Coordination in Games with Private Timing*

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Abstract

We study a class of games in which the timing of players' moves is private information, but players have the option to disclose their actions by incurring a small cost. Payoffs net of a disclosure cost are as in a common-interest game. We show that every such dynamic game has the unique prediction that players coordinate on the efficient equilibrium while not disclosing their action if and only if the timing distribution features a form of asynchronicity and uncertainty. We allow players to choose their moving time as well and find an analogous uniqueness result.

Keywords: common-interest games, asynchronous moves, private timing, dynamic games

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1 Introduction

In numerous social and economic situations, knowledge about timing matters. A firm may want to conduct a costly investigation of the pricing strategy of its competing firm only if the rival already had an internal meeting to determine such a strategy. A salesperson at an appliance store may change her sales pitch depending on whether the customer has already visited another store and could benefit from communication from that store. Investors may want to condition their funding decisions for a start-up company on whether other investors had enough time to make their investment decisions for the company. Knowing this, early investors may want to disclose to others—if possible—that they invested. In all these situations, choices of actions depend on what one believes about the timing of the choices by other actors, and on the communication technology available.

This paper proposes a new class of games that we call *games with private timing* to analyze how uncertainty about timing affects economic interactions. In particular, these games allow us to model dynamic situations in which the players have uncertainty about their order of moves, moves are unlikely to take place at the exact same time (timing may be fine) and players are able to communicate information about their chosen action to an opponent. These assumptions hold in various business and economic environments including the aforementioned examples.

Games with private timing have an underlying two-player normal-form game which we call the *component game*. In particular, we focus on component games that are of *common-interest*. These are games that have a "best action profile," which strictly Pareto-dominates all other action profiles.[1] Each of two players learns privately the time at which they are able to choose an action, once and for all, in the component game. In doing so, they face uncertainty regarding the time at which the opponent is able to choose her action. If there is no information disclosure between periods, then these games are strategically equivalent to simultaneous-move games. We assume, however, that each player has the option to disclose her own action to the opponent at a small cost, and that the opponent may use the information if he has not yet moved. The main model assumes

[1]A common example of a common-interest game is the Stag Hunt game.

that players *do not* choose their time to move. Rather, the moving times are exogenously given.

Our main result shows that, under certain timing assumptions that we explain below, all such games have a unique equilibrium outcome when the component game is of common-interest. In this unique equilibrium outcome, players coordinate on their best action profile *implicitly*, i.e., without communicating anything to their opponent. In contrast, in the simultaneous move version of the game, it is possible, in equilibrium, that players coordinate on a Pareto-dominated equilibrium of the component game. Interestingly, a game with private timing may look "in practice" as a simultaneous move game, as players do not exchange information in equilibrium. However, once one accounts for the underlying timing of moves and informational structure explicitly the predictions of the game can be starkly different.

More specifically, we characterize the set of timing distributions D such that:

$$\text{timing distribution } p \in D \quad \Longleftrightarrow \quad \begin{array}{l} \forall \text{common-interest game } S, \quad \exists \text{small enough cost } c \\ \text{s.t. the dynamic game } (S, p, c) \text{ has a unique PBE.} \end{array}$$

Furthermore, in the unique equilibrium, *players choose the best coordination equilibrium of the component game and do not disclose their action.*

The distributions of timing that we identify (i.e., those in the set D) satisfy two conditions. The first condition is what we call *strong asynchronicity*, which requires that players' moving times be asynchronous—allocating zero probability weight to simultaneous moves—and rule out a strong type of correlation of moving times that is akin to synchronicity. Second, players must have sufficient uncertainty about their opponent's moving time. To be precise, we require that a player cannot know with certainty that they are the last player to move (*potential leader property*), or, if a player does, she must assign positive probability to the event that the opponent has thought he would be a second-mover with a very high probability (*unlikely leader condition*). Intuitively, such conditions hold in settings in which time is "fine" and there is any, however small, level of inherent uncertainty regarding a player's ability to act at a given time. For example, in real-world settings, players are subject to uncertain response times, their attention is scarce, drawn by multiple competing sources, or else they need an uncertain amount of time to process information that allows them to make a decision.[2]

The second set of results analyzes games in which players may choose their moving times but, as before, their choice of action and timing is private unless it is disclosed at a small cost. We define a concept of trembling hand perfect equilibrium with uncertainty about timing (THPE) in which the players are relatively more likely to err in their choice of moving time than they are in their action and disclosure decisions. We show that the dynamic game with an endogenous timing has a unique THPE if the players' first possible moving times do not coincide and, for each possible moving time of a player, the opponent has the option to choose a later moving time. In the unique equilibrium, the players take their Pareto dominant action and do not pay to disclose it. Our results in this section do not require that the players be unable to move simultaneously as in the exogenous move case. Under a strategy profile in which players move simultaneously and play an inefficient action profile, *players would choose to move asynchronously* instead, in order to coordinate on the efficient action profile.

To illustrate our model and results, consider the problem of two firms deciding on a product's concept and design. Each firm has its own strength, and hence would like to target, through its design, one specific segment which differs from the other firm's preferred segment. If they target different segments, they can avoid direct competition. The game between the two firms can be expressed as a common-interest game in which the preferred equilibrium is that each firm targets their preferred segment.

The firms face uncertainty about the time at which the opponent decides on its product design. Such uncertainty may arise exogenously or endogenously. In some situations, the opponent itself may be facing uncertainty about its own timing for some exogenous reasons. For example, the time necessary to develop prototypes may be stochastic, and delaying decisions may not be an option if the firm needs to secure future funding or needs to have cash in hand. In other settings, however, firms may have control over timing so uncertainty arises endogenously from a

[2] In our arguments, asynchronicity implies less strategic uncertainty about the opponent's choice than synchronicity when the players have the choice to disclose their actions. Uncertainty about the timing distribution implies, in practice, less freedom on allowable beliefs at information sets. More specifically, it rules out instances of equilibrium multiplicity produced by freedom on the off-path beliefs.

strategic reason. In either case, our result then predicts that if the firms can talk to the opponent firm at a small cost,[3] and as long as the timing and information structure features the conditions as explained above, they will each target the segment they prefer without resorting to communications.

The result sheds light on the implementation of antitrust law. Explicit collusive agreements, such as price-fixing agreements, are *illegal per se*. In addition, in some cases, informal practices that do not involve explicit communication about prices or other decisions may be deemed illegal as well. Our result shows that the mere existence of the possibility of communication, even if it is not used, can induce firms to coordinate in a unique equilibrium, hence suggesting a limitation of sole reliance on evidence of explicit collusion. Note that the uniqueness makes this claim strong, compared to, say, an argument that relies on a repeated-games reasoning where collusion is just one equilibrium out of many possibilities.

The present paper is part of the literature that tries to understand the relationship between timing and economic behavior. As discussed, asynchronicity and uncertainty are the keys to our results. The role of asynchronicity in equilibrium selection is present in the literature. Lagunoff and Matsui (1997) consider asynchronous repeated games and show a uniqueness result for games in which all players have the same payoff function.[4]Caruana and Einav (2008) consider a finite-horizon model with switching costs and show that there is a unique equilibrium under asynchronicity. Calcagno et al. (2014) show uniqueness of equilibrium in a finite-horizon setting with asynchronicity and a stage game that is a (not necessarily perfect) coordination game.[5] Similar to our work, in these papers asynchronicity is important for the selection of an equilibrium. However, their informational setting differs from ours: while those papers assume perfect information, we assume that players may not observe the actions taken by their opponent. In fact, our result shows that if the component game is a common-interest game, under our assumptions, players do not observe any action by the opponent on the equilibrium path.

Our paper adds to the literature on uncertainty about timing. Kreps and

[3]This cost may arise from a small probability of detection by the anti-trust agency.

[4]See also Yoon (2001), Lagunoff and Matsui (2001), and Dutta (1995).

[5]Ishii and Kamada (2011) further examine the role of asynchronicity by considering a model with a mix of asynchronous and synchronous moves.

Ramey (1987) provide an example of an extensive-form game in which players do not have a sense of calendar time and do not know which player moves first. They argue that such situations may naturally arise in reality and show that they may give rise to a new issue in specifying players' beliefs at off-path information sets. Matsui (1989) considers a situation involving private timing in a context quite different from ours: He considers an espionage game in which, with a small probability prior to the infinite repetition of the stage game, a player can observe the opponent's supergame strategy and revise her own supergame strategy in response to it, but whether there has been such a revision opportunity is private information. The equilibrium strategies in his model have a similar flavor to the costly-disclosure option in our model, in that they use a supergame strategy in which a player signals to the opponent that he has been able to observe the opponent's strategy, by taking an action that is costly in terms of instantaneous payoffs. Our paper adds also to the more recent literature in which players have uncertainty about the extensive form of a game. Nishihara (1997), Salcedo (2017), and Doval and Ely (2020) find in their respective settings that allowing the timing to be private may, together with a cleverly-crafted information structure, expand the set of equilibria. In contrast, we prove uniqueness under private timing.

Our model studies how timing affects behavior, but some papers have analyzed how behavior affects timing. Ostrovsky and Schwarz (2005, 2006) consider models in which players can target their activity times but their choices are subject to exogenous noise, which results in uncertainty. Park and Smith (2008) consider a timing game in which players choose their timing to be on the right "rank" in terms of moving times, and the equilibrium strategies entail mixing. Thus uncertainty about timing endogenously arises as a result of mixing by the players. Our paper differs in that players can change their actions depending on their exogenously given moving time and observation at that point. Such conditioning, which seems to fit to the real-life examples that we mentioned, is not present in the aforementioned papers. We note that we also consider the case of endogenous moving times in Section 4. In particular, our results imply that in a game in which players choose their moving times, every trembling hand perfect equilibrium with uncertainty about timing features coordination on the best Nash equilibrium of the component game without action disclosure.

The organization of the paper is as follows. Section 2 provides the model. Section 3 presents our main results. In Section 4, we consider the case where players have a choice of moving time. Section 5 provides discussions to deepen the understanding of our results. Section 6 concludes. The Appendix contains proofs that are not provided in the main text and the Online Appendix provides additional discussions.

2 Model

Component Game The component game is a strategic-form game $S = (N, (A_i)_{i\in N}, (g_i)_{i\in N})$, where $N = \{1, 2\}$ is the set of players, A_i is player i's finite action space, and $g_i : A \to \mathbb{R}$ is player i's payoff function, where $A := A_1 \times A_2$.[6] In the bulk of the paper, we focus on component games that are *common-interest games*. These are normal-form games in which there is a Nash equilibrium a^* such that $g_i(a^*) > g_i(a)$ for all $a \neq a^*$. The action profile a^* in this case is called *best action profile*.

Dynamic Game In the dynamic game, time progresses in an ascending manner. There is a countable set of times $\mathcal{T} \subset \mathbb{R}$, and each player moves once at a stochastic time $T_i \in \mathcal{T}$ which is drawn by Nature according to a commonly-known probability mass function $p(T_1, T_2)$. A countable $\mathcal{T}$ can accommodate settings in which time is fine, if for example $\mathcal{T} = \mathbb{Q}$, or discrete, if for example $\mathcal{T} = \mathbb{Z}$.[7] For any pair of events E and F such that F has positive probability, let $\text{Prob}^p(E|F)$ be the conditional probability of E given F induced by p. Let $\mathcal{T}_i = \text{supp}(T_i)$. Given a realization of times (t_1, t_2), player i chooses an element from $A_i \times \{\text{pay}, \text{not}\}$ at time t_i, after observing her own t_i, and additionally (a_j, t_j) if $t_j < t_i$ and the opponent

[6] In most of the paper, we focus on the two-player case except in some of the analysis, such as the existence result (Proposition 5), where we consider n players. See footnote 12 for a further discussion on the case of n players.

[7] We chose this countable-time setting instead of one with continuous time in order to avoid non-essential technical burdens associated with measure theoretic issues. We note that we do not restrict $\mathcal{T}$ to be of the form $\mathcal{T} = \{t_k\}_{k\in\mathbb{N}}$ with $t_{k_1} < t_{k_2}$ if $k_1 < k_2$. For example, $\mathcal{T}$ may be dense in $\mathbb{R}$ like the rational numbers. A further discussion on this can be found in Section 5.5. Also, our analysis is independent of how $\mathcal{T}$ is "embedded" into $\mathbb{R}$. However, viewing $\mathcal{T}$ as a subset of $\mathbb{R}$ helps us understand and interpret our conditions and results, so we let $\mathcal{T} \subset \mathbb{R}$.

j chose (a_j, pay) at t_j.[8] If player i chooses (a_i, pay) for some $a_i \in A_i$, then she pays the cost $c > 0$. We denote by $\Gamma = (S, \mathcal{T}, p, c)$ the complete specification of the dynamic game. We will omit the reference to Γ whenever there is no room for ambiguity.

Strategies A history h is composed of a sequence of times and action choices by all players:

$$h = (t_i, a_i, d_i)_{i \in N} \in \times_{i \in N} [\mathcal{T}_i \times A_i \times \{\text{pay}, \text{not}\}]$$

where t_i is the moving time of player i and $(a_i, d_i) \in (A_i \times \{\text{pay}, \text{not}\})$ is the profile of the action and disclosure decision of player i at that time. If $h = (t_i, a_i, d_i)_{i \in N}$ is such that $p(t_1, t_2) > 0$, we say that h is **feasible**. Let $\mathcal{H}$ be the set of feasible histories.

Player i's private history at her moving time, h_i, is defined as an element in

$$[(\mathcal{T}_{-i} \times A_{-i}) \cup \{\emptyset\}] \times \mathcal{T}_i.$$

The interpretation is that a history takes the form of $((t_{-i}, a_{-i}), t)$ when i moves at time t and observes $-i$'s disclosure of his action a_{-i} at time t_{-i}, while if it takes the form of $(\emptyset, t)$, then i moves at time t and observes no disclosure.

We say that a history $\tilde{h} = (\tilde{t}_j, \tilde{a}_j, \tilde{d}_j)_{j \in N}$ is **compatible** with a private history $\hat{h}_i$ if (i) $\hat{h}_i = (\emptyset, \tilde{t}_i)$ and either $\tilde{t}_i \le \tilde{t}_{-i}$ or $\tilde{d}_{-i} = \text{not}$, or (ii) $\hat{h}_i = ((\tilde{t}_{-i}, \tilde{a}_{-i}), \tilde{t}_i)$, $\tilde{t}_{-i} < \tilde{t}_i$ and $\tilde{d}_{-i} = \text{pay}$. That is, a history is compatible with a player's private history if it is not ruled out by the player's own observation, contained in her private history.

The set of all possible private histories that have some feasible history compatible with them is denoted $\mathcal{H}_i := \{h_i | \exists h \in \mathcal{H} \text{ s.t. } h \text{ is compatible with } h_i.\}$.

Player i's strategy, $\sigma_i : \mathcal{H}_i \to \Delta(A_i \times \{\text{pay}, \text{not}\})$, is a map from private histories to probability distributions over A_i and disclosure decisions. Let Σ_i be the set of all strategies for player i. Define $\Sigma = \times_{i \in N} \Sigma_i$.

[8] We assume that disclosures succeed with probability 1. Online Appendix B.1 discusses the case in which disclosures fail with positive probability.

Payoffs To summarize the specifications of payoffs described so far, if the chosen action profile is a and i chooses $d_i \in \{\text{pay}, \text{not}\}$, then her overall payoff is

$$g_i(a) - c \times \mathbb{I}_{d_i=\text{pay}}$$

with $c > 0$ which we assume to be common across players.[9] The expected payoff for player i from strategy profile σ is denoted by $u_i(\sigma)$.[10] A *belief* $\mu \in \Delta(\mathcal{H})$ is a probability measure over histories. A continuation payoff $u_i(\sigma|\mu, t)$ is well defined given that (i) the distribution of the past play at times strictly before t is given by the belief μ, (ii) the distribution of moving times at and after time t is given by μ, and (iii) the play at and after time t is given by σ.[11]

Equilibrium Notion A strategy profile σ induces a probability distribution over the set of histories $\mathcal{H}$. Let $\mathcal{H}(\sigma)$ be the set of histories that have positive probability given σ. The strategy profile σ is a *weak perfect Bayesian equilibrium* (henceforth we simply call this a "*PBE*") if, for each player i, the following two conditions hold:

1. (On-path best response) $u_i(\sigma) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Sigma_i$.

2. (Off-path best response) For each $h_i \in \mathcal{H}_i$ such that i moves at t, there exists

[9] As will become clear, the assumption that the disclosure cost c does not vary across players is imposed only for notational simplicity and is not crucial for any of our results, which pertain to small costs. Also, if $c = 0$, then we would obtain multiple equilibria because, when the opponent is playing a_{-i}^*, (a_i^*, pay) would now yield the same payoff as (a_i^*, not). To make our exposition succinct, we assume $c > 0$ throughout the paper.

[10] Player i's expected payoff given strategy σ is given by

$$u_i(\sigma) = \sum_{(t,t')\in\mathcal{T}^2} p(t,t')\mathbb{E}_\sigma \left[g_i(a^{(t,t')}) - c \times \mathbb{I}_{d_i^{(t,t')}=\text{pay}} \middle| T_1 = t, T_2 = t' \right],$$

where $(a^{(t,t')}, (d_1^{(t,t')}, d_2^{(t,t')})) \in A \times \{\text{pay}, \text{not}\}^2$ denotes a choice in the support of σ at moving times (t, t'). The expectation is taken over probabilities over actions and disclosure choices induced by σ at (t, t').

[11] Note that, absent a belief, a pair of a strategy profile and a private history (together with Bayes rule) does not necessarily determine the continuation payoff. For example, it may be that a given strategy profile assigns probability one to the event that j discloses before time t and the given private history of i at time t does not include a disclosure by j.

$\mu \in \Delta(\mathcal{H})$ such that every $h \in \text{supp}(\mu)$ is feasible and compatible with h_i, and $u_i(\sigma|\mu, t) \geq u_i(\sigma_i', \sigma_{-i}|\mu, t)$ for all $\sigma_i' \in \Sigma_i$.

That is, we require optimality on the equilibrium path of play, while off the path we only require optimality against *some* (possibly correlated) distribution over the strategy profile of the opponents and Nature's moves that is compatible with the observation.[12] Note that condition 1 implies that players best-respond to the beliefs computed by Bayes rule on the equilibrium path. In Section 5.4, we discuss what would happen if we did not impose condition 2. Existence of a PBE is not trivial because the support of the times of play, $\mathcal{T}$, may not be finite. We show existence in Section 5.1.

3 Unique Equilibrium in Common-Interest Games

3.1 An Example

Here we consider a simple example that provides part of the intuition for our uniqueness result. For each $\varepsilon > 0$, consider a timing distribution p^ε such that there exists a probability mass function f^ε over the possible moving times $\mathcal{T} = \mathbb{Z}$ with an associated cumulative distribution function F^ε, satisfying[13] (i) $p^\varepsilon(t_1, t_2) = f^\varepsilon(t_1) \cdot f^\varepsilon(t_2)$ for all $t_1, t_2 \in \mathcal{T}$ and (ii) $0 < \frac{f^\varepsilon(t)}{1-F^\varepsilon(t-1)} \leq \varepsilon$ for every $t \in \mathcal{T}$. That is, the moving times are independently and identically distributed across players, and each player's distribution of moving times has full support and a hazard rate that is bounded from above by ε. For example, f^ε could be a geometric distribution on

[12] This allows for correlated beliefs over Nature's moves and the opponent's deviations. This weak notion is enough to establish uniqueness in the two-player games that we consider in our main analysis.

[13] One may wonder if it is plausible to have a support that is unbounded on the left, which may suggest there is no "beginning" of the dynamic game. However, it is for simplicity that we define the support to be $\mathbb{Z}$. In fact, the support of the timing distribution does not have to be unbounded on the left or the right for the argument to go through. This is because we can equivalently define a dynamic game in which there are countably many moving times in a bounded interval. For instance, the example works with a support of moving times $\{\ell(x)|x \in \mathbb{Z}\}$ defined by a mapping $\ell : \mathbb{Z} \to (-1, 1)$ with $\ell(x) = 1 - e^{-x}$ for $x \geq 0$ and $\ell(x) = e^x - 1$ for $x < 0$. See footnote 7 as well.

	A	B
A	$2,2$	$0,0$
B	$0,0$	$1,1$

Figure 1: A coordination game $\hat{S}$

both the positive and negative integers, with hazard rate ϵ.

Consider the payoff matrix in Figure 1 and let $\hat{S}$ denote this coordination game. We can show the following:

Proposition 1. *There exist $\bar{\varepsilon} > 0$ and $\bar{c} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and $c \in (0, \bar{c})$, there is a unique PBE in the game $(\hat{S}, \mathbb{Z}, p^{\varepsilon}, c)$. On the path of the unique PBE, each player i takes (A, not) for any realization of T_i.*

Proposition 1 implies that by having the option to disclose their actions, players are able to coordinate on the best action profile (A, A) without exercising the option to disclose.

The result requires that the disclosure cost $c > 0$ be sufficiently small. In fact, the proof we will present goes through as long as $0 < c < g_i(A, A) - g_i(B, B) = 1$. Remark 4 discusses the case of large disclosure cost.

To illustrate the subtleties of the result in Proposition 1, let us provide three examples of alternative timing distributions that yield multiple equilibria.[14]

Example 1. [Simultaneous-Move Game]

Suppose that $\mathcal{T} = \{1\}$ and $p(1, 1) = 1$. That is, it is common knowledge that, at time 1, both players take actions with probability one. The component game is as in Figure 1 and c is any strictly positive real number. First, no player pays the disclosure cost in any PBE because even if i pays, $-i$ does not have a chance to move after observing it. Hence, the game is strategically equivalent to the static simultaneous-move game. There are three Nash equilibria in such a game, namely (A, A), (B, B) and a mixed equilibrium, and each of them corresponds to a PBE of the dynamic game. □

[14] We note that the constructions of multiple equilibria do not rely on the wide freedom in belief choice allowed by condition 2 of the definition of PBE. Indeed, the PBE we construct are sequential equilibria as well.

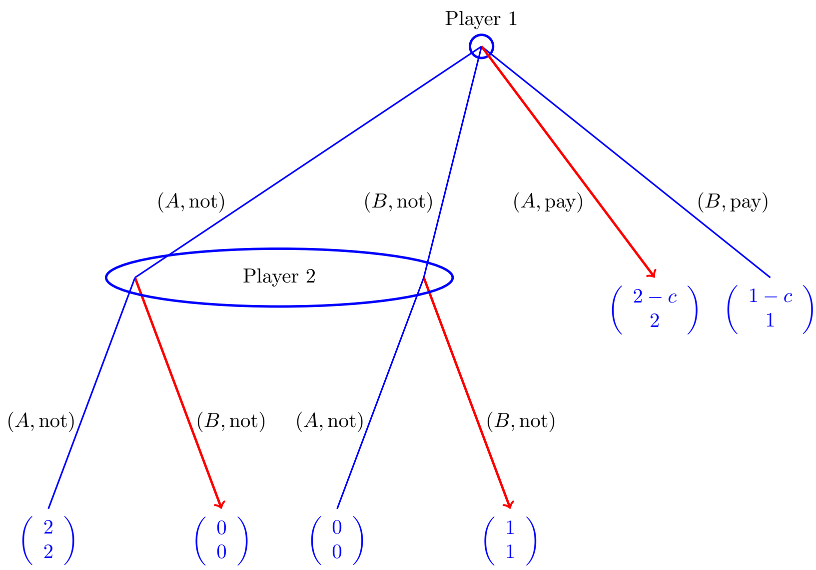

Figure 2: A forward-induction argument

Example 2. [Deterministic Sequential-Move Game (and Forward Induction)]

Suppose $\mathcal{T} = \{1, 2\}$, $c < 1$, and $p(1, 2) = 1$. That is, it is common knowledge that player 1 moves at time 1 and player 2 moves at time 2 with probability one. The component game is as in Figure 1. There are at least two PBE in this game. In the first PBE, each player plays (A, not) on the path of play. In the event that 2 observes 1's action, 2 takes a static best response. The second PBE is what we call the *pessimist equilibrium*. In this equilibrium, player 1 plays (A, pay), and player 2 plays a static best response if 1 discloses her action, while he plays (B, not) if 1 does not.

Let us check that this second strategy profile constitutes a PBE. First, player 1 takes a best response given 2's strategy. Also, 2's strategy obviously specifies a best response after 1's disclosure. After no disclosure, (B, not) is a best response under the belief that 1 has played (B, not).

Let us note that this pessimist equilibrium would be ruled out by a so-called "forward induction" argument. To see this point, consider the extensive-form representation in Figure 2 of the game in consideration. Note that we omitted actions corresponding to player 2's "pay," as they are obviously suboptimal. We also omitted 2's actions following 1's payment, as it is a unique best response to

	A	B
A	$3,3$	$-3,0$
B	$0,-3$	$1,1$

Figure 3: A risky common-interest game

play (A, not) after (A, pay) and to play (B, not) after (B, pay). The payoffs after 1's payment are written assuming 2's best response. In this game, for player 1, (B, not) yields a payoff of at most 1 and thus is dominated by (A, pay) when $c > 0$ is small, as the latter gives $2 - c$. On the other hand, (A, not) is not dominated. A forward induction argument would then dictate that at player 2's information set, his belief must assign probability 0 to the right node. Given this belief, player 2's unique best response at the information set is to play (A, not). Hence, 1 can obtain the best feasible payoff in the game by playing (A, not), and thus it must be the unique action that can be played by player 1.

Our private-timing game rules out such an outcome *without* resorting to a "forward induction" argument. Still, we will see that the proof relies on a similar idea.[15] $\square$

Example 3. [Correlated-Move Game]

Suppose that $\mathcal{T} = \mathbb{Z}$. For all $t_1 \in \mathbb{Z}$, p satisfies $\sum_{t_2 \in \mathbb{Z}} p(t_1, t_2) = \frac{1}{2}\frac{1-r}{r} r^{\frac{|t_1|+1}{2}}$ with $r \in (0,1)$ for odd t_1, and $\sum_{t_2 \in \mathbb{Z}} p(t_1, t_2) = 0$ for even t_1. We also assume that $\text{Prob}^p(T_2 = t_1 - 1 | T_1 = t_1) = \text{Prob}^p(T_2 = t_1 + 1 | T_1 = t_1) = \frac{1}{2}$. That is, T_1 is positive with probability $\frac{1}{2}$, negative with probability $\frac{1}{2}$, and follows a geometric distribution with rate r over the odd integers on each side of zero. Player 2's moving time is either right before or right after player 1's, with equal probability. These conditions imply that $\text{Prob}^p(T_1 = t - 1 | T_2 = t) : \text{Prob}^p(T_1 = t + 1 | T_2 = t) = 1 : r$ for all even $t \geq 2$ and an analogous condition holds for all even $t \leq -2$ (the ratio is $1:1$ if $t = 0$). The component game is given in Figure 3.

There are at least two PBE in this game when $r < 1$ is sufficiently close to 1. In the first PBE, each player plays (A, not) on the path of play. In the event that player i observes the opponent j's action, i takes the static best response. The second PBE is one in which each player plays (B, not) on the path of play. In the event that i observes j's action, i takes the static best response.

[15] See Remark 2 for discussion on this similarity.

The second strategy profile is a PBE because a deviation by player 1 to A can only be profitable if it involves disclosure, in which case she would succeed in coordination with probability 1/2 and miscoordinate with player 2 with probability 1/2. Hence, the expected payoff from the deviation is $\frac{1}{2}\cdot 3+\frac{1}{2}\cdot(-3)-c=-c$, which is less than the payoff of 1, obtained under no deviation. A similar reasoning applies to player 2's incentive when $r<1$ is sufficiently close to 1 that the probability that 1 moves before or after 2 is close to $\frac{1}{2}$ at every possible moving time of hers.

The key difference relative to the game in Proposition 1 is that, at any realization of a player's moving time, the probability that the player assigns to being the first mover is only (close to) 1/2 . In the game in Proposition 1, however, this probability becomes arbitrarily close to 1 as $t \to -\infty$, due to the independence of the players' moving times.[16] $\square$

Proof Sketch of Proposition 1

Here we provide a rough sketch of the proof of Proposition 1. For a formal proof we refer to that of Theorem 3, as Proposition 1 is a corollary of that theorem.[17] The proof consists of two steps. In the first step, we prove that players must play A on the path of play of every PBE. The second step shows that no player pays the disclosure cost.

- *First Step: A is chosen at every moving time.* Suppose that player i gets her move at time t and she has observed no action disclosed. The assumption that the timing distribution is independent across players and the probability of simultaneous moves is small implies that, if t is early enough, the probability that she assigns to the event that the opponent moves later is close to 1. At such t, the expected payoff from playing (A, pay) is close to $2-c$, while that from playing (B, pay) or (B, not) is at most 1. Hence, at t, player i does not play B upon no observation in any PBE as long as $c<1$. By the

[16] In this example, any private information about a player's own moving time does not reveal sufficiently precise information about the order of moves. Online Appendix B.2 makes this point even clearer by considering an extreme case in which players do not have a sense of calendar time.

[17] In the language in Theorem 3, S is $\frac{1}{2}$-common for each player and is q-dispersed for any $q \leq 1-\varepsilon$, where ε is the upper bound of the hazard rate at a moving time, as defined in this section.

independence of the timing distribution, this is true for all times before t as well.

Now, consider the incentives of player $-i$ at time $t+1$ when he does not observe any disclosure. Player $-i$ is on the path of play because i moves after t with positive probability, and hence he uses Bayes rule to conclude that i must have chosen A at times before $t+1$ and calculate the probability of i moving at $t+1$. Specifically, this latter probability is at most $\frac{f(t+1)}{1-F(t)} \leq \varepsilon$ for any strategy profile that the players may follow.[18] Note that, since

- we have already concluded that i plays A at a moving times before $t+1$ and
- playing (A, pay) at $t+1$ guarantees that i will play A at times after $t+1$,

player $-i$'s payoff from choosing (A, pay) at $t+1$, upon observing no disclosure, is no less than $2(1-\varepsilon)-c$ against any equilibrium strategy of i. Since ε and c are small, this payoff is greater than 1. Since the payoff from playing (B, pay) or (B, not) is at most 1, at $t+1$, player $-i$ does not play B upon no observation in any PBE as long as ε and c are small enough. Applying this argument iteratively shows that, for any time, the moving player chooses A.

- *Second Step: players do not disclose their action.* Suppose that player i moves at time t, and has not observed any disclosed action. From the first step we know that i plays A at time t. If she plays (A, pay), her expected payoff is at most $2-c$. However, by the first step, regardless of i's belief about $-i$'s moving time, i knows that if she does not pay, $-i$'s action is A with probability 1.[19] Thus, the expected payoff from i's playing (A, not) is

[18] This probability is highest, and equal to $\frac{f(t+1)}{1-F(t)}$, if the opponent pays at every opportunity before $t+1$, in which case $-i$ is certain, in equilibrium, that i has not moved yet. We note that the fact that the probability is at most $\frac{f(t+1)}{1-F(t)}$ does not depend on $-i$ being on the path of play. This is because, if he were off the path of play, the conditional probability would be 0 as the only way for $-i$ to be off-path is that i deviated before time $t+1$. In such an event, the conditional probability of i moving *again* at $t+1$ must be 0.

[19] Note that it is important that there is common knowledge about which action is the "good" action. If there is incomplete information about which action profile brings about a larger payoff, then players might pay the cost to disclose when the disclosure cost is small. We analyze one

2. Hence, it is a unique best response to play (A, not). Since the choice of t was arbitrary, this shows that on the path of any PBE, at any t, player i does not pay the disclosure cost.

□

Remark 1. [**Relation to Examples**]

How does this proof relate to the three examples that we examined? In Example 1, we demonstrated that simultaneity may prevent uniqueness. In the first step of the Proof Sketch, we used the fact that the probability of simultaneous moves is small to conclude that $-i$ never plays B at time $t+1$. To obtain this conclusion, we could effectively ignore the possibility that i moves at $t+1$ because the conditional probability of such an event is small.

Example 2 illustrated the effect of a deterministic order of moves on multiplicity. In the first step of the Proof Sketch, we used the fact that the players are unsure about the opponent's moving time, to conclude that $-i$ plays A at each moving time as he knows he is on the equilibrium path. Such a conclusion cannot be obtained in Example 2: If player 2 has not had any observation, he is off the equilibrium path under the pessimist equilibrium.

Example 3 showed that multiplicity is possible under a highly correlated timing distribution. In particular, in the second PBE in Example 3, for any realized moving time, the moving player assigns a nontrivial probability to the event that the other player has already moved. We used the fact that the timing distribution is independent in the first step of the Proof Sketch, where we argued that, at early enough times when i has not observed the opponent's disclosure, she assigns only a small probability to the event that the opponent has already moved. □

Remark 2. [**Similarity to the Forward Induction Argument**]

Step 1 resembles the logic of forward induction, described in Example 2, in that action (A, pay) is used to eliminate B. However, in forward induction, the fact that 2's information set after no disclosure is reached is interpreted as containing sure information regarding 1's choice. In our model, on the other hand, the reasoning relies on uncertainty about the timing of moves. Due to the recursive argument in

such model in Online Appendix. B.3.

Step 1, player 1 chooses A on the path of play, and thus, when player 2 receives a move in the absence of a disclosure, he still entertains the possibility that 1 has not moved yet and is, therefore, on the path of play. $\square$

Remark 3. [**Lack of Lower Hemicontinuity**]

Suppose $\mathcal{T} = \mathbb{Z}$. Recall the definition of f^ε and consider a distribution parameterized by $\xi \in (0,1)$, $p^{(\xi)}$, such that

$$p^{(\xi)}(t_1,t_2) := [((1-\xi)\cdot \mathbb{I}_{\{t_1=1\}} + \xi f^\varepsilon(t_1)] \cdot [((1-\xi)\cdot \mathbb{I}_{\{t_2=2\}} + \xi f^\varepsilon(t_2)].$$

That is, under $p^{(\xi)}$, the players' moving times are independently distributed, and with probability $(1-\xi)^2$ the order of moves is as in the deterministic-move game in Example 2. When the disclosure cost $c > 0$ and hazard rate ε are small enough, for any $\xi \in (0,1)$, the same logic as in the above Proof Sketch applies to show that there is a unique PBE, and in that unique PBE each player plays (A, not).[20]

These timing distributions converge pointwise to the distribution with $p(1,2) = 1$ as $\xi \to 0$. However, as Example 2 shows, there are multiple equilibria under this limit distribution. Thus, there is a lack of lower hemicontinuity with respect to the timing distribution.[21] One additional PBE that obtains in the limit is the pessimist equilibrium. The reason for the lack of lower hemicontinuity is that in the approximating timing distributions, at each of her moving times, player i attaches positive probability to $-i$ moving after her. This condition will be formalized as the "potential leader condition" in Definition 3 in Section 3.2. $\square$

Remark 4. [**Large Disclosure Cost**]

What happens if the disclosure cost is large? In that case, there exists a PBE in which each player plays (B, not) upon no observation. The Proof Sketch of

[20]To be precise, the proof does not go through as is because player 2 expects a large conditional probability of simultaneous moves if he is called upon to move at time 1. However, we can show that player 1 chooses A at time 1 just as in step 1 of the Proof Sketch, and therefore, so does player 2. Analogously, although player 1 expects a large conditional probability of simultaneous moves at time 2, she still continues to plays A at that time.

[21]Consider, for example, the sup norm: For two timing distributions p and p', the distance between them is $d(p,p') = \sup_{t,t'\in\mathbb{R}} |p(t,t') - p'(t,t')|$. The equilibrium correspondence is lower hemicontinuous if for every equilibrium σ under p and every $\varepsilon > 0$ if $d(p,p')$ is sufficiently small, then there is an equilibrium σ' under p' such that $|\sigma - \sigma'| < \varepsilon$, where $|\sigma - \sigma'|$ is also defined as a sup norm over private histories defined both under p and p'.

Proposition 1 uses the assumption that c is small. It says that if the time is early enough, the payoff from playing (A, pay) is close to $2 - c$ and that is greater than the payoff from playing $(B, \cdot)$. But this conclusion fails if c is large. That is, for the non-communication coordination to be the unique outcome of the game, it is important that communication not be prohibitively expensive.

3.2 General Common-Interest Games

Examples 1, 2 and 3 illustrate how simultaneous moves, the lack of uncertainty about timing, and a high correlation between the timing of moves lead to multiplicity of equilibria. In this section, we ask what are exactly the timing distributions that guarantee that, in equilibrium, players uniquely coordinate in the Pareto dominant outcome. That is to say, we characterize the set of timing distributions such that the best action profile with no disclosure is the unique PBE outcome when the component game is a common-interest game.

The set is characterized by three conditions that we call the *strong asynchronicity property*, *potential leader property* and *unlikely leader property*. Let us define these properties before stating our main theorem.

Given a time $t'' \in \{-\infty\} \cup \mathbb{R}$, a set $B_j \subseteq \mathcal{T}_j$ and $q_i > 0$, let[22]

$$\mathscr{T}_i(B_j, t'', q_i) := \{t \in \mathcal{T}_i | \text{Prob}^p(T_j \in [t'', t] \cap B_j | T_i = t, T_j \geq t'') \geq q_i\}$$

That is, $\mathscr{T}_i(B_j, t'', q_i)$ is the set of i's moving times t such that, conditional on i moving at time t and j moving no earlier than t'', the probability that j has moved in B_j and in between t'' and t is no less than q_i. Thus, supposing that j does not disclose her actions when she moves in $[t'', \infty) \cap B_j$, at each time $t \in \mathscr{T}_i(B_j, t'', q_i)$, player i believes, conditional on knowing that j moved no earlier than t'', that j moved in $B_j \cap [t'', t]$ with probability at least q_i.

To illustrate, suppose $B_j = \mathcal{T}_j$, and T_1 and T_2 are independently distributed. Then, $\mathscr{T}_i(B_j, t'', q_i)$ is the set of times at which, conditional on player j moving no earlier than t'', j moves between t'' and t with a probability of at least q_i. If T_1 and T_2 are not independent, the latter probability conditions also on $T_i = t$. If

[22] We use the convention that for any two events A and E, $\text{Prob}^p(A|E) = 0$ if $p(E) = 0$, and for $a, b \in \mathbb{R}$, $[a, b] = \emptyset$ if $a > b$. .

$B_j = \{\hat{t}\}$ for some $\hat{t} \in \mathcal{T}_j$, then $\mathscr{T}_i(B_j, t'', q_i)$ is the set of times $t \geq \hat{t}$ at which the probability that j moves at time $\hat{t}$, conditional on i moving at time t and j moving no earlier than t'', is at least q_i.

Definition 1. The distribution of the timing of moves, p, satisfies the **strong asynchronicity property (SAP)** if for all $t'' \in \mathbb{R} \cup \{-\infty\}$, $q_1, q_2 > 0$ and $B_1 \subseteq \mathcal{T}_1$, $B_2 \subseteq \mathcal{T}_2$,

$$\emptyset \neq B_i \subseteq \mathscr{T}_i(B_{-i}, t'', q_i) \implies B_{-i} \not\subseteq \mathscr{T}_{-i}(B_i, t'', q_{-i}),$$

for each $i \in \{1, 2\}$.

In words, assuming players do not disclose their actions, there cannot be two sets of players' moving times B_1, B_2, a time t'' and fixed probabilities $q_1, q_2 > 0$ satisfying the following property: Conditional on i moving at any time $t \in B_i$ and $-i$ moving weakly after t'', the probability that $-i$ moves at a time in $B_{-i} \cap [t'', t]$ is bounded below by q_i.

Thus, two players, i and j, cannot simultaneously think that, at times in any pair of sets B_i and B_j, the opponent moved "before" (or at the same time as) them with a probability that is bounded away from zero. Moreover, by setting $B_i = B_j = \{t''\}$, one can see that SAP implies that players move asynchronously, i.e. $p(t'', t'') = 0$ for every $t'' \in \mathcal{T}$.

The SAP further excludes some situations in which two players move simultaneously in a certain approximate sense. To see this, first recall Example 3. The SAP fails in this example as both players think it likely that the opponent has just moved at each moving time. Formally, it fails by taking $t'' = -\infty$, $B_i = \mathcal{T}_i$, and $q_i = \frac{r}{1+r}$ for each i. Now, consider mapping this example with $\mathcal{T} = \mathbb{Z}$ to a model with $\mathcal{T} \subset [-1, 1]$ by, for instance, associating each t in the former model with time $1 - e^{-t}$ if $t \geq 0$ and with time $e^t - 1$ if $t < 0$.[23] Then, players' moving times are correlated in a way that is as if the two players move "close to simultaneously at time -1" with positive probability. The reason is that with a probability that is bounded away from zero, conditional on a player moving at time t in the transformed model, the moving time of the opponent must become closer and closer to

[23]This is a transformation that we discussed in footnote 13.

-1 as t approaches -1.

In our arguments we show that if sets B_i and B_j that make SAP fail existed, then there would be a common-interest component game in which a Pareto dominated equilibrium is played with positive probability because players believe with enough probability that the opponent already moved and played the dominated action, and therefore, coordination on the dominated equilibrium cannot be excluded from the set of PBE outcomes.

As hinted by the previous discussion, if an opponent is unlikely to move just before a player, SAP is satisfied. More formally, the following easier-to-interpret condition, which we view to be a mild and natural assumption in settings with asynchronous moving times, implies SAP. This point is shown in Proposition 2 below.

Definition 2. The timing distribution p has **the dispersed potential moves property** if for each $t'' \geq \inf(\mathcal{T})$, there is a player $i \in \{1, 2\}$ such that

$$\lim_{t \to t''} \mathrm{Prob}^p(T_{-i} \in [t'', t] | T_i = t, T_{-i} \geq t'') = 0.$$

That is, the dispersed potential moves property requires that at every time t'', there is a player i and a small enough interval $[t'', t')$, such that for every $t \in [t'', t')$, i attaches a small probability to the event that the other player moves within $[t'', t]$, conditional on i moving at time t. Thus, at least one player must find it unlikely that the opponent moved in a short time interval just before their moving time.

The dispersed potential moves property also requires that there is at least one player who assigns a small probability to the event that she is the second mover if she moves early enough (note that this property corresponds to the dispersed potential moves property when t'' is equal to $\inf \mathcal{T}$). That is, when it is early in the game, at least one player believes it is unlikely that the other player has already moved. We refer to this property as the "knowing that it's early when it's early."[24] This property is not satisfied in the timing distribution of Example 3. The "knowing that it's early when it's early" condition is satisfied if the distributions of moving times are independent across players and asynchronous.

[24]More formally, the distribution p satisfies "knowing that it's early when it's early" if the dispersed potential moves property holds for $t'' = \inf \mathcal{T}$.

Under asynchronous moves, the dispersed potential moves property holds true for $t'' > -\infty$ when we impose $\mathcal{T} \subseteq \mathbb{Z}$ (or whenever $\mathcal{T}$ is comprised of isolated points). Thus, if $\mathcal{T} \subseteq \mathbb{Z}$, the dispersed potential moves property is equivalent to requiring, in addition to asynchronous moves, that one player satisfies "knowing that it's early when it's early." SAP, dispersed potential moves and asynchronous moves are all equivalent to each other if $\mathcal{T} \subseteq \mathbb{N}$.

An important setting where the dispersed potential moves property holds is one in which moving times are asynchronous and independently distributed. The intuition is that what makes SAP fail is i's inference from her moving time that j moved right before her with sufficiently high probability (as in Example 3). When moving times are independently distributed, however, a player's moving time does not contain information about the opponent's moving time.

The following proposition formalizes the relationship between SAP and the dispersed potential moves property.

Proposition 2. *1. If p has the dispersed potential moves property, then it satisfies SAP.*

2. There are distributions that satisfy SAP that do not satisfy dispersed potential moves.

Part 2 of this proposition is shown by Example 5 that we provide in Appendix A.2.

The discussion so far is summarized in Figure 4.

Define $T_i^< = \{t \in \mathcal{T}_i | \text{Prob}^p(T_{-i} \geq t | T_i = t) = 0\}$. That is, $T_i^<$ is the set of times at which i is sure she moves after the opponent.

Definition 3. Player i is a **potential leader under** p if $T_i^< = \emptyset$.

Player $-i$ is an **unlikely leader under** p if $T_i^<$ is non-empty and for every non-empty $E \subseteq T_i^<$,[25]

$$\inf\{r > 0 |\, r = \text{Prob}^p(T_i \in E | T_{-i} = \tilde{t}) \text{ for } \tilde{t} \in \mathcal{T}_{-i}\} = 0.$$

[25]Note that, by the definition of $T_i^<$, for any non-empty $E \subseteq T_i^<$, there exists $\tilde{t} \in \mathcal{T}_{-i}$ such that $\text{Prob}^p(T_i \in E | T_{-i} = \tilde{t}) > 0$.

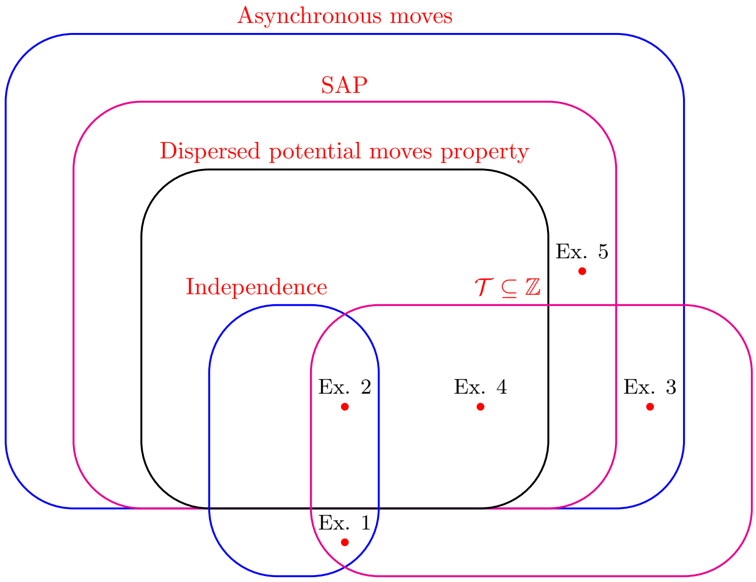

Figure 4: Comparison of timing distributions. The distributions in the examples in the paper are represented by dots.

The potential leader property is satisfied when, without any observation, no player can ever be sure that she is the follower (the second mover). This condition is not satisfied in the timing distribution of Example 2, as in the example, player 2 assigns probability 0 to being the first mover (he is not a "potential leader"). The unlikely leader condition holds if for any event in which player i is certain that she is the follower, there is a moving time of $-i$ in which $-i$ thinks it possible but very unlikely that such an event will happen, i.e., it is unlikely that $-i$ is the leader (the first mover) of such i. Note that if there is i who is an unlikely leader under p, then there must be infinitely many points in $\mathcal{T}$ before the time $\inf T_i^<$.[26]

Define the set $D \subseteq \Delta(\mathcal{T}^2)$ as follows. The distribution p is an element of D if and only if each of the following two conditions hold:

1. p satisfies SAP.

2. For each i, either one of the following two conditions holds.

   (a) i is a potential leader under p.

[26]This claim holds because, in the definition of unlikely leader, we must be able to take a sequence of $\tilde{t}$ such that $\text{Prob}^p(T_i \in T_i^< | T_{-i} = \tilde{t})$ is strictly positive and converges to 0.

(b) $-i$ is an unlikely leader under p.

Theorem 1. *Fix $\mathcal{T}$. The timing distribution p is in the set D if and only if, for any common-interest game S, there exists $\bar{c} > 0$ such that for all $c \in (0, \bar{c})$, there is a unique PBE outcome in the game $(S, \mathcal{T}, p, c)$.*[27] *In the unique PBE outcome, each player i plays (a_i^*, not) at any realization of T_i.*

Let us give an intuition for the necessity and sufficiency of SAP. For a given time t'', suppose the sets $B_i \subseteq \mathcal{T}_i \cap [t'', \infty)$ and $B_j \subseteq \mathcal{T}_j \cap [t'', \infty)$ represent the set of times at which actions other than a_i^* and a_j^* are taken with positive probability (upon no disclosure) by players i and j, respectively, starting at some $t'' \in \mathbb{R} \cup \{\infty\}$. In order for players to choose those actions, they must deem it sufficiently likely that the opponent will choose one of those actions. Under SAP, the latter can never be true. To see this, suppose that for each $k \in \{1, 2\}$, k must choose an action other than a_k^* with probability at least q_k for an action other than a_{-k}^* to be a static best response. Then, $\mathscr{T}_i(B_j, t'', q_i)$ is the set of i's moving times at which she deems it at least likely with probability q_i that j moved in B_j between t'' and the present time t. Therefore, we must have $B_i \subseteq \mathscr{T}_i(B_j, t'', q_i)$. The SAP requires that $B_j \not\subseteq \mathscr{T}_j(B_i, t'', q_j)$. That is to say, it requires that there is at least one time $\tilde{t}$ in B_j at which player j believes that i had moved in B_i between t'' and $\tilde{t}$ with probability strictly less than q_j—implying that j would not be willing to choose $a_j \neq a_j^*$ at time $\tilde{t}$. This is true for every q_i and q_j. Thus, every pair of sets, B_i and B_j, is an inadequate candidate to be the sets of times in which players should play an action other than a_i^* and a_j^*.

Conversely, if SAP does not hold, one can construct a game in which players choose an action other than a_i^* or a_j^* at the corresponding sets B_i and B_j. The idea is to construct a game in which miscoordination is sufficiently costly that there is a PBE in the dynamic game in which each player i plays $a_i \neq a_i^*$ at times in B_i because she deems it sufficiently likely that her opponent chose $a_j \neq a_j^*$ right before (or at the exact same time) at a time in B_j. The construction covers the case in which $B_i = B_j = \{t''\}$, i.e., when the players move simultaneously with positive probability.

[27] Although the outcome is unique, there may exist multiple PBE. The reason is that at off-path information sets in which the first mover chooses an action $a_i \neq a_i^*$ and discloses it, the opponent $-i$ may have multiple best responses against a_i.

The purpose of condition 2 is to rule out beliefs after off-path play that can force a player i to choose (a_i^*, pay) as in Example 2. To understand why these conditions rule out those "undesirable" beliefs, notice that failure to observe disclosure can be off-path only at times in which player i knows for sure that $-i$ has already moved (i.e., times in $T_i^<$). In such cases it is possible, as in Example 2, that upon no disclosure i chooses an action other than a_i^* in equilibrium. Condition 2a requires that the set $T_i^<$ be empty. Condition 2b, in contrast, guarantees that in the event that player i moves at a time in $T_i^<$, "no disclosure" is on the equilibrium path in every PBE: it requires that there be times in which $-i$ thought it so unlikely that i would move in any subset of $T_i^<$ that the cost of disclosure was not justified. Example 4 below illustrates the role of condition 2b.

The discussions so far imply that, if $\mathcal{T} \subseteq \mathbb{N}$, SAP is equivalent to asynchronicity, and no $-i$ can be an unlikely leader. Thus, Theorem 1 reduces to the following:

Corollary 3. *Fix $\mathcal{T} \subseteq \mathbb{N}$. The timing distribution p satisfies asynchronicity and each i is a potential leader if and only if, for any common-interest game S, there exists $\bar{c} > 0$ such that for all $c \in (0, \bar{c})$, there is a unique PBE outcome in the game $(S, \mathcal{T}, p, c)$. In the unique PBE outcome, each player i plays (a_i^*, not) at any realization of T_i.*

Remark 5. Let us make some remarks regarding Theorem 1.

1. The argument in the proof for sufficiency of the two conditions in this corollary (asynchronicity and potential leader) has some analogies with those in the starting example in Section 3.1. In the example, however, we were only concerned with sufficiency. The asynchronicity condition was not satisfied in the example but the result held because the timing distribution was "close to" the one satisfying the asynchronicity condition for a fixed component game. Section 5.2 explores the connection between the degree of commonality of the players' interest in the component game and the level of "dispersion" of the two players' moving times, where the degree of commonality is measured by the difference between each player's best payoff and her second-best payoff.

2. The proof for necessity is by construction. That is, for each timing distribution violating any of the conditions characterizing the set D, we construct an

example of a component game such that the corresponding dynamic game has multiple PBE. For example, for a distribution that puts positive probability on synchronous moves at some time t, we construct a component game and a strategy profile for the corresponding dynamic game such that an action other than a_i^* may be a best response for each player i at time t. The existence of such a strategy profile, i.e., the one featuring $a_i \neq a_i^*$ at time t and specifying a fully contingent plan at other times, is established using an argument that is analogous to the one for the general existence result in Section 5.1.

3. Notice that condition 2a requires that $t < \sup_{t' \in \mathcal{T}} t'$ hold for every $t \in \mathcal{T}$, while condition 2b requires that there be infinitely many times before t if $t = \sup_{t' \in \mathcal{T}} t'$. At least one of these conditions is necessary to guarantee uniqueness. In particular, if $\mathcal{T}$ is finite, then the uniqueness result does not hold. We make this point clear in Section 5.3.

4. A possible setting in which the theorem can be applied is one in which the analyst only knows that the players face a common-interest game, that the disclosure costs are small, and that the structure of the game is common knowledge among the players, but she does not know the exact cardinal utility of the players. The theorem identifies the conditions under which the analyst can be certain that the Pareto efficient outcome (i.e., a^* is played and no payment for disclosure takes place) is obtained. It is possible that the analyst's interest is only in the actions in the component game and not in the disclosure behavior. In the proof of Theorem 1, it is shown that SAP is a necessary and sufficient condition on the timing distribution p for a^* to be played with probability one in all PBE.

□

Example 4. [Second-Mover Game]

Suppose that the component game S is as in Figure 1. The timing distribution p over $\mathcal{T} = \mathbb{Z}$ is given by the following rule: With probability $\frac{1}{2}$, T_1 follows a geometric distribution over positive integers with parameter p, while T_2 follows a geometric distribution over nonpositive even integers with parameter p.[28]

With the complementary probability $\frac{1}{2}$, T_1 follows a geometric distribution over negative odd integers with parameter $p' < p$, while $T_2 = T_1 + 1$.

Note that the distribution p does not satisfy the potential leader condition, because player 1 always knows which event is drawn—she either moves at positive or negative integers—and in the first event she is the second mover with probability one. Player 2 does not know which event he is at because his moving time is always a negative even time. However, $p' < p$ implies that the likelihood of him being in the second event (and hence being the second mover) becomes arbitrarily close to 1 as time goes to $-\infty$. Thus, for any common-interest game, there exists $\bar{t}$ such that for all $t < \bar{t}$, it is not worthwhile for player 2 to pay the disclosure cost because player 2 is so certain that he is the second mover. This implies that, in every PBE, even though player 1 knows she is the second mover at positive times, she believes she is on the path of equilibrium play even if she does not observe any past disclosure.

Formally, the distribution p satisfies condition 2b, and by Theorem 1, the dynamic game $(S, \mathcal{T}, p, c)$ has a unique PBE outcome. $\square$

4 Choice of Moving Times

We have so far assumed that players do not have any control over when to move. However, in some situations, players may have some control over the timing of their moves. In this section, we consider games in which players can choose their timing of moves within a fixed set. We show that, under mild conditions on the set, the efficient action profile of the component game is played without disclosure under a trembling-hand perfect equilibrium refinement in which trembles on moving times are sufficiently more likely than those on actions.

To begin, let us first consider a setting in which players can choose their moving time but are not subject to uncertainty about timing. For each $i = 1, 2$, fix a

[28]If a random variable T is distributed according to a geometric distribution with parameter p over a sequence of times $\{t_k\}_{k=1}^{\infty}$, then $\text{Prob}(T = t_k) = p(1-p)^{k-1}$ for each $k \in \mathbb{N}$. In the first event described in the text, T_1 has support over $\{\hat{t}_k\}_{k=1}^{\infty}$ with $\hat{t}_k = k$ and T_2 has support over $\{\hat{\hat{t}}_k\}_{k=1}^{\infty}$, with $\hat{\hat{t}}_k = -2 \cdot k$.

countable set of times $\mathcal{T}_i \subset \mathbb{R}$. Consider an extensive-form game in which the following occurs.

1. First, each agent i simultaneously chooses an element of $\mathcal{T}_i$. Each agent does not observe the other agent's choice.

2. Second, the private-timing game is played. That is, if an agent chooses t in the previous stage, the agent moves at t, choosing her action in the component game S and her disclosure decision, where disclosure costs $c > 0$. When doing so, she observes the opponent's action if it is played and disclosed at a time strictly before t.

We call this game the **game with moving-time choice**. It is characterized by $(S, (\mathcal{T}_i)_{i\in N}, c)$.

In this game, given the choices of the times at the first stage, the histories and the strategies for the second stage can be defined in the exact same manner as in the main model. For this reason, we use the same notation for the space of histories. We use π_i to denote player i's strategy. Note that π_i specifies i's first-stage action as well. Since i makes a choice of the moving time without any information in the first stage, we denote by $\emptyset$ the null history that i faces at the first stage, i.e., $\pi_i(\emptyset)(t)$ is the probability that i assigns to time t at the first stage. Let Π_i be the set of behavioral strategies of player i, and $\Pi = \times_{i\in N}\Pi_i$. We extend the definition of the payoff function in a natural way by letting $u_i : \Pi \to \mathbb{R}$.

Note first that, if each $\mathcal{T}_i$ has the same minimum, there may exist a PBE in which two players choose to play a Pareto-inefficient Nash equilibrium action at that minimum time. The following Lemma shows that, in all other cases, only a^* is played on the equilibrium path.

We assume without loss of generality that $\inf \mathcal{T}_1 \cup \mathcal{T}_2 = 0$, and $\sup \mathcal{T}_1 \cup \mathcal{T}_2 = 1$.

Lemma 4. *Let $(S, (\mathcal{T}_i)_{i\in N}, c)$ be a game with moving-time choice, with S being a two-player common-interest component game, and suppose $0 \notin \mathcal{T}_1 \cap \mathcal{T}_2$. Then there is $\bar{c} > 0$ such that for all $c \leq \bar{c}$, on the path of play of every PBE each player i plays $(a_i^*, \cdot)$.*

The idea is that if the two players do not share the minimum time, there is a player i that "always has an earlier moving time," and that player can choose to play the efficient action and pay a small disclosure cost c, to ensure the payoff of $u_i(a^*) - c$.

Lemma 4 does not preclude equilibria in which players disclose their action. This is because players can choose to move in a deterministic order in equilibrium. For example, without loss of generality, take $t_1 \in \mathcal{T}_1$ and $t_2 \in \mathcal{T}_2$ such that $t_1 < t_2$. Then, there is a PBE in which player 1 chooses $T_1 = t_1$ in the first stage and (a_1^*, pay) in the second stage, while player 2 chooses $T_2 = t_2$ and B in the absence of a disclosure (as in Example 2).

To formally introduce uncertainty about timing to the endogenous-timing environment, let us consider a notion of Trembling Hand Perfect Equilibrium in which players are more prone to make a mistake about the choice of their timing of moves than they are to jointly make a mistake about their choice of a component-game action and disclosure decision.

A totally mixed behavioral strategy profile $\pi^\varepsilon \in \Pi$ is an **ε-constrained equilibrium** if there are functions $\varepsilon : \cup_{i\in\{1,2\}} (\mathcal{H}_i \times A_i) \times \{\text{pay}, \text{not}\} \to (0, \varepsilon)$, $\varepsilon_i : \mathcal{T}_i \to (0, \varepsilon)$ for each $i = 1, 2$, such that, for each $i = 1, 2$,

$$\pi_i^\varepsilon \in \arg\max_{\pi_i' \in \Pi_i} \left\{ u_i(\pi_i', \pi_{-i}^\varepsilon) \;\middle|\; \begin{array}{l} \pi_i'(\emptyset)(t) \geq \varepsilon_i(t), \pi_i'(h_i)(\tilde{a}_i, \tilde{d}_i) \geq \varepsilon(h_i, \tilde{a}_i, \tilde{d}_i), \\ \text{for every } h_i\,, t \in \mathcal{T}_i,\ \tilde{a}_i \in A_i, \text{ and } \tilde{d}_i \in \{\text{pay}, \text{not}\} \end{array} \right\}.$$

The functions $\varepsilon(\cdot)$ and $\varepsilon_i(\cdot)$ are called *trembles*.

Thus, in an ε-constrained equilibrium each player i must assign a positive weight on all their available actions and moving times.

Definition 1. A strategy profile π^* is an **extensive-form trembling-hand equilibrium with uncertainty about timing (THPE)** if there are sequences $(\varepsilon_n)_{n=1}^\infty \subseteq \mathbb{R}_{++}$ and $(\pi^n)_{n=1}^\infty \subseteq \Pi$, with associated trembles $\varepsilon^n(\cdot)$ and $\varepsilon_i^n(\cdot)$, such that the following hold.

1. π^n is an ε_n-constrained equilibrium for each $i \in N$ and each $n = 1, 2, \ldots$, and $\varepsilon_n \to 0$.
2. $\pi_i^*(h_i) = \lim_{n\to\infty} \pi_i^n(h_i)$ for each private history $h_i \in \mathcal{H}_i$ and $i \in \{1, 2\}$.
3. For every $t', \tilde{t} \in \mathcal{T}_i$ and $a_i \in A_i$, $d_i \in \{\text{pay}, \text{not}\}$, $\lim_{n\to\infty} \sum_{t>t'} \varepsilon_i^n(t)/\varepsilon_i^n(t') = \lim_{n\to\infty} \sum_{t>t'} \varepsilon_i^n(t)/\varepsilon^n(\tilde{t}, a_i, d_i) = \infty$.

Condition 3 requires that the moving time be the primary source of uncertainty and the uncertainty about the choice of actions be secondary—the situation that we want to capture. In fact, for each time interval in $(t, 1) \subseteq [0, 1]$, the chance that a player moves in $(t, 1)$ is assumed to vanish slower, as $\varepsilon_n \to 0$ than the chance that she erroneously moves at time t or takes a wrong action at any given time. Thus, compared to erring into a time t or taking a wrong action, erring in moving after t is more likely to occur.

Theorem 2. *Let $(S, (\mathcal{T}_i)_{i \in N}, c)$ be a game with moving-time choice, with S being a two-player common-interest component game. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be such that (a) $0 \notin \mathcal{T}_1 \cap \mathcal{T}_2$ and (b) for each $i \in \{1, 2\}$, $\sup \mathcal{T}_i = 1$ and $1 \notin \mathcal{T}_i$. Then, the game has a THPE, and there exists $\bar{c} > 0$ such that for all $c < \bar{c}$, on the path of play of every THPE, each player i plays (a_i^*, not).*

Theorem 2 says that if there is a player who always has an earlier moving time, and there is no time that is the last moving time of a player, then the unique equilibrium outcome is (a_i^*, not) for each player i. The result relies on Lemma 4, which under condition (a) guarantees that the Pareto optimal action profile is realized in every equilibrium. Condition (b) guarantees that in an ε-constrained equilibrium there is enough uncertainty about timing that players do not pay for disclosure. Notice that the condition is akin to the potential leader condition, and analogously to the arguments in Theorem 1, it guarantees that players are always on the path of play.

Theorem 2 does not restrict the joint distribution of the players' moving times to not being synchronous or "close to" synchronous, as SAP does in the exogenous timing case. Such restrictions are not necessary for two reasons. First, if there is a risk of miscoordination due to simultaneous moves, the players would *choose* to change their moving times to prevent it. The requirement that a player i can always choose an earlier time guarantees that this is possible. Second, since an equilibrium is a strategy profile, the distributions of the players' moving times are independent. Moreover, trembles on moving times are not correlated across players.

5 Discussion

5.1 Existence

So far, we have only studied common-interest games, but is our framework applicable to other settings of interest? As a first step in answering this question, we present a general existence result pertaining to general component games. Specifically, we show that there exists a PBE in a private timing game with *any* component game. This result is encouraging, as it suggests that it may be worthwhile to study other settings featuring private timing.[29]

Proposition 5. *Every game* $(S, \mathcal{T}, p, c)$ *has a PBE.*

In our game, the support of the times of play can be infinite, so the finite-dimensional fixed-point argument does not apply. Moreover, since we deal with general component games, there is no obvious way to design a constructive proof as was possible in the main sections of the paper. We overcome these difficulties by reformulating a game with private timing as a two-period stochastic game in which the first-period state is comprised of the first player to move and her moving time, and the second-period state is the second player's moving time. At each player i's moving time, i's signal consists of her moving time, and $-i$'s action and moving time if the latter moved earlier and chose to disclose. Unless a player has observed the opponent's move, she does not receive any information (besides her own moving time) regarding whether she is in the first or second period of the stochastic game. With this reformulation, we can use Moroni (2022)'s result on the existence of a Trembling Hand Perfect Equilibrium, which applies due to the finiteness of the action sets and the countability of the state and signal space. As shown in Moroni (2022), a Trembling Hand Perfect Equilibrium is also a PBE. Note that, since we show the existence of a Trembling Hand Perfect Equilibrium, the fact that our definition of PBE is not stringent does not play a key role in proving existence.

[29] Indeed, Kamada and Moroni (2023) study some other types of component games, such as opposing-interest games in which there are multiple Pareto unranked Nash equilibria.

5.2 q-Dispersed Timing Distribution and s_i-Common-Interest Games

Although Theorem 1 requires asynchronicity of moves, Proposition 1 proves uniqueness of PBE allowing for a small degree of synchronicity. This difference is due to the fact that the former result applies to any common-interest games, while the latter to a fixed common-interest game. This suggests that there may be a relationship between the type of common-interest game that we fix, the degree of synchronicity of the timing distribution, and uniqueness of a PBE. This section provides one way to formalize such a relationship.

Given a common-interest game S with more than two action profiles, let $g_i^* := g_i(a^*)$ be player i's payoff from the best action profile. We also let $\underline{g}_i := \min_{a \in A} g_i(a)$ be the minimum payoff, and $g_i^S := \max\{g_i(a) | a \in A,\ g_i(a) \neq g_i^*\}$ be the second-highest payoff for player i. Notice that $g_i^* > g_i^S, \underline{g}_i$ holds because the component game is a common-interest game.

Definition 2. For any $s_i > 0$, a common-interest game is s_i-**common** for i if $\frac{g_i^* - g_i^S}{g_i^* - \underline{g}_i} = s_i$.

Note that $s_i \in (0, 1]$, and it measures how good the best payoff is, in relative terms, for player i.

Definition 3. The timing distribution p is q-**dispersed** if for each $t'' \geq \inf(\mathcal{T})$, there is a player $i \in \{1, 2\}$ such that

$$\limsup_{t \to t''} \text{Prob}^p(T_{-i} \in [t'', t] | T_i = t, T_{-i} \geq t'') < 1 - q.$$

From this definition, q-dispersion implies that the probability that the two players move at the same time at any one time t is less than $1-q$. If a distribution is q-dispersed for every $q \in (0, 1)$ then it has dispersed potential moves. In particular, q-dispersion is satisfied when T_1 and T_2 are independent and their hazard rate is strictly less than $1 - q$ (this includes our starting example in Section 3.1 with $\varepsilon \leq 1 - q$). Notice that in Example 3, the assumption fails for $t'' = \inf \mathcal{T} = -\infty$, when $q < 1$ is sufficiently close to 1. This is because for any $t \in \mathcal{T}$, the probability that the opponent moves earlier than the player moving at t is at least $\frac{r}{1+r}$.

We now provide a sufficient condition on the joint distributions of T_1 and T_2 and the component game S such that (a^*, not) is the only outcome of the private-timing game when the cost of disclosure is small enough. In order to simplify the statement and proof, we do not consider the unlikely leader condition but we impose the potential leader condition.

Theorem 3. *Fix a dynamic game $(S, \mathcal{T}, p, c)$ with S being a two-player common-interest game. Suppose that there exist $(s_1, s_2) \in \mathbb{R}^2_{++}$ and $\varepsilon > 0$ such that S is s_i-common for each $i \in N$ and p is $(1+\varepsilon-\min_{i\in N} s_i)$-dispersed. Then there exists $\bar{c} > 0$ such that for all $c < \bar{c}$, a^* is assigned probability one under any PBE of $(S, \mathcal{T}, p, c)$. Moreover, if each player i is a potential leader under p, then there is a unique PBE outcome. On the path of this unique PBE outcome, each player i takes (a_i^*, not) for any realization of $T_i \in \mathcal{T}_i$.*

This theorem implies Proposition 1.

5.3 Horizon Length

As we noted when stating Theorem 1 (Remark 5-3), it is important that the support of the moving times is infinite in at least one direction. We formalize this claim here. To avoid notational complexity, we restrict $\mathcal{T}$ to be a subset of $\mathbb{Z}$. The results can be readily extended to more general settings.

Proposition 6.

1. *For any $t^* \in \mathbb{Z}$, there exist $\mathcal{T} \subseteq \mathbb{Z}$ with $\min_{t\in\mathcal{T}} t = t^*$ and p such that for every two-player common-interest game S, there is $\bar{c} > 0$ such that for every $c < \bar{c}$, the dynamic game $(S, \mathcal{T}, p, c)$ has a unique PBE. On the path of the unique PBE, each player i plays (a_i^*, not) at any realization of T_i.*

2. *For any $t^* \in \mathbb{Z}$, there exist $\mathcal{T} \subseteq \mathbb{Z}$ with $\max_{t\in\mathcal{T}} t = t^*$ and p such that for every two-player common-interest game S, there is $\bar{c} > 0$ such that for every $c < \bar{c}$, the dynamic game $(S, \mathcal{T}, p, c)$ has a unique PBE. On the path of the unique PBE, each player i plays (a_i^*, not) at any realization of T_i.*

3. *For any $t^*, t^{**} \in \mathbb{Z}$, for all $\mathcal{T} \subseteq \mathbb{Z}$ with $\min_{t\in\mathcal{T}} t = t^*$ and $\max_{t\in\mathcal{T}} t = t^{**}$ and for every p, there exists a common-interest game S such that there is $\bar{c} > 0$ such that for every $c < \bar{c}$, the dynamic game $(S, \mathcal{T}, p, c)$ has multiple PBE.*

The proofs in this section are omitted as they are straightforward.[30] The proposition implies that a game in which the distribution of moving times has finite support must have multiple PBE. This seeming discontinuity of the set of PBE outcomes occurs due to the order of limits[31]: we fix a timing distribution and then consider all possible common-interest games. If we flip the order of limits, then we retain continuity. The next proposition makes this point clear.

Proposition 7. *Consider a family of pairs* $\{(\mathcal{T}_K, p_K)\}_{K\in\mathbb{N}}$ *defined by* $\mathcal{T}_K = \{1, \ldots, K\}$ *and let* p_K *be the uniform distribution over* $\mathcal{T}_K$*, independent across players.*

1. *For any* $K \in \mathbb{N}$*, there exists a two-player common-interest game* S *and* $\bar{c} > 0$ *such that, for all* $c < \bar{c}$*, the dynamic game* $(S, \mathcal{T}_K, p_K, c)$ *has multiple PBE.*
2. *For any two-player common-interest game* S*, there exists* $\bar{K} < \infty$ *and* $\bar{c} > 0$ *such that, for all* $K > \bar{K}$ *and* $c < \bar{c}$*, the dynamic game* $(S, \mathcal{T}_K, p_K, c)$ *has a unique PBE. On the path of the unique PBE, each player* i *plays* (a_i^*, not) *at any realization of* T_i*.*

The first part of the proposition is a corollary of the third part of Proposition 6, while the second part shows that for any game one can find a sufficiently dispersed distribution that yields the Pareto optimal outcome as the unique equilibrium outcome. We omit the proof of Proposition 7 as it follows from our previous results.[32]

[30]For part 1, consider the timing distribution p such that player 1's moving time has a full support over positive odd integers, and player 2's moving time has a full support over positive even integers, and the two players' moving items are independent. The desired distribution can be obtained by "shifting" the support by $t^* - 1$. For part 2, modify Example 4 by letting $T_1 = 1$ if and only if the example specifies that T_1 is positive. Again, "shifting" the support by $t^* - 1$ would yield the desired distribution. Part 3 follows by applying Theorem 1 because condition 2 cannot hold.

[31]Consider, e.g., a metric $d(p, p') = \sum_{(t_1,t_2)} |\text{Prob}^p(T_1 = t_1, T_2 = t_2) - \text{Prob}^{p'}(T_1 = t_1, T_2 = t_2)|$.

[32]Specifically, part 1 of Proposition 7 follows from part 3 of Proposition 6, and part 2 of Proposition 7 follows from Theorem 3.

5.4 Bayes Nash Equilibrium

In the unique PBE for common-interest games, even if we did not assume optimality after deviations have been detected, deviations would not be optimal. This may suggest uniqueness might hold even under the notion of Bayes Nash equilibrium (BNE) which requires only condition 1 in the definition of PBE. But this is not the case. That is, there may exist multiple BNE. To see this, consider the component game as in Figure 1 and an independent timing distribution such that $\mathcal{T}_1$ is the set of odd natural numbers and $\mathcal{T}_2$ is the set of even natural numbers. By inspection, one can verify that the strategy profile in which each player plays (B, not) under all private histories is a BNE. This BNE can be supported by an off-path strategy specification in which each player chooses a different action than the opponent's once the opponent deviates to disclose his action.

The reason why we do not obtain uniqueness is that, without off-path optimality, which requires players to best-respond to an observed action, Step 1 of the Proof Sketch for Proposition 1 does not go through. Thus, off-path optimality plays a key role in eliminating inefficient outcomes.

5.5 Discrete Time vs. Continuous Time

In this paper, we analyzed a discrete-time rather than continuous-time settings. We do not take a stance on which of discrete- and continuous-time models are more realistic but focused on the former. This is because we wanted to avoid non-essential technicalities associated with conditional measures. Recall that we imposed SAP to show Theorem 1. A wide class of atomless full-support distributions on continuous time satisfying certain regularity conditions would satisfy a version of SAP.[33] Although an atomless full-support distribution is not necessarily a consequence of continuous time, it is somewhat a standard assumption. Additionally, as we have seen in the formulation of SAP, our assumptions allow for a countable set of times $\mathcal{T}$ that is dense on the continuum, such as the rational numbers. Therefore, one can interpret our conditions on asynchronicity and uncertainty as being akin to the properties of many commonly-studied distributions in continuous-time settings.

[33] For example, one could assume that a density is strictly positive and continuous on $[0, 1]^2$.

6 Conclusion

This paper studied games with private timing. These games satisfy the often realistic assumption that the timing of moves in a strategic interaction can be private information. Many questions arise if additionally players have an option to disclose information between moves: Do players want to disclose their own actions to the opponent? Do they want to commit to monitor the opponent's actions? What if actions are disclosed with some exogenously given probability? How do these possibilities affect the players' choice of the component-game actions? In this paper, we focus on one particular information revelation mechanism that highlights the non-triviality of these problems. Namely, when the component game is a coordination game and players have an option to disclose their actions at a small cost, we proved that uniqueness of the Pareto optimal outcome in every PBE holds if and only if the game satisfies conditions that require, in rough terms, a strong form of asynchronicity of moves and enough uncertainty about the opponent's moving time conditional on one's information. We also provided a number of discussions and extensions to further understand those results.

Our paper demonstrates that the class of games with private timing is simple but involves non-trivial strategic interactions. Let us discuss a number of other questions that we believe are worth tackling in the context of private timing, both in theory and in applications. For example, we focused on common-interest games, but one could consider a wider class of component games. Kamada and Moroni (2023) considers various other component games, such as opposing-interest games, constant-sum games, games with a dominant action for each player, and a game that is solvable by iterated dominance. One prominent example that we do not cover is the Cournot quantity-competition game. It is straightforward to show that there exists a PBE in which the first mover plays the Stackelberg action and pays the disclosure cost, but it is an open question whether it is the unique equilibrium.

Beyond the examination of different component games, there are other avenues for future research in the framework of games with private timing. First, one could investigate the effect of monitoring options. With common-interest games with costly monitoring, the best action profile may not be the unique outcome. One can construct examples in which a Pareto-dominated action profile is played and no monitoring takes place. Second, one could consider a cost of secrecy.[34] Third, one can consider settings in which disclosure produces an informative yet imperfect signal about the action taken, and examine the effect of the noisiness of the signal on the set of equilibrium outcomes. In Online Appendix B.1, we consider a setting where a signal is probabilistically sent, but it is correct whenever it is generated. Another possibility that we do not study is that there be a signal that is always generated but may be incorrect. Fourth, it would be interesting to examine how private information about timing interacts with private information about the payoff functions. Online Appendix B.3 considers a simple case where two players are uncertain about which of two possible coordination games is the true one and shows that, in that setting, players pay the disclosure cost. Finally, the present paper concentrated on the case in which each of two players moves only once before an action profile is determined. One may want to extend this setting to the case where there are more players and/or each player moves more than once.

[34] We thank Drew Fudenberg for suggesting this possibility.

A Appendix

Before we start our proofs, let us introduce the following notation. For $t \in \mathcal{T}_i$, a time-t private history of player i is a private history of the form $((t_j, a_j), t) \in \mathcal{H}_i$ or $(\emptyset, t) \in \mathcal{H}_i$. The set of time-$t$ private histories of player i is denoted $\mathcal{H}_{i,t}$.

A.1 Proof of Theorem 1

The "only if" direction:

Fix a component game that is a common-interest game S, and denote by $(g_i)_{i \in N}$ the payoff functions. Fix a timing distribution $p \in D$ with support $\mathcal{T}$. For each disclosure cost $c > 0$, fix a PBE σ^c of the associated dynamic game $(S, \mathcal{T}, p, c)$.

Given $\sigma \in \Sigma$, let $\mathcal{H}_i(\sigma) \subseteq \mathcal{H}_i$ be the collection of i's private histories that have positive probability under strategy profile σ.

The proof consists of several steps.

Step 1-1 (a_i^* is played on the path):

For each $i = 1, 2$, let

$$B_i := \{t \in \mathbb{R} | \sigma_i^c(h_i)(\{(a_i^*, \text{pay}), (a_i^*, \text{not})\}) < 1, h_i = (\emptyset, t) \in \mathcal{H}_i(\sigma^c)\}.$$ [35]

That is, B_i is the set of times at which player i plays an action other than a_i^* with positive probability when there is no observation of disclosure at an on-path private history.

Suppose for a contradiction that at least one out of B_1 and B_2 is nonempty. Let

$$t'' := \inf (B_1 \cup B_2). \tag{1}$$

That is, t'' is the infimum time at which some player i does not play a_i^*.

For each $i \in \{1, 2\}$ and $q \in \mathbb{R}$, consider the following inequality:

$$(1-q) g_i(a^*) + q \min_{a_{-i} \in A_{-i}} g_i(a_i^*, a_{-i}) - c \geq \max_{a_i \neq a_i^*, a_{-i} \in A_{-i}} g_i(a_i, a_{-i}).$$

Notice that the left-hand side of the above inequality is continuous and strictly

[35] It is important that $h_i \in \mathcal{H}_i(\sigma^c)$ holds. See footnote 36 to see this point.

decreasing in q by the definition of a^*. Let $\bar{c}_i = g_i(a^*) - \max_{a_i \neq a_i^*, a_{-i} \in A_{-i}} g_i(a_i, a_{-i})$. Notice that $\bar{c}_i > 0$ by the definition of a^*. Then, for any $c \in (0, \bar{c}_i)$, there exists $\bar{q}_i^c > 0$ such that for all $q \in (0, \bar{q}_i^c]$, the above inequality is satisfied. Let $q_i^c = \min\{\bar{q}_i^c, 1\}$. With this notation, we have that some $a_i \neq a_i^*$ is a best response for i at a given time only if she assigns probability no less than q_i^c to her opponent j playing actions other than a_j^* weakly before i moves. Hence, by the definition of $\mathscr{T}_i$,

$$B_i \subseteq \mathscr{T}_i(B_{-i}, t'', q_i^c) \tag{2}$$

holds for each $i \in \{1, 2\}$. This contradicts SAP if $B_i \neq \emptyset$, for some $i \in \{1, 2\}$. The latter is true by hypothesis.[36]

Step 1-2 (a_i^* is played off the path):

The argument so far proves that, under σ^c, at a private history such that i has not observed a disclosure *and is on the path of play under* σ^c, i plays a_i^* at any time $t \in \mathcal{T}_i$ (by "i is on the path of play under $\sigma \in \Sigma$," we mean that i is at some private information set h_i such that $h_i \in \mathcal{H}_i(\sigma)$. And "off the path" refers to the complementary event). Let us now show that i plays a_i^* even if she is off the path of σ^c (the proof eventually implies, however, i cannot be off the path of σ^c).

Suppose that under σ^c, i's realized moving time is t and she does not observe disclosure while she is off the path of play under σ^c.

Note that the only way in which player i can be off the path of play at time t under σ^c is that $t \in T_i^<$, i.e., $\text{Prob}^p(T_{-i} \geq t | T_i = t) = 0$, because otherwise by Bayes rule player i assigns positive probability to the event that $T_{-i} \geq t$ and thus she is on the path of play under σ^c. This implies that the set $T_i^<$ is nonempty.

Let $E_i \subseteq T_i^<$ be the set of times such that $t' \in E_i$ if and only if $\sigma_i^c(h_i)(\{(a_i^*, \text{pay}), (a_i^*, \text{not})\}) < 1$ for $h_i = (\emptyset, t')$.

Claim 8. *There is $\bar{c} > 0$ such that for every $c \in (0, \bar{c})$, E_i is an empty set.*

[36]Notice that the conclusion in (2) does not follow if, in the definition of B_i, we allow h_i to be outside of $\mathcal{H}_i(\sigma^c)$. For example, consider the game in Example 2, and let $j = 1$ and $i = 2$, and σ^c be the strategy profile described in the example in which player 2 chooses B upon no disclosure. Then, without the modification, $B_1 = B_2 = \emptyset$, and hence $\mathscr{T}_2(B_1, t'', q_2^c) = \emptyset$ as well, for every t''. With the modification, however, $B_2 = \{2\}$ because under σ^c player 2 plays action B if player 1 does not pay in the first period. However, $B_1 = \emptyset$ holds because under σ^c player 1 always plays A, which implies $\mathscr{T}_2(B_1, 2, q_2^c) = \emptyset$. Hence, $B_2 \not\subseteq \mathscr{T}_2(B_1, 2, q_2^c) = \emptyset$ holds.

Proof. Suppose for contradiction that $E_i \neq \emptyset$. We will show that there is $t' \in E_i$ and $\tilde{t} \in \mathcal{T}_{-i}$ with $\text{Prob}^p(T_{-i} = \tilde{t} | T_i = t') > 0$ such that player $-i$ assigns probability 1 to (a^*_{-i}, not) at $\tilde{t}$ under σ^c_i. Fix $\epsilon \in \left(0, \frac{c}{\max_{a\in A}(g_{-i}(a^*)-g_{-i}(a))}\right)$. The unlikely leader condition implies that there exists $\tilde{t} \in \mathcal{T}_{-i}$ with $\text{Prob}^p(T_i \in E_i | T_{-i} = \tilde{t}) \in (0, \epsilon)$. The expected payoff for player $-i$ at $\tilde{t}$ from playing (a^*_{-i}, not) is at least

$$\text{Prob}^p(T_i \notin E_i | T_{-i} = \tilde{t}) \cdot g_{-i}(a^*) + \text{Prob}^p(T_i \in E_i | T_{-i} = \tilde{t}) \cdot \left(\min_{a\in A} g_{-i}(a)\right). \tag{3}$$

This is due to the definition of E_i.

By the definition of $\tilde{t}$, the value (3) is strictly greater than

$$(1-\epsilon) \cdot g_{-i}(a^*) + \epsilon \cdot \left(\min_{a\in A} g_{-i}(a)\right).$$

The definition of ϵ then implies that this is strictly greater than

$$\left(1 - \frac{c}{\max_{a\in A}\left(g_{-i}(a^*) - g_{-i}(a)\right)}\right) \cdot g_{-i}(a^*) + \frac{c}{\max_{a\in A}\left(g_{-i}(a^*) - g_{-i}(a)\right)} \cdot \left(\min_{a\in A} g_{-i}(a)\right).$$

It is straightforward to see that this is equal to $g_{-i}(a^*) - c$. Any action $a_{-i} \neq a^*_{-i}$ of $-i$ will give him a payoff no greater than $\max_{a_i \neq a^*_i, a_{-i}\in A_{-i}} g_{-i}(a_i, a_{-i})$ at time $\tilde{t}$. Hence, for all $c < \min_i \bar{c}_i =: \bar{c}$, player $-i$ assigns probability 1 to playing (a^*_{-i}, not).

Now, notice that $\text{Prob}^p(T_i \in E_i | T_{-i} = \tilde{t}) > 0$ implies there exists $t' \in E_i$ such that $\text{Prob}^p(T_{-i} = \tilde{t} | T_i = t') > 0$. Hence, by Bayes rule, at i's moving time t', if i does not observe disclosure, then she is on the path of play under σ^c. This implies that, by our earlier conclusion, if i does not observe disclosure at time t' and is on the path of play under σ^c, she assigns probability one to $\{(a^*_i, \text{pay}), (a^*_i, \text{not})\}$. This contradicts $t' \in E_i$ by the definition of E_i. Hence, we conclude that E_i is an empty set. $\square$

Claim 8 implies that at every time in $T^<_i$, i assigns probability one to $\{(a^*_i, \text{pay}), (a^*_i, \text{not})\}$ when she observes no disclosure *even if she is off the path of play under* σ^c if $c \in (0, \bar{c})$. Hence, each player i must assign probability one to $\{(a^*_i, \text{pay}), (a^*_i, \text{not})\}$ at every moving time on and off the path of play under σ^c if $c \in (0, \bar{c})$.

Step 2 (Disclosure does not occur):

Now, take an arbitrary player j and $t \in \mathcal{T}_j$, j expects the payoff of $g_j(a^*)$ by playing (a_j^*, not) if $-j$ follows σ_{-j}^c when $c \in (0, \bar{c})$. Since any other action of j will give her a strictly lower payoff than $g_j(a^*)$, this proves that, under σ_j^c, j assigns probability one to (a_j^*, not) at t when there is no disclosure. Thus, under σ^c, each player j plays (a_j^*, not) with probability one at any private history at which there is no observation.[37] In particular, this means that there is a unique PBE outcome in which each player i plays (a_i^*, not) at any realization of T_i. This completes the proof of the "only if" direction of the theorem.

The "if" direction:

We are going to show that, if a given time distribution p (which uniquely pins down $\mathcal{T}$) is not in the set D, then we can construct a component game S such that there is $\bar{c} > 0$ such that for all $c < \bar{c}$, there are multiple PBE in the dynamic game $(S, \mathcal{T}, p, c)$. This claim implies the necessity of the conditions in the theorem for the uniqueness of a PBE.

Specifically, we first assume that SAP fails and show the existence of such $\bar{c}$. Then we assume that both the potential leader condition and the unlikely leader condition fail and show the existence of $\bar{c}$.

1. Necessity of SAP:

Fix $(\mathcal{T}, p)$, and suppose that SAP does not hold. In what follows, $i, j \in \{1, 2\}$ and $i \neq j$. There are $t'' \in \{-\infty\} \cup \mathbb{R}$, $q_1, q_2 > 0$ and sets $B_k \subseteq \mathcal{T}_k \cap [t'', \infty)$ for $k \in \{1, 2\}$ such that, for $i \in \{1, 2\}$,

$$\emptyset \neq B_i \subseteq \mathscr{T}_i(B_j, t'', q_i) \text{ and } B_j \subseteq \mathscr{T}_j(B_i, t'', q_j).$$

Note that these conditions imply

$$\text{Prob}^p(T_k \geq t'') > 0 \text{ and } B_k \cap [t'', \infty) \neq \emptyset \text{ for } k \in \{1, 2\}. \tag{4}$$

Consider the component game in Figure 5, where $\nu > 0$. Denote it by S. We

[37] Note that this implies that no player can be off the path of σ^c if there is no observation of disclosure.

	a_2	b_2
a_1	q_1, q_2	$-(1-q_1)-\nu, 0$
b_1	$0, -(1-q_2)-\nu$	$0, 0$

,

Figure 5: Game S: A counterexample showing the necessity of SAP

consider a dynamic game $(S, \mathcal{T}, p, c)$ for an arbitrary $c > 0$. Notice that in the component game S, if i thinks that player j plays b_j with probability at least q_i, then i's best response must assign probability 1 to b_i. Now, in the dynamic game, conditional on choosing an action from $\{(b_i, \text{pay}), (b_i, \text{not})\}$, there is no gain from inducing the opponent to choose (b_j, not). This is because b_i yields a payoff of 0 regardless of the opponent's play. Therefore, the unique best response when i believes b_j is chosen with probability at least q_i is to assign probability 1 to (b_i, not).

Notice that for each $i \in \{1, 2\}$ and $t \in B_j$,

$$\text{Prob}^p\left(T_i \in B_i \cap [t'', t) | T_j = t, T_i \geq t''\right) \geq q_j$$

holds because $B_j \subseteq \mathscr{T}_j(B_i, t'', q_j)$.

Therefore, it is a best response for j to play (b_j, not) at every time in B_j if i plays (b_i, not) with probability one at all times in B_i.

Now, consider a modification of $(S, \mathcal{T}, p, c)$ in which, for each player i, if i's moving time realizes in B_i, then i is restricted to assign probability one to (b_i, not), while for all other realized moving time, the action set is still $\{(a_i, \text{pay}), (a_i, \text{not}), (b_i, \text{pay}), (b_i, \text{not})\}$. In this new dynamic game, a PBE exists (Moroni (2022)), which we denote by σ.

We check that σ is also a PBE in the original dynamic game $(S, \mathcal{T}, p, c)$. First, it is immediate that each i takes a best response at all times in $\mathcal{T}_i \setminus B_i$. Second, for each player i and for times in B_i, we have concluded that, given that j takes (b_j, not) in B_j with probability one, (b_i, not) is a best response for i. Since σ_j assigns probability one to (b_j, not) in B_j, it is indeed a best response for i to play (b_i, not) at times in B_i. Hence, σ is a PBE in $(S, \mathcal{T}, p, c)$.

Next, consider another strategy profile that we denote by σ', in which each player i plays (a_i, not) at all times under no observation of a disclosure, and takes

	A	B
A	$1,1$	$-M,0$
B	$0,-M$	$0,0$

Figure 6: Counterexample showing the necessity of the potential leader condition or the unlikely leader condition

a static best response after an observation. This is obviously a PBE.

Since (4) holds, $B_j \neq \emptyset$ for each player j, and thus $\sigma \neq \sigma'$. Hence, there are multiple PBE for every $c > 0$. This completes the proof of this part.

2. Necessity of the potential leader condition or the unlikely follower condition:

Fix a new timing structure $(\mathcal{T}, p)$. Suppose, for a contradiction, that there is a player i and a nonempty set $E \subseteq T_i^<$ such that

$$\inf\{r > 0 | r = \text{Prob}^p(T_i \in E | T_{-i} = \tilde{t}), \text{ for } \tilde{t} \in \mathcal{T}_{-i}\} = \delta > 0. \tag{5}$$

For each $c > 0$, consider a dynamic game $(S, \mathcal{T}, p, c)$. Define:

$$E_{-i} = \{\tilde{t} \in \mathcal{T}_{-i} | \text{Prob}^p(T_i \in E | T_{-i} = \tilde{t}) > 0\}, \text{ and}$$

$$\tilde{E}_i = \{t \in T_i^< | \text{Prob}^p(T_{-i} \in E_{-i} | T_i = t) = 1\}.$$

Thus, $t \in E_{-i}$ if (i) $t \in \mathcal{T}_{-i}$ and (ii) conditional on $-i$ moving at time t, there is a positive probability that i moves at a time in set E. Notice that E_{-i} is non-empty by equation (5). Also, $t \in \tilde{E}_i$ if (i) $t \in T_i^<$ and (ii) the probability that $-i$ moves at E_{-i} given that i moves at time t is 1.

For the following proof, we show the following claim:

a. $E \subseteq \tilde{E}_i$.

b. $\text{Prob}^p(T_i \in \tilde{E}_i | T_{-i} = t) = 0$, for $t \in \mathcal{T}_{-i} \setminus E_{-i}$.

Notice that by the definition of E_{-i}, $\text{Prob}^p(T_i \in E | T_{-i} = t) = 0$ for $t \in \mathcal{T}_{-i} \setminus E_{-i}$. Therefore, $\text{Prob}^p(T_i \in E, T_{-i} \in \mathcal{T}_{-i} \setminus E_{-i}) = 0$ and $\text{Prob}^p(T_{-i} \in E_{-i} | T_i = t) = 1$ for $t \in E$, implying (a). Also, at $t \in T_i^< \setminus \tilde{E}_i$, $\text{Prob}^p(T_{-i} \in E_{-i} | T_i = t) < 1$. By the definition of $\tilde{E}_{-i}$, $\text{Prob}^p(T_{-i} \in \mathcal{T}_{-i} \setminus E_{-i} | T_i = t) = 0$ for $t \in \tilde{E}_i$, which implies $\text{Prob}^p(T_{-i} \in \mathcal{T}_{-i} \setminus E_{-i}, T_i \in \tilde{E}_i) = 0$, and, therefore, we have (b).

We will show that for any $M \in [0, \infty)$, we can find $\bar{c} > 0$ such that if the component game S is the common-interest game in Figure 6 and $c \in (0, \bar{c})$, then σ^*, defined as follows, is a PBE of $(S, \mathcal{T}, p, c)$. For each $t \in \mathcal{T}_{-i}$ and $h_{-i,t} \in \mathcal{H}_{-i,t}$, let

$$\sigma^*_{-i}(h_{-i,t}) := \begin{cases} (A, \text{pay}) & \text{if } t \in E_{-i} \text{ and } h_{-i,t} = (\emptyset, t) \\ (A, \text{not}) & \text{if } t \notin E_{-i} \text{ and } h_{-i,t} = (\emptyset, t) \\ (A, \text{not}) & \text{if } h_{-i,t} = ((t', A), t) \text{ for some } t' \in \mathcal{T}_i \\ (B, \text{not}) & \text{if } h_{-i,t} = ((t', B), t) \text{ for some } t' \in \mathcal{T}_i \end{cases}.$$

Also, for each $t \in \mathcal{T}_i$ and $h_{i,t} \in \mathcal{H}_{i,t}$, let

$$\sigma^*_{i}(h_{i,t}) := \begin{cases} (B, \text{not}) & \text{if } t \in \tilde{E}_{i} \text{ and } h_{i,t} = (\emptyset, t) \\ (A, \text{not}) & \text{if } t \notin \tilde{E}_{i} \text{ and } h_{i,t} = (\emptyset, t) \\ (A, \text{not}) & \text{if } h_{i,t} = ((t', A), t) \text{ for some } t' \in \mathcal{T}_{-i} \\ (B, \text{not}) & \text{if } h_{i,t} = ((t', B), t) \text{ for some } t' \in \mathcal{T}_{-i} \end{cases}.$$

Note that, because $E_{-i} \neq \emptyset$, we have $\sigma^* \neq \sigma'$ where σ' takes (A, not) at any private history without observation.

Now we specify beliefs. First, for each player $j = 1, 2$, if a private history is $((t', a_{-j}), t)$ for some time t', then j's belief is an arbitrary probability distribution that assigns probability one to the set of histories $\{((t, a_j, d_j), (t', a_{-j}, \text{pay})) | a_j \in \{A, B\}, d_j \in \{\text{pay}, \text{not}\}\}$. For each private history $(\emptyset, t)$ of player $-i$, $-i$'s belief is computed by Bayes rule. Also, except at times in $\tilde{E}_i$, for each private history of player i, i's belief is computed by Bayes rule. If the private history is $(\emptyset, t)$ and $t \in \tilde{E}_i$, take an arbitrary element $t^*(t)$ of $\{t' \in \mathcal{T}_{-i} \mid t' < t,\ p(t', t) > 0\}$. We define player i's belief at private histories at time $t \in \tilde{E}_i$ to be an arbitrary probability distribution that assigns probability 1 to the set of histories $\{((t^*(t), B, \text{not}), (t, a_i, d_i)) \,| a_i \in \{A, B\}, d_i \in \{\text{pay}, \text{not}\}\}$. Thus, in the off-path private histories at times in $\tilde{E}_i$ at which player i does not observe (A, pay), she believes that $-i$ played (B, not) at time $t^*(t)$.

We now check that each player takes a best response at each private history.

First, it is straightforward to check that σ_i^* and σ_{-i}^* specify best responses after private histories in which there has been an observation of an action taken by the opponent. In what follows, we consider each player's action after a private history in which there has not been any observation. In the off-path private history at times in $\tilde{E}_i$ at which player i has not observed A, player i's belief is that $-i$ played B and, therefore, she best-responds with (B, not). At all other private histories, player i believes that player $-i$ has played or will play $(A, \cdot)$ and best-responds with (A, not). For player $-i$, if there has been no observation at $t' \in E_{-i}$, the payoff of playing (A, pay) is $1 - c$, while the payoff of playing (A, not) is at most $\delta \cdot (-M) + (1 - \delta) \cdot 1$ because (a) implies that i chooses B at times in E. Thus, for $c \in (0, \min\{1, \delta(M + 1)\})$, $-i$'s best response is to play (A, pay) at every time $t' \in E_{-i}$. At every time $\tilde{t} \notin E_{-i}$, choosing (A, not) is a best response for player $-i$ as, by (b), player i's strategy and Bayes rule imply that player $-i$'s belief at such a time must assign probability 1 to the event that player i plays (A, not) if $-i$ chooses (A, not).

Hence, σ^* is a PBE of $(S, \mathcal{T}, p, c)$ for every $c \in (0, \min\{1, \delta(M + 1)\})$. Since σ' is also a PBE for such c as before and we already concluded $\sigma^* \neq \sigma'$, there are multiple PBE in $(S, \mathcal{T}, p, c)$ for every $c \in (0, \min\{1, \delta(M + 1)\})$. This completes the proof of this part. □

A.2 Proof of Proposition 2

Part 1:

Let us show by contradiction that the dispersed potential moves property (henceforth, "DPM") implies SAP. Suppose that p satisfies DPM but there are t'', $B_j, B_i \subseteq [t'', \infty)$, and $q_1, q_2 > 0$ such that

$$\emptyset \neq B_i \subseteq \mathscr{T}_i(B_j, t'', q_i) \text{ and } B_j \subseteq \mathscr{T}_j(B_i, t'', q_j). \tag{6}$$

Let $\tilde{t} = \inf(B_i \cup B_j)$, and suppose that i satisfies the limit condition in DPM at time $\tilde{t}$. By (6), for every $t \in B_i$,

$$\text{Prob}^p(T_j \in [t'', t] \cap B_j | T_i = t, T_j \geq t'') \geq q_i, \tag{7}$$

and for every $t \in B_j$,

$$\mathrm{Prob}^p(T_i \in [t'', t] \cap B_i | T_j = t, T_i \geq t'') \geq q_j.$$

By the limit condition in DPM, there is $\hat{t} > \tilde{t}$ such that

$$\mathrm{Prob}^p(T_i \in [\tilde{t}, t] \cap B_i | T_j = t, T_i \geq \tilde{t}) < q_j$$

for every $t \in [\tilde{t}, \hat{t}]$. Therefore, for every $t \in [\tilde{t}, \hat{t}]$,

$$\mathrm{Prob}^p(T_i \in [t'', t] \cap B_i | T_j = t, T_i \geq t'') = \\ \mathrm{Prob}^p(T_i \in [\tilde{t}, t] \cap B_i | T_j = t, T_i \geq \tilde{t}) \cdot \frac{\mathrm{Prob}^p(T_j = t, T_i \geq \tilde{t})}{\mathrm{Prob}^p(T_j = t, T_i \geq t'')} < q_j,$$

where the equality follows from $B_i \cap [t'', \tilde{t}) = \emptyset$ by the definition of $\tilde{t}$, and the inequality follows from $\frac{\mathrm{Prob}^p(T_j = t, T_i \geq \tilde{t})}{\mathrm{Prob}^p(T_j = t, T_i \geq t'')} \leq 1$.

This implies $\mathscr{T}_j(B_i, t'', q_j) \cap [\tilde{t}, \hat{t}] = \emptyset$. Therefore, by (6), we obtain $B_j \subseteq (\hat{t}, \infty)$, which implies, by (7), $B_i \subseteq (\hat{t}, \infty)$. This contradicts the definition of $\tilde{t}$.

<u>Part 2:</u>

The following example shows that SAP does not imply DPM.

Example 5. [Distribution that satisfies SAP and not DPM]

Let $\mathcal{T}_1 = \left\{ \frac{1}{n} | n = 1, 4, 5, 8, 9, 12, \dots \right\}$ and $\mathcal{T}_2 = \left\{ \frac{1}{n} | n = 2, 3, 6, 7, 10, 11, \dots \right\}$, and define p by, for each $k = 0, 1, 2, \dots$,

$$\begin{aligned} \mathrm{Prob}^p\left(T_2 = \tfrac{1}{4k+2} | T_1 = \tfrac{1}{4k+1}\right) &= 1, \\ \mathrm{Prob}^p\left(T_2 < T_1 | T_1 = \tfrac{1}{4k+4}\right) &= 0, \\ \mathrm{Prob}^p\left(T_1 = \tfrac{1}{4k+4} | T_2 = \tfrac{1}{4k+3}\right) &= 1, \\ \mathrm{Prob}^p\left(T_1 < T_2 | T_2 = \tfrac{1}{4k+2}\right) &= 0. \end{aligned} \tag{8}$$

To see that this distribution of moves does not satisfy DPM, let $t'' = 0$. Then, for any $t' > 0$ and each player $i = 1, 2$, there is a moving time $t \in [0, t']$ such that

i assigns probability 1 to the opponent moving in $[0, t]$.[38]

To see that the distribution of moves satisfies SAP, take $B_2 \subseteq \mathcal{T}_2$, $B_2 \neq \emptyset$ and $B_1 \subseteq \mathscr{T}_1(B_2, t'', q_1)$ for $t'' \in \mathbb{R}$, and $q_1 > 0$. By (8) and the definition of $\mathscr{T}_1$, $B_1 \subseteq \left\{\frac{1}{4k+1}\right\}_{k=1}^{\infty}$. Therefore, from (8), $\mathscr{T}_2(B_1, t'', q_2) = \emptyset$ for every $q_2 > 0$. A similar argument applies to every $B_1 \subseteq \mathcal{T}_1$, $B_1 \neq \emptyset$, and $B_2 \subseteq \mathscr{T}_2(B_1, t'', q_2)$. This shows that SAP holds. $\square$

A.3 Proofs of Section 4

Proof of Lemma 4

Fix a PBE π^* of a game with moving-time choice and take $\bar{c} < g_j(a^*) - \max_{a \neq a^*} g_j(a)$ for $j \in \{1, 2\}$. Take i such that for any $t' \in \mathcal{T}_{-i}$, there is $t'' < t'$ with $t'' \in \mathcal{T}_i$. Such i exists by assumption.

Suppose that there is a time t and an action $a_i \neq a_i^*$ such that (i) π_i^* assigns positive probability to t, (ii) it is with positive probability that i does not observe $-i$'s action at t, and (iii) conditional on no observation at t, it assigns positive probability to a_i. Notice that by (ii), player i is on the path of play at time t upon no observation.

First, notice that the payoff from π_i^* is at most $\max_{a' \neq a^*} g_i(a')$ because player i must be indifferent between all the possibilities that she assigns positive probability to. Second, since player i always has an earlier moving time, there is a time $t' \in \mathcal{T}_i$ such that, for all $c < \bar{c}$,

$$\left(\sum_{t' \geq \tilde{t}} \pi_{-i}^*(\emptyset)(\tilde{t})\right)\left(\min_a g_i(a) - c\right) + \left(\sum_{\tilde{t} > t'} \pi_{-i}^*(\emptyset)(\tilde{t})\right)(g_i(a^*) - c) > \max_{a \neq a^*} g_i(a).$$

The left-hand side of this inequality is a lower bound of the payoff from choosing t' with probability one and then playing (a_i^*, pay) at t' with probability one (notice that we are using the fact that the players' moving-time distributions are independent of each other). This is a contradiction.

Now, by our previous argument, when player $j \neq i$ is on the equilibrium path,

[38] More precisely, for $i = 1$, take $t = \frac{1}{4k+1}$ for any positive integer k such that $\frac{1}{4k+1} < t'$, and for $i = 2$, take $t = \frac{1}{4k+3}$ for any positive integer k such that $\frac{1}{4k+1} < t'$.

he knows that player i has chosen a_i^* at all previous times. Therefore, at any time t, on the equilibrium path, the payoff from (a_j^*, pay) is at least $g_j(a^*) - c$, while the payoff from $a_j \neq a_j^*$ is at most $\max_{a \neq a^*} g_j(a)$. Therefore, for all $c < \bar{c}$, j must choose a_j^* at any t that he assigns positive probability to. $\square$

Proof of Theorem 2

Lemma 9. *A THPE exists.*

Proof: Let us first show that one can construct a function $\varepsilon_i(t)$ that satisfies condition (b). To see this, note first that since $\mathcal{T}_i$ is countable, we can write $\mathcal{T}_i = \{t_j\}_{j \in \mathbb{N}}$ and, by (b), we can extract a sequence from $\mathcal{T}_i$, $\{\tilde{t}_k\}_{k=\lceil 1/\varepsilon \rceil}^{\infty}$, such that $1 - \tilde{t}_k < 1/k < \varepsilon$. If we set $\varepsilon_i(t_j) = \varepsilon^2/2^j$ for $t_j \notin \{\tilde{t}_k\}_{k=\lceil 1/\varepsilon \rceil}^{\infty}$, and $\varepsilon_i(\tilde{t}_k) = 1/k(k+1)$ for each $k \geq \lceil 1/\varepsilon \rceil$, then, for each $t' \in \mathcal{T}_i$ there is a constant $\kappa > 0$ such that $\sum_{t > t'} \varepsilon_i(t)/\varepsilon_i(t') \geq \lceil 1/\varepsilon \rceil^{-1}/\varepsilon^2 \cdot \kappa \to \infty$ as $\varepsilon \to 0$ (where in the previous inequality we used the fact that $\sum_{j=k}^{\infty} 1/(j(j+1)) = 1/k$).

To see that a THPE exists, fix $\bar{t}_i \in \mathcal{T}_i$ for each i and notice that the following is a THPE:

$$
\begin{aligned}
&\pi_i^*(\emptyset)(\bar{t}_i) = 1 \\
&\pi_i^*(\emptyset, t)(a_i^*, \text{not}) = 1 && \text{for all } t \in \mathcal{T}_i \\
&\pi_i^*((t', a_{-i}, \text{pay}), t)(a_i, \text{not}) = 1 && \text{for some } a_i \in \arg\max_{a_i \in A_i} g_i(a_i, a_{-i}) \\
& && \text{for every } a_{-i} \in A_{-i}, t \in \mathcal{T}_i, t' \in \mathcal{T}_{-i}.
\end{aligned}
$$

To see that this is a THPE, let $\varepsilon_n \to 0$, and consider a sequence, $\{\pi_n\}_{n \in \mathbb{N}}$, of ε_n-constrained equilibria, with trembles satisfying $\varepsilon^n((\emptyset, t), a_i, \tilde{d}_i) = \varepsilon_n$ for $t \neq \bar{t}_i$ and $a_i \neq a_i^*$, and $\varepsilon^n((\emptyset, \bar{t}_i), a_i, \tilde{d}_i) = \varepsilon_n^2$ for $a_i \neq a_i^*$, for each $i \in \{1, 2\}$. Suppose also that $\sum_{t > \bar{t}_i} \varepsilon_i^n(t)$ is of order $\sqrt{\varepsilon_n}$ for $i \in \{1, 2\}$.[39] Along the sequence of ε_n-constrained equilibria π_n, it is a best response for player i to allocate the maximum possible weight on $\bar{t}_i$ and to choose a_i^* with the maximum possible weight at each time t in the absence of a disclosure. Hence, π_n converges to π^*. To see that each i has no profitable deviation under such a strategy, note that the probability that

[39] Let $\{t_j\}_{j \in \mathbb{N}} = \{t \in \mathcal{T}_i, t > \bar{t}_i\}$, we can set $\varepsilon_i^n(t_j) = \frac{1}{(j+k-1)(j+k)}$ with $k = \lceil (1/\sqrt{\varepsilon_n}) \rceil$. This yields $\varepsilon_i^n(t_j) \leq \varepsilon_n$, and $\sum_{j \in \mathbb{N}} \varepsilon_i^n(t_j) = \sum_{j \geq k} \frac{1}{j(1+j)} = \frac{1}{k}$.

i trembles on her own action at time $\bar{t}_i$ is of order ε_n^2, while if she chooses $t \neq \bar{t}_i$ the probability is of order ε_n. At the same time, the probability that the opponent trembles to a move at a time after $\bar{t}_i$ and trembles in his choice of action is bounded by a function of order of at most $\sqrt{\varepsilon_n} \cdot \varepsilon_n$. Hence the benefit from moving later to learn whether the opponent trembles is at most of that order. Therefore, π_n specifies a best response.

$\square$

Let $\bar{c} < \min_{j\in\{1,2\}} \left[g_j(a^*) - \max_{a\neq a^*} g_j(a)\right]/2$ and let π be a THPE in game $(S, (\mathcal{T}_i)_{i\in N}, c)$ with $c < \bar{c}$, and let $\{\pi^m\}_m$ be a sequence of approximating ε_m-constrained equilibria with $\varepsilon_m \to 0$ where $\varepsilon_m \in (0,1)$ for each $m \in \mathbb{N}$. Let π^m's associated tremble functions be $\varepsilon^m(\cdot)$, and $\varepsilon_i^m(\cdot)$.

Claim 10. *If no observation by player j at time t is on the path of play of π_{-j} for player $j \in \{1,2\}$ (i.e. $\sum_{\tilde{t}<t, a_{-j}\in A_{-j}} \pi_{-j}(\emptyset)(\tilde{t})\pi_{-j}(\emptyset,\tilde{t})(a_{-j}, pay) < 1$), then there is $\bar{m} < \infty$ such that for each $m \geq \bar{m}$, either $\pi_j^m(\emptyset)(t) = \varepsilon^m(t)$ or $\pi_j^m(\emptyset,t)(a_j,d_j) = \varepsilon^m((\emptyset,t),a_j,d_j)$ for all $a_j \neq a_j^*$ and $d_j \in \{pay, not\}$.*

Proof. Suppose, to the contrary, that there is a time t, a player $j \in \{1,2\}$ and an action $a_j \neq a_j^*$[40] such that for every $\bar{m} < \infty$, there is $m > \bar{m}$ such that (a) $\pi_j^m(\emptyset)(t) > \varepsilon^m(t)$ and $\pi_j^m(\emptyset,t)(a_j,d_j) > \varepsilon^m((\emptyset,t),a_j,d_j)$ for some $d_j \in \{\text{pay}, \text{not}\}$, (b) it is with positive probability that j does not observe $-j$'s action at t under π_{-j}.

Suppose first that $j = i$ where i is such that for any $t' \in \mathcal{T}_{-i}$, there is $t'' < t'$ with $t'' \in \mathcal{T}_i$. Such i exists by assumption.

Since π^m converges to π, there is $\bar{m}_1 \in \mathbb{N}$ such that player j does not observe $-j$'s action at time t with a probability that is bounded away from zero under π^m for every $m \geq \bar{m}_1$.

Since player j always has an earlier moving time, there is a time $t' \in \mathcal{T}_j$ such that, for all $c < \bar{c}$,

$$\left(\sum_{\tilde{t}\leq t'} \pi_{-j}(\emptyset)(\tilde{t})\right)\left(\min_a g_j(a) - c\right) + \left(\sum_{\tilde{t}>t'} \pi_{-j}(\emptyset)(\tilde{t})\right)(g_j(a^*) - c) > \max_{a\neq a^*} g_j(a).$$

[40] Action a_j can be chosen to be constant in m because the action space is finite.

Since π^m converges to π, there is $\bar{m}_2 \geq \bar{m}_1$ such that

$$\left(\sum_{\tilde{t}\leq t'} \pi^m_{-j}(\emptyset)(\tilde{t})\right)\left(\min_a g_j(a) - c\right) + \left(\sum_{\tilde{t}>t'} \pi^m_{-j}(\emptyset)(\tilde{t})\right)(g_j(a^*) - c) > \max_{a\neq a^*} g_j(a),$$

for every $m \geq \bar{m}_2$. This is a contradiction because the left-hand side is $O(\varepsilon^m)$ away from i's payoff from (a^*_j, pay) at time t' and the payoff from $a_j \neq a^*_j$ is at most $\max_{a\neq a^*} g_j(a)$, which is the right-hand side. Therefore, there is $\bar{m}_3 \geq \bar{m}_2$ such that for $m \geq \bar{m}_3$, player j would strictly prefer to reduce $\pi^m_j(\emptyset)(t)$ in favor of $\pi^m_j(\emptyset)(t')$ and at time t' choose a^*_j with the maximum possible probability under the ε^m-constrained game.

Now suppose that (a) and (b) hold for player $j = -i$, and let

$$\tilde{B} = \{t' \in \mathcal{T}_{-j} | \pi_{-j}(\emptyset)(t') > 0, \pi_{-j}(\emptyset, t')(a_{-j}, \cdot) > 0, a_{-j} \neq a^*_{-j}\}.$$

From Lemma 4, as a THPE is a PBE (Moroni, 2022), we know that player $-j = i$ must be off the path of play of π when she chooses $a_{-j} \neq a^*_{-j}$. Therefore, since $\pi_{-j}(\emptyset)(t) > 0$ for any $t \in \tilde{B}$, if player $-j$ moves at time $t \in \tilde{B}$, the probability that the opponent, j, puts probability 1 on moving times strictly before time t is 1 under π. Notice that this observation implies that $t > \inf \mathcal{T}_j$ for every $t \in \tilde{B}$.

Therefore, there is $\hat{t} \in \mathcal{T}_j$ such that

$$\left(\sum_{\tilde{t}\leq \hat{t}, \tilde{t}\in\tilde{B}} \pi^m_{-j}(\emptyset)(\tilde{t})\right)\left(\min_a g_j(a) - c\right) + \left(\sum_{\tilde{t}>\hat{t} \text{ or } \tilde{t}\notin\tilde{B}} \pi^m_{-j}(\emptyset)(\tilde{t})\right)(g_j(a^*) - c) > \max_{a\neq a^*} g_j(a).$$

At such time $\hat{t}$, the payoff of $a_j \neq a^*_j$ is at most $\max_{a\neq a^*} g_j(a)$, while the payoff from a^*_j is at least $O(\varepsilon^m)$ away from the left-hand side of the previous equation. This is a contradiction as there is $\bar{m} \geq \bar{m}_3$ such that player j would be better off by shifting weight from time t to $\hat{t}$ and choosing a^*_j with the maximum possible probability at time $\hat{t}$, for every $m \geq \bar{m}$. $\square$

Now we show that there is no THPE such that a player i plays $a_i \neq a^*_i$ under no observation. Suppose towards a contradiction, that there is a THPE π such that the set $\tilde{B} = \{t \in \mathcal{T}_i | \pi_i(\emptyset, t)(a_i, \cdot), \pi_i(\emptyset)(t) > 0, a_i \neq a_i^*\}$—of times such that i chooses $a_i \neq a_i^*$ with positive probability upon no disclosure under π—is non-empty. As before, from Lemma 4, as a THPE is a PBE, we know that player i must be off the path of play of π when she chooses $a_i \neq a_i^*$. Therefore, since $\pi(\emptyset)(t) > 0$ for any $t \in \tilde{B}$, if player i moves at time $t \in \tilde{B}$, the probability that $-i$ puts probability 1 on moving times strictly before time t is 1 under π. Notice that this observation implies that $t > \inf \mathcal{T}_{-i}$ for every $t \in \tilde{B}$.

Since the distributions of players' moves are independent, there is $\tilde{t} \in \tilde{B}$ and $\bar{m}_1 \in \mathbb{N}$ such that for all $\hat{t} < \tilde{t}$ and $m \geq \bar{m}_1$,

$$\left(1 - \gamma_m(\hat{t})\right)\left(\min_{a \in A} g_{-i}(a) - c\right) + \gamma_m(\hat{t})(g_{-i}(a^*) - c) > \max_{a \neq a^*} g_{-i}(a),$$

where $\gamma_m(\hat{t}) = (1 - \varepsilon_m) \sum_{\hat{t} < t' \text{ or } t' \notin \tilde{B}} \pi_i^m(\emptyset)(t')$ is a lower bound on the probability that i moves after $\hat{t}$ and does not tremble in her best response. Therefore, at every time $\hat{t} \in \mathcal{T}_{-i}$ such that $\hat{t} < \tilde{t}$, player $-i$ puts the minimum possible weight on $a_{-i} \neq a_{-i}^*$, under π^m.

Thus, there is $\bar{m} \geq \bar{m}_1$ such that for $m \geq \bar{m}$, the probability that player $-i$ chose $a_{-i} \neq a_{-i}^*$ at a time weakly before $\tilde{t}$, denoted $b_m^{\text{true}}(\tilde{t})$, is at most

$$\sum_{t \leq \tilde{t}, a_{-i} \neq a_{-i}^*} \pi_{-i}^m(\emptyset)(t)\varepsilon^m(t, a_{-i}, \text{not}) + \varepsilon_{-i}^m(\tilde{t}) \leq \max_{t \in \mathcal{T}_i, a_{-i} \neq a_{-i}^*} \varepsilon^m(t, a_{-i}, \text{not})|A_{-i}| + \varepsilon_{-i}^m(\tilde{t}) =: b_m^{\text{bound}}(\tilde{t}),$$

where in the previous expression we used the fact that, since $\pi_i(\emptyset)(\tilde{t}) > 0$, if $\tilde{t}$ happens to be an element of $\mathcal{T}_{-i}$, $-i$ is on the path of play of π at time $\tilde{t}$. Hence, by Claim 10, there is $\bar{m} \geq \bar{m}_1$ such that $\pi_{-i}^m(\emptyset, \tilde{t})(a_i, \cdot) > \varepsilon^m((\emptyset, \tilde{t}), a_i, \cdot)$ for all $a_i \neq a_i^*$, implies $\pi_{-i}^m(\emptyset)(\tilde{t}) = \varepsilon_{-i}^m(\tilde{t})$ for every $m \geq \bar{m}$.

The probability that $-i$ chose or will choose a_{-i}^* if i plays (a_i^*, pay) at time $\tilde{t}$, denoted $a_m^{\text{true}}(\tilde{t})$, is at least

$$a_m^{\text{bound}}(\tilde{t}) := \sum_{t < \tilde{t}} \pi_{-i}^m(t, a_{-i}^*, \text{not})\pi_{-i}^m(\emptyset)(t) + \sum_{t > \tilde{t}} \pi_{-i}^m(\emptyset)(t)(1 - \varepsilon_m)$$

for $m \geq \bar{m}$, where we used the fact that $-i$ puts the maximum possible weight on a_{-i}^* upon observing a_i^*. Note that $a_m^{\text{bound}}(\tilde{t})$ is strictly positive for any m because

$\tilde{t} < 1$ (because $\tilde{t} \in \tilde{B} \subseteq \mathcal{T}_i$ and $1 \notin \mathcal{T}_i$) and $\sup \mathcal{T}_{-i} = 1$ by assumption.

Now, note that $0 < a_m^{\text{bound}}(\tilde{t}) \leq a_m^{\text{true}}(\tilde{t})$ and $b_m^{\text{true}}(\tilde{t}) \leq b_m^{\text{bound}}(\tilde{t})$. This implies that $\frac{b_m^{\text{true}}(\tilde{t})}{a_m^{\text{true}}(\tilde{t})} \leq \frac{b_m^{\text{bound}}(\tilde{t})}{a_m^{\text{bound}}(\tilde{t})}$. Hence,

$$\frac{a_m^{\text{bound}}(\tilde{t})}{a_m^{\text{bound}}(\tilde{t}) + b_m^{\text{bound}}(\tilde{t})} = \frac{1}{1 + \frac{b_m^{\text{bound}}(\tilde{t})}{a_m^{\text{bound}}(\tilde{t})}} \leq \frac{1}{1 + \frac{b_m^{\text{true}}(\tilde{t})}{a_m^{\text{true}}(\tilde{t})}} = \frac{a_m^{\text{true}}(\tilde{t})}{a_m^{\text{true}}(\tilde{t}) + b_m^{\text{true}}(\tilde{t})}.$$

Therefore, i's continuation payoff from (a_i^*, pay) at time $\tilde{t}$ is at least

$$\begin{aligned}
&\frac{a_m^{\text{true}}(\tilde{t})}{a_m^{\text{true}}(\tilde{t}) + b_m^{\text{true}}(\tilde{t})}(g_i(a_i^*) - c) + \frac{b_m^{\text{true}}(\tilde{t})}{a_m^{\text{true}}(\tilde{t}) + b_m^{\text{true}}(\tilde{t})}(\min_a g_i(a) - c) \\
\geq &\frac{a_m^{\text{bound}}(\tilde{t})}{a_m^{\text{bound}}(\tilde{t}) + b_m^{\text{bound}}(\tilde{t})}(g_i(a_i^*) - c) + \frac{b_m^{\text{bound}}(\tilde{t})}{a_m^{\text{bound}}(\tilde{t}) + b_m^{\text{bound}}(\tilde{t})}(\min_a g_i(a) - c).
\end{aligned}$$

for $m \geq \bar{m}$. On the other hand, the continuation payoff from $(a_i, \cdot)$ is at most $\max_{a \neq a^*} g_i(a)$. The former dominates the latter for large enough m and c, since $\sum_{t > \tilde{t}} \pi_{-i}^m(\emptyset)(t) \geq \sum_{t > \tilde{t}} \varepsilon_{-i}^m(t)$ (which implies $a_m^{\text{bound}}(\tilde{t})/b_m^{\text{bound}}(\tilde{t}) \to \infty$ as $m \to \infty$ by condition 3 in the definition of THPE). So i would be better off playing a_i^* at $\tilde{t}$ which is in $\tilde{B}$, which is a contradiction, and so $\tilde{B}$ is empty.

Now, at every time t in which $-i$ receives no disclosure with positive probability under π_i, player $-i$ believes that i will choose or has chosen a_i^* with probability 1 due to Bayes rule. Therefore, $-i$ chooses (a_{-i}^*, not) with probability 1 at those times. Since the identity of player i is arbitrary, this implies that at every t with $\pi_j(\emptyset)(t) > 0$, each player $j \in \{1, 2\}$ chooses (a_j^*, not) on the path of play. $\square$

A.4 Proof of Theorem 3

Step 1:

Step 1-1: Fix S, a common-interest game that is s_i-common for each $i \in N$, $\varepsilon \in (0, \min_{i \in N} s_i)$, and a timing structure p that is $(1 + \varepsilon - \min_{i \in N} s_i)$-dispersed. Fix a PBE, σ, and take $c \in (0, \min_{i \in N} \left[\varepsilon(g_i^* - \underline{g}_i)\right])$. Notice that, by the definition of s_i and g_i^S, if the probability that the opponent does not choose a_j^* is at most $s_i - \varepsilon$, then the difference in payoff between a_i^* and $a_i \neq a_i^*$ is at least

$$(1 - (s_i - \varepsilon))g_i^* + (s_i - \varepsilon)\underline{g}_i - g_i^S = \varepsilon(g_i^* - \underline{g}_i) > 0, \tag{9}$$

where the equality follows from S being s_i-common for each $i \in N$.

Let $N_i \subseteq \mathcal{T}_i$ be the set of times $t \in \mathcal{T}_i$ such that there exists a time-t private history of player i, without observation, such that, under σ, i assigns positive probability to an action other than a_i^* at that private history. For contradiction, we suppose that N_i is nonempty for some $i \in N$. Let $t^* := \inf(N_i \cup N_{-i})$.[41] If $t^* \in \mathcal{T}_i$, any player i who moves at time t^* must choose a_i^*. In fact, the probability that any opponent j chooses an action other than a_j^* before time t^* is zero. The probability that the opponent chooses $a_j \neq a_j^*$ if player i chooses (a_i^*, pay) is bounded above by $\text{Prob}^p(T_{-i} = t^* | T_i = t^*, T_{-i} \geq t^*) \leq s_i - \varepsilon$, where the inequality follows from $(1 + \varepsilon - \min_{i \in N} s_i)$-dispersion. Therefore, by equation (9), a lower bound on the difference between i's payoff from (a_i^*, pay) and the one from $a_i \neq a_i^*$ at time t^* is $\varepsilon(g_i^* - \underline{g}_i) - c$, which is strictly positive because $\varepsilon(g_i^* - \underline{g}_i) > c$.

Step 1-2: Similarly, by the definition of $(1 + \varepsilon - \min_{i \in N} s_i)$-dispersion, there must exist $i \in N$ and $t' > t^*$ such that for $j \neq i$ and $t \in (t^*, t'] \cap \mathcal{T}_i$, $\text{Prob}^p(t^* < T_j \leq t | T_i = t, T_j \geq t^*) < s_i - \varepsilon$. Therefore, by the definition of t^* and equation (9), our choice of c implies that, at any time in $(t^*, t']$, (a_i^*, pay) would give such i a strictly higher payoff than playing any action other than a_i^*. Thus, i would not take an action different from a_i^* at any time in $(t^*, t']$, and we must have $(t^*, t'] \cap N_i = (t^*, t'] \cap N_j = \emptyset$. This contradicts the definition of t^*. Hence, $N_i(a^*)$ is empty for each i.

Step 2:

Assume now the potential leader condition. Suppose for contradiction that under the fixed PBE σ, there exist t and i such that there is a positive ex-ante probability with which i pays the disclosure cost at t. Player i's payoff from such σ is strictly less than g_i^*. But consider i's deviation to playing (a_i^*, not) with probability 1 at all the information sets at time t that can be reached with positive probability under σ, while no change is made to the distribution of actions conditional on other private histories. Call this strategy σ_i'. Then, for any realization of $T_j \in \mathcal{T}_j$, since the potential leader condition holds, j is at an information set that can be reached with positive probability under σ, so plays $(a_j^*, \cdot)$. Thus, (σ_i', σ_{-i}) must assign probability one to a^*. Hence, the payoff from (σ_i', σ_{-i}), starting at an

[41] This infimum is taken in $\mathbb{R}$.

information set at time t is g_i^*, and therefore, the deviation is profitable. This is a contradiction to the assumption that σ is a PBE. Therefore, there is no time at which some player pays the disclosure cost.

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