Online Supplementary Appendix to: Rationalizable Partition-Confirmed Equilibrium with Heterogeneous Beliefs

Drew Fudenberg and Yuichiro Kamada*

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*Fudenberg: Department of Economics, Massachusetts Institute of Technology, e-mail: drew.fudenberg@gmail.com Kamada: Haas School of Business, University of California, Berkeley, e-mail: y.cam.24@gmail.com

C The Choice of the Approximation Criterion

To help understand Theorem 2, we now define an alternative notion of ϵ-observational consistency:

Given $\pi \in \Pi$, let $\Pi(D_i(\pi)) := \{\pi' \in \Pi | D_i(\pi') = D_i(\pi)\}$ be the set of strategy profiles that induce the distribution $D_i(\pi)$.

Definition U2(ϵ)$'$ Given a unitary belief model U, version $u_i^k \in U_i$ is strongly ϵ-self-confirming with respect to π^* if $\min_{\tilde{\pi} \in \Pi(D_i(\pi_i^k, \pi_{-i}^*))} \max_{u_{-i} \in \text{supp}(p_i^k)} ||(\pi_i^k, \pi_{-i}(u_{-i})) - \tilde{\pi}|| < \epsilon$.

That is, a version is strongly ϵ-self-confirming with respect to π^* if there exists a single strategy profile of the opponents that (i) induces the same distribution over terminal nodes as π^* and (ii) differs from any strategy profile in the support of the version's belief by up to ϵ.

Definition U3(ϵ)$'$ Given a unitary belief model U, u_i^k is strongly ϵ-observationally consistent if $p_i^k(\tilde{u}_{-i}) > 0$ implies, for each $j \neq i$, $\tilde{u}_j$ is strongly ϵ-self-confirming with respect to $(\pi_i(u_i^k), \pi_{-i}(\tilde{u}_{-i}))$.

Remark 1

(a) There is a $K < \infty$ that does not depend on ϵ such that if the strong ϵ-self-confirming condition holds then the $K\epsilon$-self-confirming condition holds. This is because $||D_i(\pi) - D_i(\pi')|| \leq (\#A)^2 \cdot ||\pi - \pi'||$ for any $\pi, \pi' \in \Pi$ (by Claim 2 in the Appendix of the paper), so

$$||D_i(\pi_i^k, \pi_{-i}) - D_i(\pi_i^k, \pi_{-i}^*)|| = ||D_i(\pi_i^k, \pi_{-i}) - D_i(\tilde{\pi})|| \leq (\#A)^2 \cdot ||(\pi_i^k, \pi_{-i}) - \tilde{\pi}||$$

for any $\tilde{\pi} \in \Pi(D_i(\pi_i^k, \pi_{-i}^*))$.

(b) The ϵ-self-confirming condition allows for ex ante difference of strategies by ϵ, while the strong ϵ-self-confirming condition only allows for an ϵ difference conditional on each information set. To see the difference in an example, consider the game in Example 1 (Figure 1), and suppose now that there is one more player, player 4, who has a singleton action set (we suppress reference to his action in what follows) and the discrete terminal node partition. Let $\pi^* = ((1-\frac{\epsilon}{2})Out_1 + \frac{\epsilon}{2}In_1, U_2, U_3)$, and suppose that version v_4 of player 4 has a belief that assigns a unit mass to $\hat{\pi} = ((1-\frac{\epsilon}{2})Out_1 + \frac{\epsilon}{2}In_1, D_2, D_3)$. Then, v_4 is ϵ-self-confirming with respect to π^* because $D_4(\hat{\pi})$ and $D_4(\pi^*)$ differ only by $\frac{\epsilon}{2}$, as the terminal nodes following player 2's and 3's moves get probability $\frac{\epsilon}{2}$ under either π^* or $\hat{\pi}$. However, v_4 is not strongly ϵ-self-confirming with respect to π^* because $\Pi(D_4(\pi^*)) = \{\pi^*\}$, while $\hat{\pi}$ and π^* differ by 1 at player 2's and 3's information sets.

Example 6 below shows that if we strengthen the definition of unitary ϵ-RPCE to require strong ϵ-observational consistency then the conclusion of Theorem 2 does not hold. The intuition is that a version v_i can conjecture that she is the only agent who plays a particular action, and that action is the only one that leads to an information set h_j which otherwise would not be reached. Observational consistency in the heterogeneous model allows v_i's belief about play at h_j and the belief held by another player to whom v_i assigns positive probability to be different, as v_i's conjecture assigns measure zero to herself (as in Example 5) and play at a zero probability information set does not affect the distribution over terminal node partitions. However, strong ϵ-observational consistency requires that these two beliefs about play at such information sets be close to each other.

Example 7 (Distance between Observations)

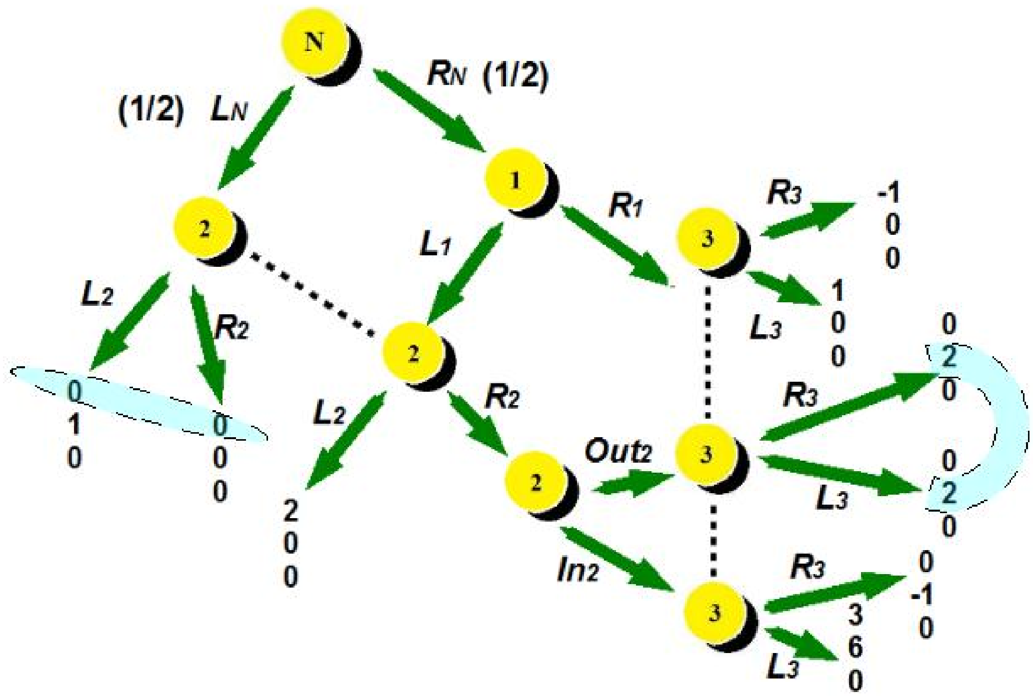

Figure 7

The game in Figure 7 modifies the game in Example 3 by adding decisions for players 2 and 3 after (R_N, L_1, R_2).

The terminal node partitions are such that everyone observes the exact terminal node reached, except that player 1 does not know 2's choice after L_N, and player 2 does not know 3's choice after Out_2.

We first show that $(R_1, (L_2, In_2), L_3)$ is a heterogeneous RPCE. The construction of the belief model is similar to the one for Example 3.[1] Each version of player 1 plays R_1 and believes that all other agents in her player role play L_1. Since in the heterogeneous belief model her conjecture assigns measure zero to the versions who play R_1 (although she herself plays R_1), she is not required (by observational consistency) to think that player 2's observe 3's action and so matches their beliefs with the actual play of player 3.

$$V_1 = \{v_1^1, v_1^2\} \quad \text{with}$$

[1]Since the belief models in this Online Supplementary Appendix are complicated, we also write in the description of each version the behavioral strategy profile in the belief of that version.

$$v_1^1 = (R_1, (L_1, (R_2, Out_2), L_3), (v_1^2, v_2^2, v_3^2)),\ v_1^2 = (L_1, (L_1, (R_2, Out_2), R_3), (v_1^2, v_2^2, v_3^3));$$

$$V_2 = \{v_2^1, v_2^2\} \quad \text{with}$$

$$v_2^1 = ((L_2, In_2), (R_1, (L_2, In_2), R_3), (v_1^1, v_2^1, v_3^3)),\ v_2^2 = ((R_2, Out_2), (L_1, (R_2, Out_2), R_3), (v_1^2, v_2^2, v_3^3));$$

$$V_3 = \{v_3^1, v_3^2, v_3^3\} \quad \text{with}\ \ v_3^1 = (L_3, (R_1, (L_2, In_2), L_3), (v_1^1, v_2^1, v_3^1)),$$

$$v_3^2 = (L_3, (L_1, (R_2, Out_2), L_3), (v_1^2, v_2^1, v_3^2));\ v_3^3 = (R_3, (L_1, (R_2, Out_2), R_3), (v_1^2, v_2^1, v_3^3));$$

$$\phi_1(v_1^1) = 1,\ \phi_2(v_2^1) = 1,\ \phi_3(v_3^1) = 1.$$

It is straightforward to check that this belief model satisfies all the conditions for heterogeneous RPCE.

Now we show that the specified distribution of strategies would not be the result of an ϵ-unitary RPCE in the associated anonymous-matching game with a large number of agents if we replace ϵ-observational consistency with strong ϵ-observational consistency. Intuitively, in the above belief model v_1^1 plays R_1 that leads to player 3's information set and at the same time conjectures that v_2^2 has positive probability. But v_1^1's belief about 3's play (L_3) and v_2^2's belief (R_3) do not coincide. If v_2^2 were to believe that 3 plays L_3 then it would be better for her to play In_2, which would lead 1 to play L_1. In the heterogeneous belief model v_1^1's belief about player 1's play assigns probability exactly equal to zero to R_1, so player 3's node after R_3 gets probability zero even though she herself plays R_1.

Formally, in the unitary model of the associated anonymous-matching game, suppose that there is a positive share of agents of player 1 who play R_1. This means that the share of player 3 who play L_3 must be at least a number close to $\frac{1}{2}$ for large T. With strong ϵ-observational consistency, this means that these versions of player 1 have a conjecture that assigns probability one to a strategy profile close to (R_2, In_2) for large T, as the convex combination of -1 and 6 that assigns to the latter a probability at least a number close to $\frac{1}{2}$ for large T dominates all other payoffs that player 2 can receive in this game. Hence R_1 cannot be a best response.

D The Strength of the Unitary Restriction

Example 8 (The Strength of the Unitary Restriction)

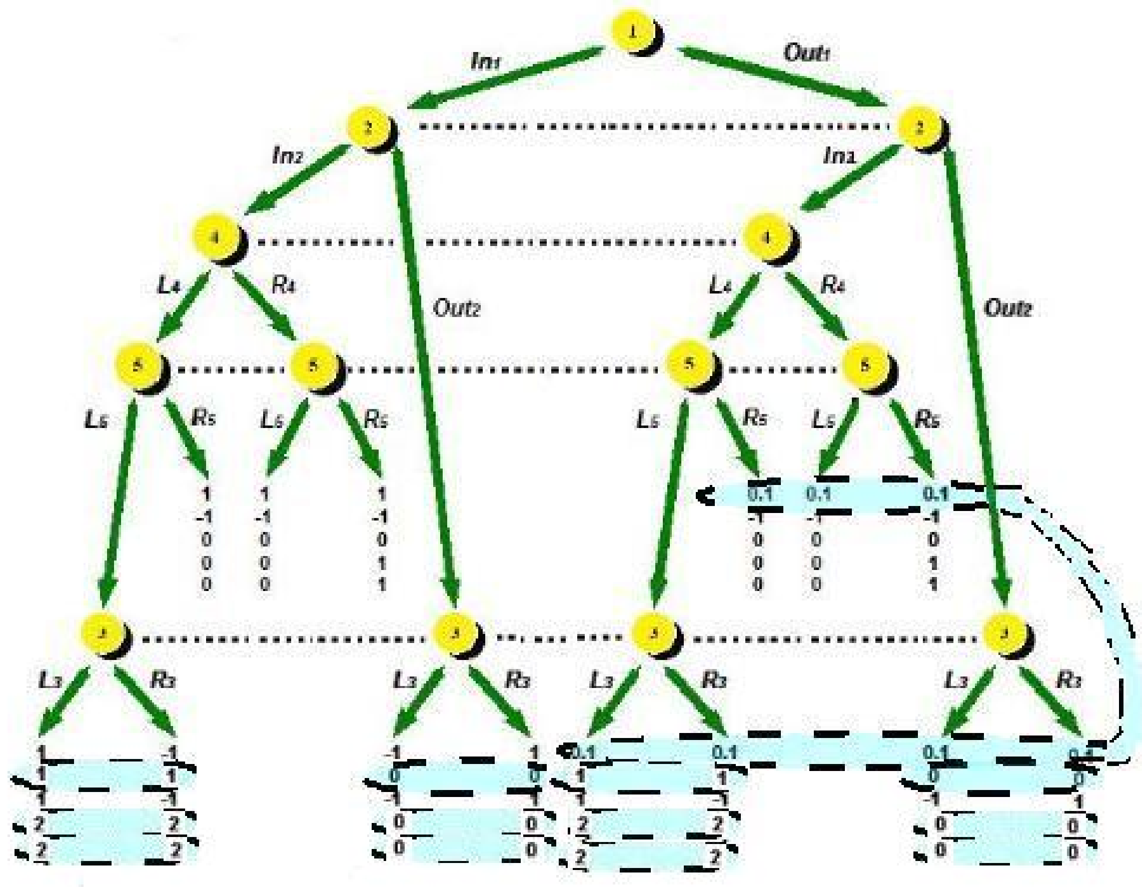

Figure 8

Examples 2 and 3 showed that a pure-strategy heterogeneous RPCE need not be a unitary RPCE. In those examples, there is a version who conjectures that there are other versions in the same player role who play differently. This example shows that even with a restriction that every version (not only actual versions) believes that other versions in the same player role play in the same way as she does, and a restriction that the actual versions' conjectures are correct, a heterogeneous RPCE can be different from a unitary RPCE, because an actual version of one player role can conjecture that different versions in *another* player role play differently.

In particular, we will show that (Out_1, Out_2, R_3) cannot be part of a unitary RPCE, while $(Out_1, Out_2, R_3, R_4, R_5)$ can be a heterogeneous RPCE.

To see that (Out_1, Out_2, R_3) cannot be played in a unitary RPCE, notice that Out_1 can be a best response only when 1 believes that at least one of (In_2, L_4, L_5, R_3) and (Out_2, L_3) is played with positive probability. However, since 3's terminal node partition is discrete and 3's best response is to match 2's action (play L_3 when In_2, and play R_3 when Out_2), observational consistency applied to an actual version of player 1, denoted u_1^k, and the best response condition for player 3 together imply that it cannot be the case that u_1^k believes either L_3 or R_3 is played with probability one, so u_1^k must believe both of L_3 and R_3 are played with positive probability. Since player 3's terminal node partition is discrete, observational consistency applied to u_1^k and the best response condition for player 3 imply that u_1^k believes that In_2 and Out_2 are played with probability $\frac{1}{2}$ each, because 3 must be indifferent between L_3 and R_3.

Since player 2's terminal node partition reveals 4 and 5's play when he plays In_2 with positive probability, u_1^k's belief about players 4 and 5's play must be such that 2 is indifferent between In_2 and Out_2, by observational consistency applied to u_1^k and the best response condition for player 2. But this is possible only when the profile (L_4, L_5) is played with probability exactly equal to $\frac{1}{2}$. Given In_2, 4 and 5's right-hand information sets lie on the path of play, so u_1^k must expect play there to correspond to a Nash equilibrium in their "subgame" (4 and 5 play a best response to each other given the opponent's play). Thus either (a) (L_4, L_5) is played with probability 1, (b) (L_4, L_5) is played with probability 0, or (c) (L_4, L_5) is played with probability $\frac{1}{9}$. Thus the probability assigned to (L_4, L_5) is not $\frac{1}{2}$, so 2 cannot be indifferent, contradicting observational consistency applied to v_1^k and the best response condition for player 2.

Now we show that $(Out_1, Out_2, R_3, R_4, R_5)$ can be a heterogeneous RPCE. To see this, consider the following belief model:

$$V_1 = \{v_1^1\} \quad \text{with}$$

$$v_1^1 = (Out_1, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, \frac{1}{2}L_3 + \frac{1}{2}R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, \frac{1}{2}v_3^3 + \frac{1}{2}v_3^4, v_4^3, v_5^3));$$

$$V_2 = \{v_2^1, v_2^2, v_2^3\} \quad \text{with} \;\; v_2^1 = (Out_2, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_2^2 = (In_2, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2)),$$

$$v_2^3 = (In_2, (Out_1, In_2, R_3, L_4, L_5), (v_1^1, v_2^2, v_3^4, v_4^2, v_5^2));$$

$$V_3 = \{v_3^1, v_3^2, v_3^3, v_3^4\} \quad \text{with} \;\; v_3^1 = (R_3, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_3^2 = (L_3, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2));$$

$$v_3^3 = (L_3, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^3, v_4^4, v_5^4));$$

$$v_3^4 = (R_3, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^3, v_3^4, v_4^5, v_5^5));$$

$$V_4 = \{v_4^1, v_4^2, v_4^3, v_4^4, v_4^5\} \quad \text{with} \;\; v_4^1 = (R_4, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_4^2 = (L_4, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2)),$$

$$v_4^3 = (L_4, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, \frac{1}{2}L_3 + \frac{1}{2}R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, \frac{1}{2}v_3^3 + \frac{1}{2}v_3^4, v_4^3, v_5^3));$$

$$v_4^4 = (L_4, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^3, v_4^4, v_5^4));$$

$$v_4^5 = (L_4, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^4, v_4^5, v_5^5));$$

$$V_5 = \{v_5^1, v_5^2, v_5^3, v_5^4, v_5^5\} \quad \text{with} \;\; v_5^1 = (R_5, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_5^2 = (L_5, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2));$$

$$v_5^3 = (L_5, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, \frac{1}{2}L_3 + \frac{1}{2}R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, \frac{1}{2}v_3^3 + \frac{1}{2}v_3^4, v_4^3, v_5^3));$$

$$v_5^4 = (L_5, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^3, v_4^4, v_5^4));$$

$$v_5^5 = (L_5, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^4, v_4^5, v_5^5));$$

$$\phi_1(v_1^1) = 1,\ \phi_2(v_2^1) = 1,\ \phi_3(v_3^1) = 1,\ \phi_4(v_4^1) = 1,\ \phi_5(v_5^1) = 1.$$

The key here is that v_1^1, the actual version of player 1, can conjecture that there are two actual versions (v_2^1 and v_2^2) in the role of player 2, each of whom conjectures that they are the only one in that role, e.g. that all player 2's reason and play as they do. Out_1 can be a best response only when player 2's action corresponds to a mixed strategy whose support is the strategies played by v_2^1 and v_2^2, but such a mixed strategy would violate the best response condition in the unitary belief model.