Online Appendix

Flash Pass

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In this appendix, we consider the general case where there are two or more types of base utility functions, which we call the **multi-type case**. Some lemmas in the multi-type case are analogous to lemmas in the single-type case. We list the correspondence between these lemmas in Table 1. With these lemmas in the multi-type case, we generalize every two-type result in Section 5. Specifically, Propositions 2–6 and Theorem 4 are generalized to Propositions 11–15 and Theorem 5, respectively. For each pair of correspondence, we later explain how the multi-type result generalizes the two-type result. We have also provided the direct proofs to these two-type results in the Supplementary Information on our personal webpages.

B The General Multi-Type Case

We first set up the multi-type case. Assume that each customer's base utility function comes from $\{u^t\}_{t=1}^T$, where t is the index for a utility type. For every $t \in \{1, \ldots, T\}$, let N^t be the number of customers with base utility function u^t, and let $N = \sum_{t=1}^T N^t$ be the total number of customers. Assume that for every $t \in \{1, \ldots, T-1\}$, $u_n^t > u_n^{t+1}$ for every $n \in \{1, \ldots, N\}$ and $u_n^t - u_{n+1}^t > u_n^{t+1} - u_{n+1}^{t+1}$ for every $n \in \{1, \ldots, N-1\}$. In addition, set $u_0^t = 0$ for all type t. Let $G((N^t)_{t=1}^T, K, p, (u^t)_{t=1}^T)$ be the strategic-form game defined analogously to that in the single-type case.

Given $((N^t)_{t=1}^T, K)$, a scheme $q = (q_0^1, \ldots, q_0^T, q_1^1, \ldots, q_1^T, \ldots, q_K^1, \ldots, q_K^T) \in (0 \cup \mathbb{N})^{(K+1)\times T}$ such that $\sum_{\tau=1}^T q_k^\tau > 0$ for every $k \in \{1, \ldots, K\}$ and $\sum_{j=0}^K q_j^t = N^t$ for every $t \in \{1, \ldots, T\}$ specifies the number of each type of customers in each priority pass, where q_k^t denotes the number of type-t customers buying θ_k. The restriction that $\sum_{t=1}^T q_k^t > 0$ for every $k \in \{1, \ldots, K\}$ ensures that every priority pass has at least one customer, which is analogous to the definition in the single-type case. Fix $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$. Fix a scheme q and a price vector p. For each $t \in \{1, \ldots, T\}$, construct the **type-specific pass-utility function** v^t from u^t.[1] Fix $j \in \{0, \ldots, K\}$

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[1]Formally, the pass-utility function in the multi-type case, $v^t(\cdot;\cdot;\cdot)$ is defined as follows. From a given scheme q in the multi-type case, we construct a scheme $\tilde{q}$ in the single-type case by letting

Description	Single-type	Multi-type
Properties of pass-utility function	Claim 1	Claim 1′
IC Reduction	Lemma 1(a)	Lemma 5
IR Reduction	Lemma 1(b)	Lemma 6
Implementability-checking price vector	Lemma 2	Lemma 7
Implication of concave base utility function	Lemma 3	Lemma 3′

Table 1: Correspondence between lemmas in the single-type and multi-type cases.

and $k \in \{1, \ldots, K\}$ such that $\sum_{t=1}^{T} q_j^t > 0$. For every $t \in \{1, \ldots, T\}$ such that $q_j^t > 0$, the **type-specific IC constraint from θ_j to θ_k with respect to customer type t** (henceforth IC_{jk}^t) is defined analogously to the single-type case with the pass-utility function changed to v^t. For every $t \in \{1, \ldots, T\}$ such that $q_k^t > 0$, the **type-specific IR constraint of θ_k with respect to customer type t** (henceforth IR_k^t) is also defined analogously to the single-type case.

Define the **set of IC constraints** to be $\{\mathrm{IC}_{jk}^t : 0 \le j \le K, 1 \le k \le K, 1 \le t \le T, q_j^t > 0\}$ and the **set of IR constraints** to be $\{\mathrm{IR}_k^t : 1 \le k \le K, 1 \le t \le T, q_k^t > 0\}$. We say that p **implements** q if (p, q) satisfies every constraint in the set of IC and IR constraints. The scheme q is said to be **implementable** if there exists a price vector that implements q.

A multi-type case $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ is **concave** if each base utility function is concave, **strictly concave** if it is strictly concave, and **linear** if it is linear. If $u^t = \beta^t u$ for some $\beta^1 > \beta^2 \cdots > \beta^T = 1$ and some base utility function u such that $u_N > 0$, then we call such a setup the **multiplicative** multi-type case.[2]

B.1 Lemmas for Multi-Type Case

The following two results are analogous to Claim 1 and Lemma 3 in the single-type case. Their proofs are omitted as the proofs are perfectly analogous to those in the single-type case.

Claim 1′. *Fix $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ and a scheme q. Fix $k \in \{1, \ldots, K\}$ and $t \in \{1, \ldots, T\}$. If $j_1, j_2 \in \{1, \ldots, k-1\}$, then*

$$v^t(\theta_k; \theta_{j_1}) = v^t(\theta_k; \theta_{j_2}) > v^t(\theta_k).$$

$\tilde{q}_j = \sum_{t=1}^{T} q_j^t$. Then, we let $v^t(\theta_j, \theta_k; q)$ take the same value as $v(\theta_j; \theta_k; \tilde{q})$ where v is the pass-utility function constructed from u^t in the single-type case. Under this definition, $v^t(\theta_j; \theta_k; q)$ is well-defined even for the case when there is no type-t customer in pass k.

[2]Here, the superscript for each β is an index, not an exponent.

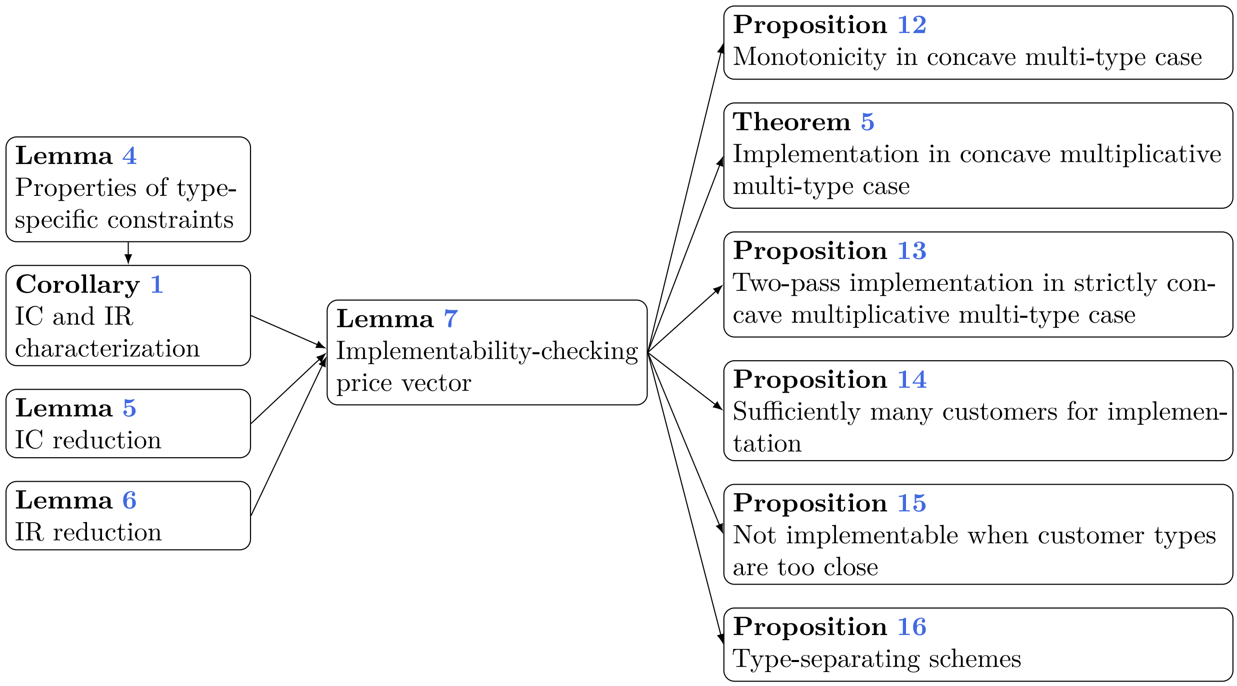

Figure 5: Roadmap of Appendix B.1.

If $l_1, l_2 \in \{0, k+1, k+2, \dots, K\}$*, then*

$$v^t(\theta_k) > v^t(\theta_k; \theta_{l_1}) = v^t(\theta_k; \theta_{l_2}).$$

Lemma 3′. *Fix* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ *such that* $K > 1$*. Fix a scheme* q *and* $t \in \{1, \dots, T\}$*. If* u^t *is concave, then for any* $j, k \in \{1, \dots, K\}$ *such that* $j < k$*,*

$$v^t(\theta_j) - v^t(\theta_k; \theta_j) \leq v^t(\theta_j; \theta_k) - v^t(\theta_k).$$

The inequality is strict if u^t *is strictly concave and either* $\max_{m \in \{j,k\}} \left(\sum_{\tau=1}^T q_m^\tau \right) > 1$ *or* $j + 1 < k$*.*

The rest of this subsection presents lemmas that are useful for implementation results in the multi-type case that will appear in Appendix B.2. Most results about implementation use Lemma 7 which we state later. A roadmap of how lemmas in this subsection contribute to the implementation results in Appendix B.2 is illustrated in Figure 5.

The following result characterizes the relations between type-specific constraints.

Lemma 4 (Properties of type-specific constraints)**.** *Fix* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$*. Fix a scheme* q *and a price vector* p*. Fix* $j, k, m \in \{1, \dots, K\}$ *such that* $j < k$ *and* $t_1, t_2 \in \{1, \dots, T\}$ *such that* $t_1 < t_2$*.*

(a) If $q_j^{t_1} > 0$ and $q_j^{t_2} > 0$, then $IC_{jk}^{t_2}$ implies $IC_{jk}^{t_1}$.

(b) If $q_k^{t_1} > 0$ and $q_k^{t_2} > 0$, then $IC_{kj}^{t_1}$ implies $IC_{kj}^{t_2}$.

(c) If $q_0^{t_1} > 0$ and $q_0^{t_2} > 0$, then $IC_{0m}^{t_1}$ implies $IC_{0m}^{t_2}$.

(d) If $q_m^{t_1} > 0$ and $q_m^{t_2} > 0$, then $IR_m^{t_2}$ implies $IR_m^{t_1}$.

Proof.

Proof for (a) If $q_j^{t_1} > 0$ and $q_j^{t_2} > 0$, then both $\mathrm{IC}_{jk}^{t_1}$ and $\mathrm{IC}_{jk}^{t_2}$ are defined. Note that (p, q) satisfies $\mathrm{IC}_{jk}^{t_2}$ if and only if

$$p_j - p_k \leq v^{t_2}(\theta_j) - v^{t_2}(\theta_k; \theta_j).$$

Since $u_n^{t_1} - u_{n+1}^{t_1} > u_n^{t_2} - u_{n+1}^{t_2}$ holds for every $n \in \{1, \dots, N-1\}$ by the definition of customer types, we have $v^{t_1}(\theta_j) - v^{t_1}(\theta_k; \theta_j) > v^{t_2}(\theta_j) - v^{t_2}(\theta_k; \theta_j)$. Therefore, if (p, q) satisfies $\mathrm{IC}_{jk}^{t_2}$, then

$$p_j - p_k \leq v^{t_2}(\theta_j) - v^{t_2}(\theta_k; \theta_j) < v^{t_1}(\theta_j) - v^{t_1}(\theta_k; \theta_j),$$

which means that (p, q) also satisfies $\mathrm{IC}_{jk}^{t_1}$.

Proof for (b) If $q_k^{t_1} > 0$ and $q_k^{t_2} > 0$, then both $\mathrm{IC}_{kj}^{t_1}$ and $\mathrm{IC}_{kj}^{t_2}$ are defined. Note that (p, q) satisfies $\mathrm{IC}_{kj}^{t_1}$ if and only if

$$p_j - p_k \geq v^{t_1}(\theta_j; \theta_k) - v^{t_1}(\theta_k).$$

Since $u_n^{t_1} - u_{n+1}^{t_1} > u_n^{t_2} - u_{n+1}^{t_2}$ holds for every $n \in \{1, \dots, N-1\}$ by the definition of customer types, we have $v^{t_1}(\theta_j; \theta_k) - v^{t_1}(\theta_k) > v^{t_2}(\theta_j; \theta_k) - v^{t_2}(\theta_k)$. Therefore, if (p, q) satisfies $\mathrm{IC}_{kj}^{t_1}$, then

$$p_j - p_k \geq v^{t_1}(\theta_j; \theta_k) - v^{t_1}(\theta_k) > v^{t_2}(\theta_j; \theta_k) - v^{t_2}(\theta_k),$$

which means that (p, q) also satisfies $\mathrm{IC}_{kj}^{t_2}$.

Proof for (c) If $q_0^{t_1} > 0$ and $q_0^{t_2} > 0$, then both $\mathrm{IC}_{0m}^{t_1}$ and $\mathrm{IC}_{0m}^{t_2}$ are defined. Note that (p, q) satisfies $\mathrm{IC}_{0m}^{t_1}$ if and only if

$$v^{t_1}(\theta_m; \theta_0) - p_m \leq 0.$$

Since $u_n^{t_1} > u_n^{t_2}$ holds for every $n \in \{1, \ldots, N\}$ by the definition of customer types, we have $v^{t_2}(\theta_m; \theta_0) < v^{t_1}(\theta_m; \theta_0)$. Therefore, if (p, q) satisfies $\text{IC}_{0m}^{t_1}$, then

$$v^{t_2}(\theta_m; \theta_0) - p_m < v^{t_1}(\theta_m; \theta_0) - p_m \leq 0,$$

which means (p, q) also satisfies $\text{IC}_{0m}^{t_2}$.

Proof for (d) If $q_m^{t_1} > 0$ and $q_m^{t_2} > 0$, then both $\text{IR}_m^{t_1}$ and $\text{IR}_m^{t_2}$ are defined. Note that (p, q) satisfies $\text{IR}_m^{t_2}$ if and only if

$$v^{t_2}(\theta_m) - p_m \geq 0.$$

Since $u_n^{t_1} > u_n^{t_2}$ holds for every $n \in \{1, \ldots, N\}$ by the definition of customer types, we have $v^{t_1}(\theta_m) > v^{t_2}(\theta_m)$. Therefore, if (p, q) satisfies $\text{IR}_m^{t_2}$, then

$$v^{t_1}(\theta_m) - p_m > v^{t_2}(\theta_m) - p_m \geq 0,$$

which means that (p, q) also satisfies $\text{IR}_m^{t_1}$. $\square$

Fix $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$, a scheme q, and a price vector p. Fix $j \in \{0, \ldots, K\}$ and $k \in \{1, \ldots, K\}$ such that $j \neq k$ and $\sum_{t=1}^T q_j^t > 0$.[3] Define IC_{jk} to be the type-specific IC constraint from $\{\text{IC}_{jk}^t : 1 \leq t \leq T, q_j^t > 0\}$ that implies every constraint in the set. Similarly, define IR_k to be the type-specific IR constraint in $\{\text{IR}_k^t : 1 \leq t \leq T, q_k^t > 0\}$ that implies every constraint in the set. Note that Lemma 4 ensures that IC_{jk} and IR_k are well-defined.

Lemma 4 immediately implies the following characterization of IC and IR constraints without superscripts. We define the following new notations for ease of characterization: For each $j \in \{0, \ldots, K\}$ such that $\sum_{t=1}^T q_j^t > 0$, define $\bar{t}_j = \max\{1 \leq t \leq T : q_j^t > 0\}$, which is the lowest customer type that chooses θ_j, and $\underline{t}_j = \min\{1 \leq t \leq T : q_j^t > 0\}$, which is the highest customer type that chooses θ_j.

Corollary 1 (IC and IR characterization)**.** *Fix $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$. Fix a scheme q and a price vector p. Fix $j, k, m \in \{1, \ldots, K\}$ such that $j < k$.*

(a) IC_{jk} is equivalent to $IC_{jk}^{\bar{t}_j}$.

(b) IC_{kj} is equivalent to $IC_{kj}^{\underline{t}_k}$.

(c) If $\sum_{t=1}^T q_0^t > 0$, then IC_{0m} is equivalent to $IC_{0m}^{\underline{t}_0}$.

[3]Analogous to footnote 16 in the main text, the condition that $\sum_{t=1}^T q_j^t > 0$ is restrictive only when $j = 0$.

(d) IR_m *is equivalent to* $IR_m^{\bar{t}_m}$.

Proof. The result is immediate from Lemma 4. □

The following lemma is analogous to Lemma 1(a) in the single-type case.

Lemma 5 (IC reduction with multiple types)**.** *Fix* $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$ *such that* $K > 1$*. Fix a scheme* q *and a price vector* p*. Fix* $j, k \in \{1, \ldots, K\}$ *such that* $j < k$*. Assume that* $\bar{t}_m \leq \bar{t}_{m+1}$ *for every* $m \in \{j, \ldots, k-1\}$*. If* (p, q) *satisfies* $IC_{m,m+1}$ *for every* $m \in \{j, \ldots, k-1\}$*, then it satisfies* IC_{jk}*.*

Proof. Assume that (p, q) satisfies $\text{IC}_{m,m+1}$ for every $m \in \{j, \ldots, k-1\}$. We have

$$\begin{aligned} p_m - p_{m+1} &\leq v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1}; \theta_m) && (\text{by IC}_{m,m+1}^{\bar{t}_m}) \\ &\leq v^{\bar{t}_j}(\theta_m) - v^{\bar{t}_j}(\theta_{m+1}; \theta_m). && (\text{by } \bar{t}_j \leq \bar{t}_m) \end{aligned} \tag{30}$$

Therefore, if (p, q) satisfies $\text{IC}_{m,m+1}$ for every $m \in \{j, \ldots, k-1\}$, then

$$\begin{aligned} p_j - p_k &= \sum_{m=j}^{k-1} (p_m - p_{m+1}) \\ &\leq \sum_{m=j}^{k-1} \left[v^{\bar{t}_j}(\theta_m) - v^{\bar{t}_j}(\theta_{m+1}; \theta_m) \right] && (\text{by } (30)) \\ &= v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_k) - \sum_{m=j}^{k-1} \left[v^{\bar{t}_j}(\theta_{m+1}; \theta_m) - v^{\bar{t}_j}(\theta_{m+1}) \right] \\ &\leq v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_k) - \left[v^{\bar{t}_j}(\theta_k; \theta_{k-1}) - v^{\bar{t}_j}(\theta_k) \right] && (\text{by Claim } 1') \\ &= v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_k; \theta_j). && (\text{by Claim } 1') \end{aligned}$$

Thus, (p, q) satisfies IC_{jk}. □

The following lemma is analogous to Lemma 1(b) in the single-type case.

Lemma 6 (IR reduction with multiple types)**.** *Fix* $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$ *such that* $K > 1$*. Fix a scheme* q *and a price vector* p*. Assume that* $\bar{t}_j \leq \bar{t}_k$ *for some* $j, k \in \{1, \ldots, K\}$ *such that* $j < k$*. If* (p, q) *satisfies* IC_{jk} *and* IR_k*, then it satisfies* IR_j*.*

Proof. Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$ and $\bar{t}_j \leq \bar{t}_k$. Assume that (p, q) satisfies

IC_{jk} and IR_k. We have

$$\begin{aligned} v^{\bar{t}_j}(\theta_j) - p_j &\geq v^{\bar{t}_j}(\theta_k;\theta_j) - p_k && (\text{by IC}_{jk}^{\bar{t}_j}) \\ &\geq v^{\bar{t}_k}(\theta_k;\theta_j) - p_k && (\text{by } \bar{t}_j \leq \bar{t}_k) \\ &\geq v^{\bar{t}_k}(\theta_k) - p_k && (\text{by Claim } 1') \\ &\geq 0. && (\text{by IR}_k^{\bar{t}_k}) \end{aligned}$$

Thus, (p, q) satisfies IR_j. □

The following result is analogous to Lemma 2 in the single-type case and provides a price vector to check implementability.

Lemma 7 (Implementability conditions in multi-type case)**.** *Fix* $\big((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\big)$. *Fix a scheme* q *such that* $v^{\bar{t}_K}(\theta_K) \geq 0$*. Assume* $\bar{t}_j \leq \bar{t}_{j+1}$ *for every* $j \in \{1, \ldots, K-1\}$*. Let* $p^* = (p_1^*, \ldots, p_K^*)$ *be such that* $p_K^* = v^{\bar{t}_K}(\theta_K)$ *and* $p_j^* - p_{j+1}^* = v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_{j+1};\theta_j)$ *for every* $j \in \{1, \ldots, K-1\}$.[4] *The following statements are equivalent:*

(a) q *is implementable.*

(b) p^* *implements* q.

(c) For any $j, k \in \{1, \ldots, K\}$ *such that* $j < k$, (p^*, q) *satisfies* IC_{kj}*. If* $\sum_{t=1}^T q_0^t > 0$, *then* (p^*, q) *satisfies* IC_{0j} *for every* $j \in \{1, \ldots, K\}$.

Proof. The following result, which generalizes Claim 3 to the multi-type case, is useful in showing that $p^* \in \mathbb{R}_+^K$. We omit its proof as it is analogous to the proof of Claim 3.

Claim 3′. *Fix* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ *and a scheme* q*. For any* $j, k \in \{1, \ldots, K\}$ *such that* $j < k$ *and every* $t \in \{1, \ldots, K\}$*, we have* $v^t(\theta_j) > v^t(\theta_k;\theta_j)$.

By Claim 3′, for every $j \in \{1, \ldots, K-1\}$, we have $p_j^* - p_{j+1}^* = v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_{j+1}) > 0$. Moreover, $p_K^* = v^{\bar{t}_K}(\theta_K) \geq 0$ by assumption. Therefore, p^* is a valid price vector.

It is clear that statement (b) implies statement (a). The proof is complete if we can show that statement (a) implies statement (c) and that statement (c) implies statement (b).

Proof for (a)$\Longrightarrow$(c) Assume that q is implementable and let p be a price vector that implements q. Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. By part (b) of Corollary 1,

[4] This definition generalizes p^* in the proof of Proposition 8.

(p, q) satisfies IC_{kj} if and only if $p_j - p_k \geq v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k)$. Note that

$$\begin{aligned} p_j - p_k &= \sum_{m=j}^{k-1} p_m - p_{m+1} \\ &\leq \sum_{m=j}^{k-1} \left[v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1}; \theta_m) \right] && \text{(by } \text{IC}^{\bar{t}_m}_{m,m+1}\text{)} \\ &= \sum_{m=j}^{k-1} p^*_m - p^*_{m+1} && \text{(by the definition of } p^*\text{)} \\ &= p^*_j - p^*_k. \end{aligned} \tag{31}$$

Thus, we have $p^*_j - p^*_k \geq p_j - p_k \geq v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k)$, that is, (p^*, q) satisfies IC_{kj}.

Assume $\sum_{t=1}^{T} q^t_0 > 0$. We can pick some $t \in \{1, \ldots, T\}$ such that $q^t_0 > 0$. Fix $j \in \{1, \ldots, K\}$. As (p, q) satisfies IC_{0j}, it satisfies IC^t_{0j}, which is equivalent to

$$v^t(\theta_j; \theta_0) - p_j \leq 0. \tag{32}$$

In (31), if we set $k = K$, we have $p_j - p^*_j \leq p_K - p^*_K$. Moreover, as (p, q) satisfies IR_K, by the definition of p^*, we have $p_K \leq v^{\bar{t}_K}(\theta_K) = p^*_K$. Therefore, $p_j \leq p^*_j$ holds and by (32), we have $v^t(\theta_j; \theta_0) - p^*_j \leq 0$, that is, (p^*, q) satisfies IC^t_{0j}. As the choice of t is arbitrary as long as $q^t_0 > 0$, we have shown that (p^*, q) satisfies IC_{0j}.

We have shown that statement (a) implies statement (c).

Proof for (c)$\Longrightarrow$(b) Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. By the definition of p^*, part (a) of Corollary 1 implies that (p^*, q) binds $\text{IC}_{m,m+1}$ for every $m \in \{j, \ldots, k-1\}$. As $\bar{t}_m \leq \bar{t}_{m+1}$ for each $m \in \{j, \ldots, k-1\}$, by Lemma 5, (p^*, q) satisfies IC_{jk}. Therefore, (p^*, q) satisfies every downward IC constraint.

By the definition of p^*, part (d) of Corollary 1 implies that (p^*, q) binds IR_K. Fix $k \in \{1, \ldots, K-1\}$. Note that we have $\bar{t}_k \leq \bar{t}_K$, and (p^*, q) satisfies both IC_{kK} and IR_K. Thus, by Lemma 6, (p^*, q) satisfies IR_k Therefore, (p^*, q) satisfies every IR constraint.

Assume that (p^*, q) satisfies IC_{kj} for any $j, k \in \{1, \ldots, K\}$ such that $j < k$. By the definition of schemes, $\sum_{t=1}^{T} q^t_0 \geq 0$ holds. If $\sum_{t=1}^{T} q^t_0 = 0$, then IC_{0j} is undefined for every $j \in \{1, \ldots, K\}$. In this case, (p^*, q) satisfies every constraint in the set of IC and IR constraints. For the case where $\sum_{t=1}^{T} q^t_0 > 0$, if additionally (p^*, q) satisfies IC_{0j} for every $j \in \{1, \ldots, K\}$, then again (p^*, q) satisfies every constraint in the set of IC and IR constraints. Overall, we have shown that p^* implements q.

We have shown that statement (c) implies statement (b). □

B.2 Implementability in the Multi-Type Case

Now we consider implementability in the multi-type case. In the two-type case, Proposition 2 shows that a scheme with exactly one customer in each priority pass is not implementable if there are more than four priority passes. We generalize this result to the multi-type case below.

Proposition 11 (One-customer passes in multi-type case)**.** *Fix* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ *and a scheme* q *where* $\sum_{t=1}^T q_k^t = 1$ *for every* $k \in \{1, \dots, K\}$*. If* $K > 2T$*, then* q *is not implementable.*

When $T = 1$ and $T = 2$, Proposition 11 is equivalent to Theorem 3 in the single-type case and Proposition 2 in the two-type case, respectively. We omit the proof of Proposition 11 since the result is straightforward given the proof of Theorem 3: when there are more than $2T$ priority passes in a scheme and each pass has exactly one customer, we can pick three customers from three different passes that have the same customer type, which implies that the scheme is not implementable by the reasoning similar to that in the proof of Theorem 3.

We next define the notion of monotonicity in Section B.2.1, which is needed for later results. We then characterize the environment in which multi-pass schemes are implementable in the multi-type case when the base utility function is concave in Section B.2.2.

B.2.1 Monotonicity

Fix $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$. Fix a scheme q and a price vector p. As in the two-type case, we cannot reduce the set of downward IC and IR constraints to the set of local downward IC constraints and IR_K as we did in Lemma 1. By Lemmas 5 and 6, such constraint reduction results can be obtained if we impose the additional condition that $\bar{t}_j \leq \bar{t}_{j+1}$ for every $j \in \{1, \dots, K-1\}$. This restriction eliminates schemes where a lower type customer buys a higher-priority pass than a higher-type customer. With this restriction, by Lemma 7, we can similarly check the implementability of a scheme by binding the lowest IR constraint and every local downward constraint. It turns out that the restriction is necessary for implementability in the concave multi-type case. To be precise, implementability in the concave multi-type case implies a condition which we call monotonicity: We say that a scheme q is **monotone** if the following two conditions hold:

(a) For any $j, k \in \{1, \dots, K\}$ such that $j < k$, $\bar{t}_j \leq \underline{t}_k$.

(b) For every $t \in \{1, \dots, \bar{t}_{K-1}\}$ such that $t < \bar{t}_K$, $q_0^t = 0$.

Condition (a) says that if two customers of different types buy some priority passes, then the higher-type customer has a weakly higher-priority pass than does the lower-type customer; condition (b) states that if a customer type is weakly higher than the lowest customer type in the second-lowest priority pass and strictly higher than the lowest customer type in the lowest priority pass, then every customer of this type has bought some priority pass in the scheme. Condition (b) is ambiguous about customer types strictly between $\bar{t}_{K-1}$ and $\bar{t}_K$ since the negative externality created when a customer of these types joins a priority pass from outside the queue could be a sufficient disincentive against joining the queue.

The following result shows that monotonicity is necessary for implementability in the concave multi-type case:

Proposition 12 (Monotonicity in concave multi-type case)**.** *Fix the concave multi-type case* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ *such that* $K > 1$*. If a scheme* q *is implementable, then each customer type is in at most two priority passes and* q *is monotone.*

Proof. Assume that a scheme q is implementable and let p be a price vector that implements q.

We first show that each customer type is in at most two priority passes. Towards a contradiction, assume that there exists $t \in \{1, \ldots, T\}$ and $j, k, m \in \{1, \ldots, K\}$ such that $j < k < m$, $q_j^t > 0$, $q_k^t > 0$, and $q_m^t > 0$. Note that (p, q) satisfies IC_{mj}^t if and only if

$$p_j - p_m \geq v^t(\theta_j; \theta_m) - v^t(\theta_m). \tag{33}$$

However, as u^t is concave, we have

$$\begin{aligned} p_j - p_m &= p_j - p_k + p_k - p_m \\ &\leq v^t(\theta_j) - v^t(\theta_k; \theta_j) + v^t(\theta_k) - v^t(\theta_m; \theta_k) && \text{(by } \text{IC}_{jk}^t \text{ and } \text{IC}_{km}^t\text{)} \\ &< v^t(\theta_j) - v^t(\theta_m; \theta_j) && \text{(by Claim } 1'\text{)} \\ &\leq v^t(\theta_j; \theta_m) - v^t(\theta_m), && \text{(by Lemma } 3'\text{)} \end{aligned}$$

which contradicts (33). Therefore, each customer type is in at most two priority passes.

We next show that q is monotone by starting with condition (a) of monotonicity. Towards a contradiction, assume that $\bar{t}_j > \underline{t}_k$ for some $j, k \in \{1, \ldots, K\}$ such that $j < k$. With this assumption, we have

$$\begin{aligned} v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_k; \theta_j) &< v^{\underline{t}_k}(\theta_j) - v^{\underline{t}_k}(\theta_k; \theta_j) && \text{(by } \bar{t}_j > \underline{t}_k\text{)} \\ &\leq v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k). && \text{(by Lemma } 3'\text{)} \end{aligned} \tag{34}$$

However, that (p,q) satisfies both IC_{jk} and IC_{kj} implies

$$v^{\underline{t}_k}(\theta_j;\theta_k) - v^{\underline{t}_k}(\theta_k) \le p_j - p_k \le v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_k;\theta_j),$$

which contradicts (34). Therefore, $\bar{t}_j \le \underline{t}_k$ for any $j,k \in \{1,\ldots,K\}$ such that $j<k$, which is condition (a) of monotonicity.

We now show condition (b) of monotonicity. Towards a contradiction, assume $q_0^t > 0$ for some $t \in \{1,\ldots,\bar{t}_{K-1}\}$ such that $t < \bar{t}_K$. That (p,q) satisfies $\text{IC}_{K-1,K}$ implies

$$v^{\bar{t}_{K-1}}(\theta_{K-1}) - p_{K-1} \ge v^{\bar{t}_{K-1}}(\theta_K;\theta_{K-1}) - p_K.$$

As u^t is concave for each $t \in \{1,\ldots,T\}$, by Lemma 3′,

$$v^{\bar{t}_{K-1}}(\theta_{K-1};\theta_K) - v^{\bar{t}_{K-1}}(\theta_{K-1}) \ge v^{\bar{t}_{K-1}}(\theta_K) - v^{\bar{t}_{K-1}}(\theta_K;\theta_{K-1}).$$

Adding up the two inequalities above, we obtain

$$v^{\bar{t}_{K-1}}(\theta_{K-1};\theta_K) - p_{K-1} \ge v^{\bar{t}_{K-1}}(\theta_K) - p_K. \tag{35}$$

Note that we have

$$\begin{aligned}
v^t(\theta_{K-1};\theta_0) - p_{K-1} &= v^t(\theta_{K-1};\theta_K) - p_{K-1} && \text{(by Claim 1′)}\\
&\ge v^{\bar{t}_{K-1}}(\theta_{K-1};\theta_K) - p_{K-1} && (\text{by } t \le \bar{t}_{K-1})\\
&\ge v^{\bar{t}_{K-1}}(\theta_K) - p_K && \text{(by (35))}\\
&\ge v^{\bar{t}_K}(\theta_K) - p_K && (\text{by } \bar{t}_{K-1} \le \bar{t}_K)\\
&\ge 0. && (\text{by IR}_K)
\end{aligned} \tag{36}$$

In (36), if $t < \bar{t}_{K-1}$, then $v^t(\theta_{K-1};\theta_K) - p_{K-1} > v^{\bar{t}_{K-1}}(\theta_{K-1};\theta_K) - p_{K-1}$; if $t = \bar{t}_{K-1}$, then $\bar{t}_{K-1} < \bar{t}_K$ and therefore $v^{\bar{t}_{K-1}}(\theta_K) - p_K > v^{\bar{t}_K}(\theta_K) - p_K$. Thus, at least one of the inequalities in (36) must be strict, which means that if $q_0^t > 0$, then (p,q) would not satisfy $\text{IC}_{0,K-1}^t$, a contradiction. Therefore, if $t \in \{1,\ldots,\bar{t}_{K-1}\}$ and $t < \bar{t}_K$, then $q_0^t = 0$, which is condition (b) of monotonicity.

We have shown that q is monotone.

The proof is complete. □

We observe that when $T=2$, the monotonicity in the multi-type case is equivalent to the monotonicity in the two-type case. Hence, Proposition 12 generalizes Proposition 3 in the two-type case.

The intuition for the necessity of condition (a) of multi-type monotonicity is very similar to that of condition (a) of two-type monotonicity in our discussion after

Proposition 3. For the necessity of condition (b) of multi-type monotonicity, the proof shows that a customer with a type weakly higher than the lowest customer type in the second-lowest priority pass has at least a weak incentive to purchase a priority pass, with the incentive further made strict if the customer type is also strictly higher than the lowest customer type in the lowest priority pass.

B.2.2 Concave Case

When there are multiple customer types, implementing a multi-pass scheme is possible if different types of customers have utility functions that are sufficiently different from each other. The following result for the multiplicative multi-type case characterizes the implementability conditions with respect to customer types in the concave case, which in particular covers both linear and strictly concave cases.

Theorem 5 (Implementation in concave multiplicative multi-type case)**.** *Fix the concave multiplicative multi-type case $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ where $K > 1$. Fix a scheme q such that $v^{\bar{t}_K}(\theta_K) \geq 0$, $\sum_{t=1}^T q_0^t = 0$, and $\bar{t}_m \leq \bar{t}_{m+1}$ for $m \in \{1, \dots, K-1\}$. There exists a profile $(b_{kj})_{j,k\in\{1,\dots,K\},j<k}$ where $b_{kj} \leq \beta^{\bar{t}_j}$ for each pair (j,k) with the inequality being strict if $j < k-1$ such that the following holds: The scheme q is implementable if and only if $\beta^{\underline{t}_k} \leq b_{kj}$ for any $j,k \in \{1,\dots,K\}$ such that $j<k$.*

Proof. Define p^* as in Lemma 7. As $\sum_{t=1}^T q_0^t = 0$, IC_{0k}^t is undefined for every $k \in \{1,\dots,K\}$ and $t \in \{1,\dots,T\}$. Since $\bar{t}_m \leq \bar{t}_{m+1}$ for $m \in \{1,\dots,K-1\}$, the conditions of Lemma 7 hold, and hence q is implementable if and only if p^* implements q. Fix $j,k \in \{1,\dots,K\}$ such that $j<k$. The definition of p^* implies that

$$p_j^* - p_k^* = \sum_{m=j}^{k-1} \left[v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1};\theta_m) \right] = \sum_{m=j}^{k-1} \beta^{\bar{t}_m} \left[v(\theta_m) - v(\theta_{m+1};\theta_m) \right].$$

Thus, (p^*, q) satisfies IC_{kj} if and only if

$$p_j^* - p_k^* = \sum_{m=j}^{k-1} \beta^{\bar{t}_m} \left[v(\theta_m) - v(\theta_{m+1};\theta_m) \right] \geq \beta^{\underline{t}_k} \left[v(\theta_j;\theta_{j+1}) - v(\theta_k) \right].$$

This is equivalent to

$$\beta^{\underline{t}_k} \leq \sum_{m=j}^{k-1} \frac{v(\theta_m) - v(\theta_{m+1};\theta_m)}{v(\theta_j;\theta_{j+1}) - v(\theta_k)} \beta^{\bar{t}_m}. \tag{37}$$

Letting b_{kj} be the right-hand side of (37), Lemma 7 and (37) together imply that q is implementable if and only if $\beta^{\underline{t}_k} \leq b_{kj}$ for any $j,k \in \{1,\dots,K\}$ such that $j<k$.

Lastly, to see that $b_{kj} \leq \beta^{\bar{t}_j}$, note that

$$\begin{aligned}\sum_{m=j}^{k-1} v(\theta_m) - v(\theta_{m+1};\theta_m) &= v(\theta_j) - v(\theta_k) - \sum_{m=j}^{k-1} [v(\theta_{m+1};\theta_m) - v(\theta_{m+1})] \\ &\leq v(\theta_j) - v(\theta_k) - [v(\theta_k;\theta_{k-1}) - v(\theta_k)] \qquad \text{(by Claim 1')} \\ &\leq v(\theta_j;\theta_{j+1}) - v(\theta_k), \qquad \text{(by Lemma 3')}\end{aligned} \tag{38}$$

which implies that b_{kj} is a convex combination of $0, \beta^{\bar{t}_j}, \beta^{\bar{t}_{j+1}}, \ldots,$ and $\beta^{\bar{t}_{k-1}}$, hence $b_{kj} \leq \max_{j \leq m \leq k-1} \beta^{\bar{t}_m} = \beta^{\bar{t}_j}$. Moreover, the inequality in (38) is strict when $j < k-1$. In this case, the weight on 0 in the convex combination for b_{kj} is strictly positive. That is, the sum of coefficients of the β's on the right-hand side of (37) is strictly less than 1. Thus, $b_{kj} < \beta^{\bar{t}_k}$ holds when $j < k-1$. □

We show how Theorem 5 implies Theorem 4 in the two-type case. Consider the setup and the scheme q in Theorem 4. As q is regular, we have $q_k^h > 0$ only if $k \in \{1, 2\}$. Moreover, if $q_2^h > 0$, then $\beta^{\bar{t}_1} = \beta$. When $q_2^h > 0$, with the assumption that u^h is linear, we can calculate b_{21} as in the right-hand side of (37) to get $b_{21} = \beta$. Thus, if we use Theorem 5, it is sufficient to focus on b_{kj} for any $j, k \in \{2, \ldots, K\}$ such that $q_k^h = 0$ and $j < k$. Denote the set of such b_{kj} by $\mathcal{B}$. By Theorem 5, q is implementable if and only if $b_{kj} \geq 1$ for every $b_{kj} \in \mathcal{B}$. Note that $q_K^h = 0$ and $q_{K-2}^l = 0$ by the regularity of q. Thus, $\beta^{\bar{t}_{K-2}} = \beta$ and $b_{K,K-2} \in \mathcal{B}$. Let $\underline{\beta}$ be the smallest value for β such that if $\beta = \underline{\beta}$, then $b_{kj} \geq 1$ for every $b_{kj} \in \mathcal{B}$. Since $b_{K,K-2} < \beta^{\bar{t}_{K-2}} = \beta$ by Theorem 5, we have $\underline{\beta} > 1$. Moreover, $\underline{\beta}$ is well-defined since every $b_{kj} \in \mathcal{B}$ is continuous in β. If $\beta < \underline{\beta}$, we have $b_{kj} < 1$ for some $b_{kj} \in \mathcal{B}$. That is, q is not implementable if $\beta < \underline{\beta}$. If $\beta \geq \underline{\beta}$, then $b_{kj} \geq 1$ for every $b_{kj} \in \mathcal{B}$ since b_{kj} is weakly increasing in β. That is, q is implementable if $\beta \geq \underline{\beta}$. We have shown that q is implementable if and only if $\beta \geq \underline{\beta}$. Thus, Theorem 4 is true.

The intuition of Theorem 5 is similar to that of Theorem 4 in the two-type case: Customer types in different priority passes need to be sufficiently different for the scheme to be implementable. Towards a straightforward intuition, consider a special case of Theorem 5 where u is linear, $K = 3$, and the scheme in consideration has m customers in each priority pass. Given p^* as defined in Lemma 7, by the proof of Theorem 2, (p^*, q) satisfies IC_{31} if and only if

$$\underbrace{\frac{\beta^{\bar{t}_1}}{2}(2m-1)}_{\substack{\text{Upper bound of} \\ p_1 - p_2}} + \underbrace{\frac{\beta^{\bar{t}_2}}{2}(2m-1)}_{\substack{\text{Upper bound of} \\ p_2 - p_3}} \geq \underbrace{\frac{\beta^{\underline{t}_3}}{2}(4m-1)}_{\substack{\text{Lower bound of} \\ p_1 - p_3}}. \tag{39}$$

Note that (39) does not hold if both $\beta^{\bar{t}_1}$ and $\beta^{\bar{t}_2}$ are too close to $\beta^{\underline{t}_3}$; but if $\beta^{\bar{t}_1}$ is sufficiently larger than $\beta^{\underline{t}_3}$, then IC_{31} holds. Intuitively, a larger difference in different types allows for a greater price difference between two priority passes, giving customers in the lower-priority less incentive to upgrade.

In the above argument, $\beta^{\bar{t}_2}$ could even be the same as $\beta^{\underline{t}_3}$ for IC_{31} to hold, as long as $\beta^{\bar{t}_1}$ is taken to be sufficiently high. This, however, is a consequence of the linearity assumption. We emphasize that, in general, for a scheme to be implementable in the concave multiplicative multi-type case, Theorem 5 implies that customer types in different priority passes, including those in passes whose priorities are close, need to be sufficiently different. For example, in the strictly concave multiplicative multi-type case, the existence of an implementable scheme where every customer buys some priority pass implies the existence of a large enough "gap" between two adjacent customer types in the queue, as illustrated below by the following result.

Proposition 13 (Two-pass implementation in strictly concave multiplicative multi-type case)**.** *Fix the strictly concave multiplicative multi-type case* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ *such that* $K = 2$ *and* $N > 2$*. Fix a scheme* q *such that* $v^T(\theta_2) \geq 0$ *and every customer buys some priority pass. The scheme* q *is implementable if and only if* $\underline{t}_2 = \bar{t}_1 + 1$ *and*

$$\frac{\beta^{\bar{t}_1}}{\beta^{\underline{t}_2}} \geq \frac{v(\theta_1;\theta_2) - v(\theta_2)}{v(\theta_1) - v(\theta_2;\theta_1)}. \tag{40}$$

Moreover, the right-hand side of (40) *is strictly larger than* 1*.*

Proof. Assume that q is implementable. Fix $t \in \{1, \ldots, T\}$. Because $N > 2$ and every customer buys some priority pass, by Theorem 1, $q_1^t > 0$ implies $q_2^t = 0$, which implies $\bar{t}_1 \neq \underline{t}_2$. Moreover, Proposition 12 implies $\bar{t}_1 \leq \underline{t}_2$. Lastly, since every customer buys some priority pass, we have $\bar{t}_1 = \underline{t}_2 - 1$.

Define p^* as in Lemma 7. Since $K = 2$ and every customer buys some priority pass, if $\bar{t}_1 = \underline{t}_2 - 1$, then the conditions of Lemma 7 hold. Therefore, by the same lemma, q is implementable if and only if (p^*, q) satisfies IC_{21}. By part (b) of Corollary 1, IC_{21} is equivalent to $\mathrm{IC}_{21}^{\underline{t}_2}$. Thus, given the multiplicative setup, that (p^*, q) satisfies IC_{21} is equivalent to

$$\frac{\beta^{\bar{t}_1}}{\beta^{\underline{t}_2}} \geq \frac{v(\theta_1;\theta_2) - v(\theta_2)}{v(\theta_1) - v(\theta_2;\theta_1)}.$$

Therefore, q is implementable if and only if $\underline{t}_2 = \bar{t}_1 + 1$ and (40) holds. Lastly, because every customer buys some priority pass and $N > 2$, by Lemma 3′, the right-hand side of (40) is strictly larger than 1. □

We show how Proposition 13 generalizes Proposition 4 in the two-type case. To see this, consider the setup and the scheme q in Proposition 4. The condition $K = 2$

is necessary for q to be implementable since the proof of Theorem 1 implies that when every base utility function is strictly concave, no customer type can be in multiple priority passes. Assume $K = 2$. Note that $\underline{t}_2 = \bar{t}_1 + 1$ is equivalent to $q_1^h = N^h$ and $q_2^l = N^l$. Thus, when $\underline{t}_2 = \bar{t}_1 + 1$ holds, we have $\beta^{\bar{t}_1} = \beta$ and $\beta^{\underline{t}_2} = 1$. Define $\underline{\beta} = \frac{v(\theta_1;\theta_2) - v(\theta_2)}{v(\theta_1) - v(\theta_2;\theta_1)}$ as in Proposition 4. By Proposition 13, q is implementable if and only if $q_1^h = N^h$, $q_2^l = N^l$, and $\beta \geq \underline{\beta}$. Moreover, $\underline{\beta} > 1$. Thus, Proposition 13 generalizes Proposition 4.

Proposition 13 implies that a multi-pass scheme may not be implementable when adjacent customer types are very close to each other even if the range of customer types (i.e., $\beta^1 - \beta^T$) is very large.

In Theorem 5 and Proposition 13, we have shown that customer types in different priorities, including those priorities that are close to each other, need to be sufficiently different in an implementable scheme. However, when the adjacent customer types are all close to each other, one may wonder whether "gaps" between customer types in different passes can be created when some types do not buy any pass. For example, suppose there are five customer types in the strictly concave multi-type case, with each customer type being very close to the nearest customer types. Consider the three-pass scheme where the first, the third, and the fifth types respectively buy the three passes, and the second and the fourth types do not buy any pass. In this scheme, there is enough difference between the customer types remaining in the queue. Proposition 12, however, implies that this particular "gap" creation is not possible in an implementable scheme, and there are restrictions to customer exclusions. We characterize some of these restrictions in the result below, which is an immediate implication of Proposition 12.

Corollary 2 (Limits to "gap" creation). *Fix the concave multi-type case* $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$. *Let q be an implementable scheme. The following hold.*

(a) $\underline{t}_1 = 1$.

(b) For every $j \in \{1, \dots, K-2\}$, *we have* $\bar{t}_j = \underline{t}_{j+1}$ *or* $\bar{t}_j + 1 = \underline{t}_{j+1}$.

(c) For every $j \in \{1, \dots, K-1\}$ *and* $t \in \{\underline{t}_j + 1, \dots, \bar{t}_j - 1\}$, *we have* $q_j^t = N^t$.

Part (a) shows that some customer of the highest customer type 1 must buy the first priority pass. Part (b) means that, for every $j \in \{1, \dots, K-1\}$, there is no gap between $\bar{t}_j$ and $\underline{t}_{j+1}$. Part (c) implies that, within a pass except for the lowest-priority pass, customer types in a pass must be "connected": If a customer's type is strictly between the highest and the lowest customer type in a pass whose priority is not the lowest, then this customer must be in that priority pass. Therefore, Corollary 2 shows that, if there is any "gap" created such that some customers do not buy any priority pass, then their types must be between those in the last two priority passes or lower than the lowest type in the last priority pass.

B.3 Queue Size and Implementability

The reader may again notice that, with the customer types fixed, (39) also holds if m is sufficiently large. In the linear multiplicative two-type case, Proposition 5 shows that a scheme is implementable if there are sufficiently many customers in each priority pass, and Proposition 6 shows that a scheme is not implementable if there are too few customers in each priority pass. The two results have generalizations to the general multi-type case. Before we introduce them, we make a definition that will be useful in the generalized results. Given a scheme q in the multiplicative multi-type case where $K > 2$, define $\underline{R}(q) = \min_{1 \leq j \leq K-2} \frac{\beta^{\bar{t}_j}}{\beta^{\underline{t}_{j+2}}}$, which gives the minimum relative difference of customer types in passes that are two priorities apart.

Given the necessary conditions for implementability in the concave case, we make the following assumption about schemes in some of the results that follow.

Definition 4 (Regular scheme)**.** Fix $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$. A scheme q is called **regular** if the following conditions hold:

(a) Every customer buys some priority pass and $v^T(\theta_K) \geq 0$.

(b) The scheme q is monotone.

(c) Every customer type is in at most two priority passes.

Condition (a) is assumed so that there exists a price that makes IR_K hold and that we do not need to consider IR_k or IC_{0k} for $k \in \{1, \ldots, K\}$; thus, we could focus on the switching incentives between different priority passes. By Proposition 12, conditions (b) and (c) are necessary for implementability in the concave multi-type case. Note that when $K > 2$, $\underline{R}(q) > 1$ holds for every regular scheme.

The following result formalizes the conjecture that sufficiently many customers lead to implementability.

Proposition 14 (Sufficiently many customers for implementation)**.** *Fix the linear multiplicative multi-type case $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$. Fix a regular scheme q. If $\sum_{t=1}^T q_k^t \geq \frac{\underline{R}(q)}{2(\underline{R}(q)-1)}$ for every $k \in \{1, \ldots, K\}$, then q is implementable.*

Proof. If $K = 1$, q is the unique regular scheme, and it is implemented by $p = v^T(\theta_1) \geq 0$.

Assume $K > 1$. By the regularity of q and Lemma 7, q is implementable if and only if p^* as defined in the lemma implements q. By the definition of p^*, for every $j \in \{1, \ldots, K-1\}$,

$$p_j^* - p_{j+1}^* = \beta^{\bar{t}_j} \left[v(\theta_j) - v(\theta_{j+1}; \theta_j) \right].$$

Thus, (p^*, q) satisfies $\mathrm{IC}_{j+1,j}$ if and only if

$$\beta^{\bar{t}_j} \left[v(\theta_j) - v(\theta_{j+1}; \theta_j) \right] \geq \beta^{\underline{t}_{j+1}} \left[v(\theta_j; \theta_{j+1}) - v(\theta_{j+1}) \right].$$

Because each base utility function is linear, by (16) and (17), $v(\theta_j) - v(\theta_{j+1}; \theta_j) = v(\theta_j; \theta_{j+1}) - v(\theta_{j+1})$. Moreover, as $\beta^{\bar{t}_j} \geq \beta^{\underline{t}_{j+1}}$ by the definition of regular schemes, (p^*, q) satisfies $\mathrm{IC}_{j+1,j}$. Thus, q is implementable if $K = 2$.

Assume $K > 2$. Fix $k \in \{1, \ldots, K-2\}$ and $j \in \{1, \ldots, k\}$. It remains to show that (p^*, q) satisfies $\mathrm{IC}_{k+2,j}$. Towards this end, note that (p^*, q) satisfies $\mathrm{IC}_{k+2,j}$ if and only if $p_j^* - p_{k+2}^* \geq v^{\underline{t}_{k+2}}(\theta_j; \theta_{k+2}) - v^{\underline{t}_{k+2}}(\theta_{k+2})$. The definition of p^* implies that this condition is equivalent to

$$\sum_{m=j}^{k+1} \left[v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1}; \theta_m) \right] - \left[v^{\underline{t}_{k+2}}(\theta_j; \theta_{k+2}) - v^{\underline{t}_{k+2}}(\theta_{k+2}) \right] \geq 0. \qquad (41)$$

Because each base utility function is linear, we have

$$\begin{aligned} v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1}; \theta_m) &= \frac{\beta^{\bar{t}_m}}{2}(u_{Q_{m-1}(q)+1} + u_{Q_m(q)}) - \frac{\beta^{\bar{t}_m}}{2}(u_{Q_m(q)} + u_{Q_{m+1}(q)}) \\ &= \frac{\beta^{\bar{t}_m}}{2}(q_m + q_{m+1} - 1) \end{aligned}$$

for each $m \in \{j, \ldots, k+1\}$, and

$$\begin{aligned} v^{\underline{t}_{k+2}}(\theta_j; \theta_{k+2}) - v^{\underline{t}_{k+2}}(\theta_{k+2}) &= \frac{\beta^{\underline{t}_{k+2}}}{2}(u_{Q_{j-1}(q)+1} + u_{Q_j(q)+1}) - \frac{\beta^{\underline{t}_{k+2}}}{2}(u_{Q_{k+1}(q)+1} + u_{Q_{k+2}(q)}) \\ &= \frac{\beta^{\underline{t}_{k+2}}}{2}\left(q_j + q_{k+2} - 1 + 2\sum_{m=j+1}^{k+1} q_m \right). \end{aligned}$$

Therefore, (41) is equivalent to

$$\sum_{m=j}^{k+1} \frac{\beta^{\bar{t}_m}}{2}(q_m + q_{m+1} - 1) - \frac{\beta^{\underline{t}_{k+2}}}{2}\left(q_j + q_{k+2} - 1 + 2\sum_{m=j+1}^{k+1} q_m \right) \geq 0. \qquad (42)$$

As the scheme q in consideration is fixed, denote $\underline{R}(q)$ by $\underline{R}$. By the definition of $\underline{R}$ and the regularity of q, $\beta^{\bar{t}_j} \geq \cdots \geq \beta^{\bar{t}_k} \geq \underline{R}\beta^{\underline{t}_{k+2}} > 1$ and $\beta^{\bar{t}_{k+1}} \geq \beta^{\underline{t}_{k+2}} \geq 1$, which together imply that the left-hand side of (42) is increasing in q_m for each $m \in \{j, \dots, k+2\}$. Thus, (42) is implied by the following inequality:

$$\frac{\underline{R}\beta^{\underline{t}_{k+2}}}{2}(k+1-j)(2\underline{m}-1) + \frac{\beta^{\underline{t}_{k+2}}}{2}(2\underline{m}-1) - \frac{\beta^{\underline{t}_{k+2}}}{2}\left[2\underline{m}(k+2-j)-1\right] \geq 0,$$

where $\underline{m} = \min_{1 \leq k \leq K} \sum_{t=1}^{T} q_k^t$. This inequality is equivalent to

$$\underline{R}(k+1-j)(2\underline{m}-1) + (2\underline{m}-1) - [2\underline{m}(k+2-j)-1] \geq 0. \tag{43}$$

Now assume $\sum_{t=1}^{T} q_k^t \geq \frac{\underline{R}}{2(\underline{R}-1)}$ for every $k \in \{1, \dots, K\}$, which is equivalent to $\underline{m} \geq \frac{\underline{R}}{2(\underline{R}-1)}$. We will show that q is implementable. Since $\underline{m} \geq \frac{\underline{R}}{2(\underline{R}-1)}$, the left-hand side of (43) is weakly decreasing in j. Therefore, the left-hand side of (43) weakly decreases if we set j to k since $j \leq k$ by definition. Thus, in this case, (43) is implied by

$$\underline{R}(2\underline{m}-1) + (2\underline{m}-1) - (4\underline{m}-1) \geq 0,$$

which holds for $\underline{m} \geq \frac{\underline{R}}{2(\underline{R}-1)}$. We have shown that (p^*, q) satisfies $\mathrm{IC}_{k+2,j}$ for any $j, k \in \{1, \dots, K-2\}$ such that $j \leq k$. Therefore, (p^*, q) satisfies IC_{kj} for any $j, k \in \{1, \dots, K\}$ such that $j < k$. Thus, by Lemma 7, q is implementable.

Overall, for every $K \geq 1$, q is implementable. This completes the proof. □

We show how Proposition 14 generalizes Proposition 5 in the two-type case. Consider the setup and the scheme q in Proposition 5. When $K = 2$, the same reasoning as in the proof of Proposition 14 for this case shows that q is implementable. When $K > 2$, we have $\underline{R}(q) = \beta$, and Proposition 5 immediately follows from Proposition 14.

As in Proposition 5, the proposition implies that any regular scheme is implementable if $\underline{R}(q) \geq 2$. Similar to Proposition 5, the lower bound of the number of customers in each priority pass in Proposition 14 tends to infinity as $\underline{R}(q)$ approaches 1, and this bound is not stated as a tight bound. This observation motivates the following proposition, which is analogous to Proposition 6.

Proposition 15 (Not implementable when customer types are too close)**.** *Fix the linear multiplicative multi-type case $\left((N^t)_{t=1}^T, K, (u^t)_{t=1}^T\right)$ such that $K > 2$. Fix a regular scheme q. If $\sum_{t=1}^{T} q_k^t < \frac{\underline{R}(q)-\frac{1}{2}}{2(\underline{R}(q)-1)}$ for every $k \in \{1, \dots, K\}$, then q is not implementable.*

Proof. Since the scheme in consideration is fixed, denote $\underline{R}(q)$ by $\underline{R}$ instead. Pick $k \in \{1, \dots, K-2\}$ such that $\frac{\beta^{\bar{t}_k}}{\beta^{\underline{t}_{k+2}}} = \underline{R}$. By Lemma 7, q is implementable if and only if p^* as defined in the lemma implements q. By the definition of p^*, (p^*, q) does not satisfy $\mathrm{IC}_{k+2,k}$ if and only if

$$\underbrace{v^{\bar{t}_k}(\theta_k) - v^{\bar{t}_k}(\theta_{k+1};\theta_k)}_{p^*_k - p^*_{k+1}} + \underbrace{v^{\bar{t}_{k+1}}(\theta_{k+1}) - v^{\bar{t}_{k+1}}(\theta_{k+2};\theta_{k+1})}_{p^*_{k+1}-p^*_{k+2}} - \left[v^{\underline{t}_{k+2}}(\theta_k;\theta_{k+2}) - v^{\underline{t}_{k+2}}(\theta_{k+2})\right] < 0.$$

By the linear multiplicative setup, the above inequality is equivalent to

$$\frac{\beta^{\bar{t}_k}}{2}(q_k + q_{k+1} - 1) + \frac{\beta^{\bar{t}_{k+1}}}{2}(q_{k+1} + q_{k+2} - 1) - \frac{\beta^{\underline{t}_{k+2}}}{2}(q_k + 2q_{k+1} + q_{k+2} - 1) < 0. \quad (44)$$

In the left-hand side of (44), the coefficient of $\beta^{\bar{t}_{k+1}}$ is $q_{k+1} + q_{k+2} - 1$, which is strictly positive; the coefficients of q_k, q_{k+1} and q_{k+2} are $\frac{\beta^{\bar{t}_k}}{2} - \frac{\beta^{\underline{t}_{k+2}}}{2}$, $\frac{\beta^{\bar{t}_k}}{2} + \frac{\beta^{\bar{t}_{k+1}}}{2} - \beta^{\underline{t}_{k+2}}$, and $\frac{\beta^{\bar{t}_{k+1}}}{2} - \frac{\beta^{\underline{t}_{k+2}}}{2}$, all of which are strictly positive by the regularity of q. Thus, if we set q_k, q_{k+1}, and q_{k+2} to $\bar{M} := \max_{1\le k\le K}\sum_{t=1}^T q^t_k$ and $\beta^{\bar{t}_{k+1}}$ to $\beta^{\bar{t}_k} = \underline{R}\beta^{\underline{t}_{k+2}}$, the left-hand side of (44) weakly increases. Therefore, (44) is implied by

$$\frac{\underline{R}\beta^{\underline{t}_{k+2}}}{2}(2\bar{M} - 1) + \frac{\underline{R}\beta^{\underline{t}_{k+2}}}{2}(2\bar{M} - 1) - \frac{\beta^{\underline{t}_{k+2}}}{2}(4\bar{M} - 1) < 0. \quad (45)$$

Now assume that $\sum_{t=1}^T q^t_k < \frac{\underline{R}-\frac{1}{2}}{2(\underline{R}-1)}$ for every $k \in \{1,\dots,K\}$, which is equivalent to $\bar{M} < \frac{\underline{R}-\frac{1}{2}}{2(\underline{R}-1)}$. Note that (45) is equivalent to $\bar{M} < \frac{\underline{R}-\frac{1}{2}}{2(\underline{R}-1)}$. Thus, (p^*, q) does not satisfy $\mathrm{IC}_{k+2,k}$. Therefore, by Lemma 7, q is not implementable. □

With the setup and the scheme q in Proposition 6, we have $\underline{R}(q) = \beta$. Hence, Proposition 15 generalizes Proposition 6 in the two-type case.

Now, we wish to explicitly analyze how the number of customers in each priority pass required for implementability would vary when the number of customer types and passes grow at the same rate, with adjacent customer types getting closer and closer. For a clear picture of this relationship and tractability, the following result considers the customer types that are equally distanced and the schemes in which each priority pass has the same number of customers, and it shows that the required number of customers grows towards infinity.

Proposition 16 (Type-separating schemes)**.** *Fix the linear multiplicative multi-type case $((N^t)_{t=1}^T, K, (u^t)_{t=1}^T)$ such that $K = T$, $N^1 = N^2 = \cdots = N^T = m$ for some m, and there exists $c > 1$ such that $\beta^t = c - \frac{t-1}{T-1}(c-1)$ for $t \in \{1,\dots,T\}$. Consider the scheme q such that $q^t_t = m$ for every $t \in \{1,\dots,T\}$, i.e., every t-th type customer is in the t-th priority pass. Assume $v^T(\theta_K) \ge 0$. Let $M(T) = 1$ if $T = 1, 2$ and $M(T) = \frac{c}{6(c-1)}(T-1) + \frac{1}{6}$ if $T \ge 3$. The scheme q is implementable if and only if $m \ge M(T)$.*

Proof. As $v^T(\theta_K) \geq 0$, q is implementable if $T = 1$. Assume $T \geq 2$. Define p^* as in Lemma 7. Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. Note that by the definition of p^*, (p^*, q) satisfies IC_{kj} if and only if

$$\left(\sum_{n=j}^{k-1} \left[v^{\bar{t}_n}(\theta_n) - v^{\bar{t}_n}(\theta_{n+1}; \theta_n) \right] \right) - \left[v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k) \right] \geq 0. \tag{46}$$

Note that (46) holds for $j = k - 1$. Thus, q is implementable if $T = 2$.

Assume that $T \geq 3$. The following lemma is useful in deriving the condition for (46) to hold for any $j, k \in \{1, \ldots, K\}$ such that $j < k$. Let $\Delta = \beta^1 - \beta^2 = \cdots = \beta^{K-1} - \beta^K = \frac{c-1}{T-1}$.

Lemma 8. *Fix $k \in \{3, \ldots, K\}$. We have that (46) holds for every $j \in \{1, \ldots, k-1\}$ if and only if $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$.*

Proof of Lemma. We first derive an equivalent representation of (46), and then prove the "only if" and "if" parts in turn.

Fix $j \in \{1, \ldots, k-1\}$. By the choice of customer types and q, for each $n \in \{j, \ldots, k-1\}$, $\beta^{\bar{t}_n} = \beta^n = \beta^k + (k-n)\Delta$ and $\beta^{\underline{t}_k} = \beta^k$. Moreover, by the linearity of the base utility functions and the choice of q, we have

$$v^{\bar{t}_n}(\theta_n) - v^{\bar{t}_n}(\theta_{n+1}; \theta_l) = \frac{\beta^k + (k-n)\Delta}{2}(2m-1),$$

and

$$v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k) = \frac{\beta^k}{2}\left[2(k-j)m - 1\right].$$

Therefore, (46) is equivalent to

$$\left(\sum_{n=j}^{k-1} \frac{\beta^k + (k-n)\Delta}{2}(2m-1) \right) - \frac{\beta^k}{2}\left[2(k-j)m - 1\right] \geq 0. \tag{47}$$

Note that (47) holds for $j = k-1$. If $j = k-2$, an algebraic manipulation shows that (47) is equivalent to $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$. Thus, the proof for the lemma is complete if $k = 3$. For the rest of the proof, assume $k > 3$.

"Only if" Part This part is an immediate consequence of our analysis of the case where $j = k-2$ above.

“If” Part Assume $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$. Let

$$\Pi := \frac{\beta^k + 2\Delta}{2}(2m-1) + \frac{\beta^k + \Delta}{2}(2m-1) - \frac{\beta^k}{2}(4m-1),$$

which is the left-hand side of (47) when $j = k-2$. Therefore, $\Pi \geq 0$ holds since $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$. Fix $j \in \{1, \ldots, k-3\}$. It remains to show that (47) holds for the fixed j and k. With the definition of Π, an algebraic manipulation shows that (47) is equivalent to

$$\Pi \geq -\sum_{n=j}^{k-3} \left[\frac{\beta^k + (k-n)\Delta}{2}(2m-1) - \beta^k m\right] = -\sum_{n=j}^{k-3} B(n), \tag{48}$$

where $B(n) := \frac{\beta^k + (k-n)\Delta}{2}(2m-1) - \beta^k m$ for every $n \in \{j, \ldots, k-3\}$. As $\Pi \geq 0$ holds, (48) holds if $B(n) \geq 0$ for every $n \in \{j, \ldots, k-3\}$. Since $B(n)$ is decreasing in n, it is minimized at $n = k-3$. The minimized value is $\frac{\beta^k + 3\Delta}{2}(2m-1) - \beta^k m$. This value is non-negative for $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$, which holds by assumption. Therefore, $B(n) \geq 0$ holds for every $n \in \{j, \ldots, k-3\}$ and thus (48) holds. This completes the proof for the case $k > 3$.

Overall, for every $k \in \{3, \ldots, K\}$, (47) holds for $j \in \{1, \ldots, k-1\}$ if and only if $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$.

The proof for the lemma is complete. □

By Lemma 8, conditional on $T \geq 3$, (46) holds for any $j, k \in \{1, \ldots, K\}$ such that $j < k$ if and only if $m \geq \frac{\beta^k}{6\Delta} + \frac{1}{2}$. As β^k is decreasing in k, we have that conditional on $T \geq 3$, (46) holds for any $j, k \in \{1, \ldots, K\}$ such that $j < k$ if and only if

$$m \geq \frac{\beta^3}{6\Delta} + \frac{1}{2} = \frac{c - 2(c-1)/(T-1)}{6(c-1)/(T-1)} + \frac{1}{2} = \frac{c}{6(c-1)}(T-1) + \frac{1}{6}.$$

To complete the proof, set $M(T) = 1$ if $T = 1, 2$ and $M(T) = \frac{c}{6(c-1)}(T-1) + \frac{1}{6}$ if $T \geq 3$. Then q is implementable if and only if $m \geq M(T)$. □

The proof shows that, with the assumptions in Proposition 16, the scheme q is implementable if and only if p^* in Lemma 7, which binds IR_K and every local downward IC constraint, satisfies IC_{31}. By the linearity of $M(T)$, we see that as the customer types get closer and the number of priority passes gets larger, the required number of customers for implementability grows towards infinity.

Note that in Proposition 16, a larger value of c, which means a wider range for customer types, helps with implementability by lowering $M(T)$. However, there is a limit to how much raising c can help: As $M(T)$ is bounded below by $T/6$, for fixed m

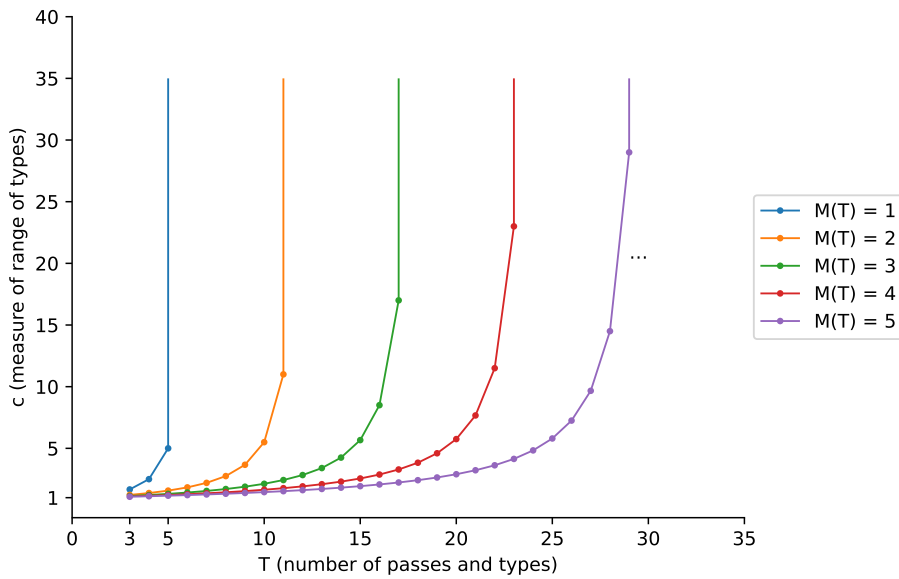

Figure 6: Level curves of $M(T)$ with respect to T and c. For any parameter pair (T, c), if the point (T, c) is to the left of the level curve for $M(T) = m$, then the scheme as described in Proposition 16 where each pass has m customers is implementable. If the point is to the right of that level curve, then such a scheme is not implementable.

and $c > 1$, the scheme q is not implementable if $T > 6m$. In fact, this observation that q is not implementable for large T is an implication of Proposition 15. To see this, note that under Proposition 16's setting, we have $\frac{\beta^1}{\beta^3} = \frac{c}{c-2(c-1)/(T-1)}$, and hence $\underline{R}(q) \leq \frac{c}{c-2(c-1)/(T-1)}$. While this upper bound on $\underline{R}(q)$ is increasing in c, it is bounded above by $\frac{T-1}{T-3}$, which converges to 1 as T tends to infinity. Thus, by Proposition 15, with the number of customers in each priority pass and the range of the customer types $(c-1)$ fixed, the scheme is not implementable if T is large enough.

Figure 6 illustrates the limitation of c's role in helping with implementability. The curves are integer-valued level curves of $M(T)$. For a parameter pair (T, c) and a level curve with value m, if the point (T, c) is to the left of the curve, then the scheme as described in Proposition 16 where every pass has m customers is implementable. In contrast, if the point is to the right of the curve, then such a scheme is not implementable. Given a level curve, we see that whenever the curve becomes vertical, a larger c no longer helps with implementability, illustrating the limited role the parameter c can play in a scheme's implementability. This limitation of c immediately leads to the following result about the special case with $m = 1$, which can be seen as a similar result to Theorem 3 where each customer is in her own pass and no two customers have the same type.

Corollary 3 (Implementation with one-customer passes)**.** *Consider the setting in*

Proposition 16, $m = 1$ *in the scheme* q*. The scheme* q *is not implementable if* $K \geq 6$*.*

Hence, although a large range of customer types $(c - 1)$ makes it possible to implement the scheme where the number of customers equals the number of passes and every pair of customers have different types when there are more than 2 priority passes, this type of scheme is not implementable for however large c when there are 6 or more priority passes.

In summary, we have shown that to implement multi-pass schemes that are not implementable under the single-type case, there need to be large enough gaps between different customer types, and sometimes even a very large gap would not make a scheme implementable. That is, with multiple types of utility functions, the difficulty of implementation is abated yet could persist.