x}(t))$ almost everywhere in $(0, T)$.

Now, note that Lemma 2 implies that if $\{t \in (0, T) | y(t) \notin [a^N, a^*]\}$ has a positive measure, then $y(t)$ cannot be optimal in X^*. Hence, $y(t) \in [a^N, a^*]$ almost everywhere. This and $\pi(y(t)) = \pi(\overline{x}(t))$ almost everywhere in $(0, T)$ imply that $y(t) = \overline{x}(t)$ almost everywhere in $(0, T)$ because π is strictly increasing on $[a^N, a^*]$.

Finally, we show $y(t) = \overline{x}(t)$ for $t = 0, T$. By the incentive constraint, $y(0) = \overline{x}(0) = a^N$. If $y(T) \neq \overline{x}(T)$, by the same argument as above (using Lemma 2), we obtain $\pi(y(T)) \neq \pi(\overline{x}(T))$. This and the fact that $y(t) = \overline{x}(t)$ almost everywhere in $(0, T)$ imply that $y(\cdot)$ and $\overline{x}(\cdot)$ achieve different payoffs, which contradicts the premise that those plans obtain the same optimal payoff. $\square$

Next, we prove the differentiability of the optimal plan. We present this result for the case $a^N < a^*$.

Proposition 7. *Under Assumptions A1-A5, the optimal plan $\overline{x}(t)$ is differentiable when $\overline{x}(t) \neq a^N, a^*$, and satisfies the differential equation $\frac{d\overline{x}}{dt} = f(\overline{x}(t))$, where*

$$f(x) := \frac{\lambda\left(d(x) + \pi(x) - \pi^N\right)}{d'(x)}$$

for $x \in (a^N, a^*)$.

Proof. Note that $\overline{x}(t) \in [a^N, a^*]$ (by Proposition 5). By Assumption A5, d has an inverse function on $(a^N, a^*]$, denoted by d^{-1}. Thus, if $\overline{x}(t) \in (a^N, a^*)$, the binding incentive constraint implies

$$\overline{x}(t) = d^{-1} \left(e^{\lambda t} \int_0^t \left(\pi(\overline{x}(s)) - \pi^N \right) \lambda e^{-\lambda s} ds \right). \tag{24}$$

The crucial step to show the differentiability of $\overline{x}(t)$ is to note the differentiability of integral $\int_0^t \left(\pi(\overline{x}(s)) - \pi^N \right) \lambda e^{-\lambda s} ds$ with respect to t. Specifically, the continuity of $\overline{x}$, established by Proposition 5, implies that $\left(\pi(\overline{x}(s)) - \pi^N \right) \lambda e^{-\lambda s}$ is continuous, and the fundamental theorem of calculus shows that $\int_0^t \left(\pi(\overline{x}(s)) - \pi^N \right) \lambda e^{-\lambda s} ds$ is differentiable with respect to t (with the derivative $\left(\pi(\overline{x}(t)) - \pi^N \right) \lambda e^{-\lambda t}$). Also, A5 implies that d^{-1} is differentiable with derivative $1/d'(a)$ (note that A5 guarantees $d'(a) \neq 0$ for $a \in (a^N, a^*]$). Therefore, the right hand side of (24) is differentiable with respect to t, and differentiating both sides of (24), we obtain the differential equation presented in the statement of Proposition 7 when $\overline{x}(t) \in (a^N, a^*)$. $\qquad\square$

We prove Theorem 2 on the Finite Time Condition.

Proof. [of Theorem 2]: By Propositions 5 and 7, the optimal plan $\overline{x}(t)$ satisfies the following conditions:

(i) it lies in $[a^N, a^*]$ for all t,

(ii) it is continuous in t,

(iii) it follows the differential equation $dx/dt = f(x)$ if $x \in (a^N, a^*)$, and

(iv) it starts with Nash action a^N at $t = 0$.

We first show that the existence of a non-trivial optimal plan implies the Finite Time Condition. Properties (i), (ii) and (iv) imply that, if $\overline{x}(t)$ is non-trivial (i.e., not equal to the Nash action a^N for all t), then $\overline{x}(t^0) = a^0 \in (a^N, a^*)$ for some $t^0 > 0$ and some a^0. At this point the optimal plan satisfies the differential equation $dx/dt = f(x)$ by (iii). By A2 and A5, $f(x) = \frac{\lambda \left(d(x) + \pi(x) - \pi^N \right)}{d'(x)} > 0$ (recall that $d(a^N) = 0$ and $\pi(a^N) = \pi^N$), when $x \in (a^N, a^*)$. Hence, once the optimal plan departs from the Nash action a^N, it is strictly increasing and never goes back to a^N. Given that the optimal plan starts with a^N, this implies the following. First, the plan stays at the Nash action for some time interval $[0, t^N]$ (this interval may be degenerate: t^N may be equal to 0). Second, after this time interval, the plan is continuous and strictly increasing with $\overline{x}(t^N) = a^N < \overline{x}(t^0) = a^0$. Therefore, on $[t^N, t^0]$, function $\overline{x}(t)$ has a continuous inverse

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that we denote by $t(x)$, and its derivative is defined on $(t^N, t^0]$ and equal to $\frac{dt}{dx} = \frac{1}{f(x)}$. This implies that $\lim_{a \downarrow a^N} \int_a^{a^0} \frac{dt}{dx} dx = \lim_{a \downarrow a^N} (t(a^0) - t(a)) = t(a^0) - t(a^N)$, where the last equality follows from the continuity of the inverse function $t(\cdot)$. By definition, $t(a^0) = t^0$ and $t(a^N) = t^N$, and therefore $\lim_{a \downarrow a^N} \int_a^{a^0} \frac{dt}{dx} dx < \infty$ holds. In addition, since $f(x)$ is Lipchitz continuous over $[a^0, a^*]$, the differential equation $\frac{dx}{dt} = f(x)$ with an initial condition $x(t^0) = a^0$ has a unique solution, and $\overline{x}$ is equal to such a solution. Hence, letting $\hat{t}$ be $\overline{x}(\hat{t}) = a^*$, we obtain $\hat{t} < \infty$. Hence, $\int_{a^0}^{a^*} \frac{1}{f(x)} dx = \hat{t} - t^0 < \infty$. Overall, we conclude that

$$\lim_{a \downarrow a^N} \int_a^{a^*} \frac{dt}{dx} dx = \left(\lim_{a \downarrow a^N} \int_a^{a^0} \frac{dt}{dx} dx \right) + \int_{a^0}^{a^*} \frac{dt}{dx} dx < \infty,$$

so the Finite Time Condition (7) holds.

Next, we show that the Finite Time Condition implies that the optimal plan is non-trivial. Choose any $a^0 \in (a^N, a^*)$. By Assumption A6 (the Lipschitz continuity), the differential equation $dx/dt = f(x)$ with boundary condition $x(0) = a^0$ has a unique solution, denoted by $x^\varepsilon(t)$, on $(a^N + \varepsilon, a^*)$ for any small enough $\varepsilon > 0$. By the same argument as above, our assumptions ensure $dx/dt = f(x) > 0$ for $x \in (a^N, a^*)$. Define

$$t^* := \lim_{\varepsilon \to 0} \int_{a^N + \varepsilon}^{a^0} \frac{1}{f(x)} dx < \infty,$$

where the finiteness follows from the Finite Time Condition (7). The above argument shows that there is a solution to the differential equation $x(t)$ such that $x(0) = a^0$ and $x(t) \downarrow a^N$ as $t \to -t^*$. Shift the origin of time and construct a new plan $y(t) := x(t - t^*)$. The new plan is also a solution to the differential equation, and it satisfies $y(t^*) = a^0$ and $y(t) \to a^N$ as $t \to 0$. Now, construct another plan $z(t)$ by suitably extending $y(t)$:

$$z(t) = \begin{cases} a^N & \text{if } t = 0 \\ y(t) & \text{if } t \in (0, t^*] \\ a^0 & \text{if } t > t^* \end{cases}$$

This plan satisfies the trigger strategy incentive constraint (3): the incentive constraint is binding on $[0, t^*]$ (because it satisfies the differential equation), and for $t > t^*$ the incentive constraint is satisfied with strict inequality. Hence, $z(t)$ is a non-

trivial trigger strategy equilibrium. This implies that the optimal trigger strategy equilibrium is non-trivial. □

Lastly, we prove Corollary 1.

Proof. [of Corollary 1]: Recall that the optimal plan $\bar{x}(t)$ satisfies conditions (i)-(iv) in the proof of Theorem 2. It turns out that there are multiple plans which satisfy those conditions. For example, *trivial constant plan* $x(t) \equiv a^N$ satisfies those conditions. In what follows, we identify all plans that satisfy conditions (i)-(iv) and find the optimal one among them.

The proof of Theorem 2 shows that there is a solution to the differential equation $x^*(t)$ that satisfies $x^*(t(a^*)) = a^*$ and $x^*(t) \to a^N$ as $t \to 0$. From $x^*(t)$, construct the following plan

$$
x_\tau(t) := \begin{cases} a^N & \text{if } t \in [0, \tau] \\ x^*(t - \tau) & \text{if } t \in (\tau, \tau + t(a^*)) \\ a^* & \text{if } t \in [\tau + t(a^*), \infty) \end{cases} .
$$

This plan $x_\tau(t)$ departs from a^N at time τ, follows the differential equation, and then hits the optimal action a^* and stays there (In a revision game with time horizon T, we must consider the restriction of $x_\tau(t)$ on $[0, T]$).

Those plans $x_\tau(t), \tau \geq 0$ obviously satisfy (i)-(iv). Next we show the converse: any plan satisfying (i)-(iv) is equal to $x_\tau(t)$ for some $\tau \in [0, \infty]$. This comes from the standard result in the theory of differential equation: $dx/dt = f(x)$ defined on an open domain $(x, t) \in (a^N, a^*) \times (-\infty, \infty)$ has a unique solution given any boundary condition, if $f(x)$ is Lipschitz continuous. The uniqueness of the solution then implies that any plan satisfying (i)-(iv) is equal to $x_\tau(t)$ for some $\tau \in [0, \infty) \cup \{\infty\}$.[50]

Among the plans $x_\tau(t), \tau \in [0, \infty]$ the one that departs from a^N immediately (i.e., $x_0(t)$) obviously has the highest payoff. Therefore, the optimal plan is given by the restriction of $x_0(t)$ on $[0, T]$, which has the stated properties in Corollary 1. □

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[50]A formal proof goes as follows: The trivial path, which satisfies (i)-(iv), is equal to x_τ with $\tau = \infty$. Consider any non-trivial path $x^0(t)$ that satisfies (i)-(iv), where $x^0(t^0) =: a^0 \in (a^N, a^*)$ for some t^0. Define $t' := t^0 - \lim_{a \downarrow a^N} \int_a^{a^0} \frac{1}{f(x)} dx$ (which is finite by the Finite Time Condition), so that $x^*(t - t')$ hits a^0 at $t = t^0$. The uniqueness of the solution to the differential equation (for boundary condition $x(t^0) = a^0$) implies $x^0(t) = x^*(t - t')$ for $t \geq t'$. If $t' \geq 0$, we obtain the desired result $x^0(t) = x_\tau(t)$ for $\tau = t'$. If $t' < 0$, $x^0(0) = x^*(-t') > a^N$ and $x^0(0)$ cannot satisfy (iv) ($x^*(-t') > a^N$ leads to a contradiction because we are considering the case $a^N < a^*$).