Supplementary Information

Flash Pass

Yuichiro Kamada[†] Zihao Zhou[‡]

First version: November 4, 2022

This version: January 7, 2025

In the Online Appendix, we have generalized and proven the results in Section 5 in the more general multi-type case. For the reader's convenience, in this Supplementary Information, which consists of Appendix C, we present the proofs for results in Section 5 without extending to the multi-type case.

C Proofs for Section 5

Appendix C.1 presents lemmas specific to the two-type case that can be used in proving results in Section 5. The correspondence between these lemmas in the two-type case and those in the multi-type case is listed in Table 2. The proofs for results in Section 5 that are specific to the two-type case appear from Appendix C.2 onward.

C.1 Lemmas for Two-Type Case

The following two results are analogous to Claim 1 and Lemma 3 in the single-type case. Their proofs are omitted as they are perfectly analogous to those in the single-type case.

Claim 1″. *Fix* $((N^h, N^l), K, (u^h, u^l))$ *and a scheme* q. *Fix* $k \in \{1, \ldots, K\}$ *and* $t \in \{h, l\}$. *If* $j_1, j_2 \in \{1, \ldots, k-1\}$, *then*

$$v^t(\theta_k; \theta_{j_1}) = v^t(\theta_k; \theta_{j_2}) > v^t(\theta_k).$$

If $l_1, l_2 \in \{0, k+1, k+2, \ldots, K\}$, *then*

$$v^t(\theta_k) > v^t(\theta_k; \theta_{l_1}) = v^t(\theta_k; \theta_{l_2}).$$

Lemma 3″. *Fix* $((N^h, N^l), K, (u^h, u^l))$ *such that* $K > 1$. *Fix a scheme* q *and* $t \in \{h, l\}$. *If* u^t *is concave, then for any* $j, k \in \{1, \ldots, K\}$ *such that* $j < k$,

$$v^t(\theta_j) - v^t(\theta_k; \theta_j) \leq v^t(\theta_j; \theta_k) - v^t(\theta_k).$$

[†]Yuichiro Kamada: Haas School of Business, University of California, Berkeley and University of Tokyo, Faculty of Economics, y.cam.24@gmail.com

[‡]Zihao Zhou: UCL School of Management, University College London, zihao.zhou@ucl.ac.uk

Description	Two-type	Multi-type
Properties of pass-utility function	Claim 1″	Claim 1′
Implication of concave base utility function	Lemma 3″	Lemma 3′
Properties of type-specific constraints	Lemma 9	Lemma 4
IC and IR characterization	Corollary 4	Corollary 1
IC reduction	Lemma 10	Lemma 5
IR reduction	Lemma 11	Lemma 6
Implementability-checking price vector	Lemma 12	Lemma 7

Table 2: Correspondence between lemmas in the two-type and multi-type cases.

The inequality is strict if u^t *is strictly concave and either* $\max_{m\in\{j,k\}}(q^h_m + q^l_m) > 1$ *or* $j+1 < k$.

The proofs for the results specific to the two-type case from Appendix C.2 onward often use a result analogous to Lemma 2 in the single-type case, which we state in this section as Lemma 12. This lemma in the two-type case itself uses a generalization of the constraint reduction results in the single-type case (parts (a) and (b) of Lemma 1), and we state them as Lemmas 10 and 11. Figure 7 provides a roadmap of how the results in this subsection contribute to proofs from Appendix C.2 onward.

We first derive a result for the relation between type-specific IC and IR constraints.

Lemma 9 (Properties of type-specific constraints)**.** *Fix* $((N^h, N^l), K, (u^h, u^l))$*. Fix a scheme* q *and a price vector* p*. Fix* $j, k, m \in \{1, \ldots, K\}$ *such that* $j < k$*.*

(a) If $q^h_j > 0$ *and* $q^l_j > 0$*, then* IC^l_{jk} *implies* IC^h_{jk}*.*

(b) If $q^h_k > 0$ *and* $q^l_k > 0$*, then* IC^h_{kj} *implies* IC^l_{kj}*.*

(c) If $q^h_0 > 0$ *and* $q^l_0 > 0$*, then* IC^h_{0m} *implies* IC^l_{0m}*.*

(d) If $q^h_m > 0$ *and* $q^l_m > 0$*, then* IR^l_m *implies* IR^h_m*.*

Proof.

Proof for (a) If $q^h_j > 0$ and $q^l_j > 0$, then both IC^h_{jk} and IC^l_{jk} are defined. Note that (p, q) satisfies IC^l_{jk} if and only if

$$p_j - p_k \leq v^l(\theta_j) - v^l(\theta_k; \theta_j).$$

Since $u^h_n - u^h_{n+1} > u^l_n - u^l_{n+1}$ holds for every $n \in \{1, \ldots, N-1\}$ by the definition of customer types, we have $v^h(\theta_j) - v^h(\theta_k; \theta_j) > v^l(\theta_j) - v^l(\theta_k; \theta_j)$. Therefore, if (p, q)

satisfies IC^l_{jk}, then

$$p_j - p_k \leq v^l(\theta_j) - v^l(\theta_k;\theta_j) < v^h(\theta_j) - v^h(\theta_k;\theta_j),$$

which means that (p,q) also satisfies IC^h_{jk}.

Proof for (b) If $q^h_k > 0$ and $q^l_k > 0$, then both IC^h_{kj} and IC^l_{kj} are defined. Note that (p,q) satisfies IC^h_{kj} if and only if

$$p_j - p_k \geq v^h(\theta_j;\theta_k) - v^h(\theta_k).$$

Since $u^h_n - u^h_{n+1} > u^l_n - u^l_{n+1}$ holds for every $n \in \{1,\ldots,N-1\}$ by the definition of customer types, we have $v^h(\theta_j;\theta_k) - v^h(\theta_k) > v^l(\theta_j;\theta_k) - v^l(\theta_k)$. Therefore, if (p,q) satisfies IC^h_{kj}, then

$$p_j - p_k \geq v^h(\theta_j;\theta_k) - v^h(\theta_k) > v^l(\theta_j;\theta_k) - v^l(\theta_k),$$

which means that (p,q) also satisfies IC^l_{kj}.

Proof for (c) If $q^h_0 > 0$ and $q^l_0 > 0$, then both IC^h_{0m} and IC^l_{0m} are defined. Note that (p,q) satisfies IC^h_{0m} if and only if

$$v^h(\theta_m;\theta_0) - p_m \leq 0.$$

Since $u^h_n > u^l_n$ holds for every $n \in \{1,\ldots,N\}$ by the definition of customer types, we have $v^l(\theta_m;\theta_0) < v^h(\theta_m;\theta_0)$. Therefore, if (p,q) satisfies IC^h_{0m}, then

$$v^l(\theta_m;\theta_0) - p_m < v^h(\theta_m;\theta_0) - p_m \leq 0,$$

which means that (p,q) also satisfies IC^l_{0m}.

Proof for (d) If $q^h_m > 0$ and $q^l_m > 0$, then both IR^h_m and IR^l_m are defined. Note that (p,q) satisfies IR^l_m if and only if

$$v^l(\theta_m) - p_m \geq 0.$$

Since $u^h_n > u^l_n$ holds for every $n \in \{1,\ldots,N\}$ by the definition of customer types, we have $v^h(\theta_m) > v^l(\theta_m)$. Therefore, if (p,q) satisfies IR^l_m, then

$$v^h(\theta_m) - p_m > v^l(\theta_m) - p_m \geq 0,$$

which means that (p,q) also satisfies IR_m^h. $\square$

Fix $((N^h, N^l), K, (u^h, u^l))$, a scheme q, and a price vector p. Fix $j \in \{0, \dots, K\}$ and $k \in \{1, \dots, K\}$ such that $j \neq k$ and $q_j^h + q_j^l > 0$. Define IC_{jk} to be the type-specific IC constraint in $\{\mathrm{IC}_{jk}^t : t \in \{h, l\}, q_j^t > 0\}$ that implies every constraint in the set; define IR_k to be the type-specific IR constraint in $\{\mathrm{IR}_k^t : t \in \{h, l\}, q_k^t > 0\}$ that implies every constraint in the set. Lemma 9 ensures that IC_{jk} and IR_k are well-defined.

Lemma 9 immediately implies the following characterization of IC and IR constraints without the type superscripts. We define the following new notations for ease of characterization: Given a scheme q, for each $j \in \{0, \dots, K\}$ such that $q_j^h + q_j^l > 0$, define $\bar{t}_j = l$ if $q_j^l > 0$ and otherwise $\bar{t}_j = h$; define $\underline{t}_j = h$ if $q_j^h > 0$ and otherwise $\underline{t}_j = l$.

Corollary 4 (IC and IR characterization)**.** *Fix $((N^h, N^l), K, (u^h, u^l))$. Fix a scheme q and a price vector p. Fix $j, k, m \in \{1, \dots, K\}$ such that $j < k$.*

(a) IC_{jk} is equivalent to $IC_{jk}^{\bar{t}_j}$.

(b) IC_{kj} is equivalent to $IC_{kj}^{\underline{t}_k}$.

(c) If $q_0^h + q_0^l > 0$, then IC_{0m} is equivalent to $IC_{0m}^{\underline{t}_0}$.

(d) IR_m is equivalent to $IR_m^{\bar{t}_m}$.

Proof. The result is immediate from Lemma 9. $\square$

The following two lemmas provide conditions under which the downward IC and IR constraint reductions are valid in the two-type case.

Lemma 10 (IC Reduction with two types)**.** *Fix $((N^h, N^l), K, (u^h, u^l))$ such that $K > 1$. Fix a scheme q and a price vector p. Fix $j, k \in \{1, \dots, K\}$ such that $j < k$. Assume that $q_j^l = 0$ or $q_m^l > 0$ for every $m \in \{j, \dots, k-1\}$. If (p, q) satisfies $IC_{m,m+1}$ for every $m \in \{j, \dots, k-1\}$, then it satisfies IC_{jk}.*

Proof. Assume $q_j^l = 0$, which implies that IC_{jk} is equivalent to IC_{jk}^h. Assume that (p, q) satisfies $\mathrm{IC}_{m,m+1}$ for every $m \in \{j, \dots, k-1\}$. By $\mathrm{IC}_{m,m+1}$ and the definition of customer types, we have

$$p_m - p_{m+1} \leq \max_{t \in \{h,l\}} \left[v^t(\theta_m) - v^t(\theta_{m+1}; \theta_m) \right] = v^h(\theta_m) - v^h(\theta_{m+1}; \theta_m). \tag{49}$$

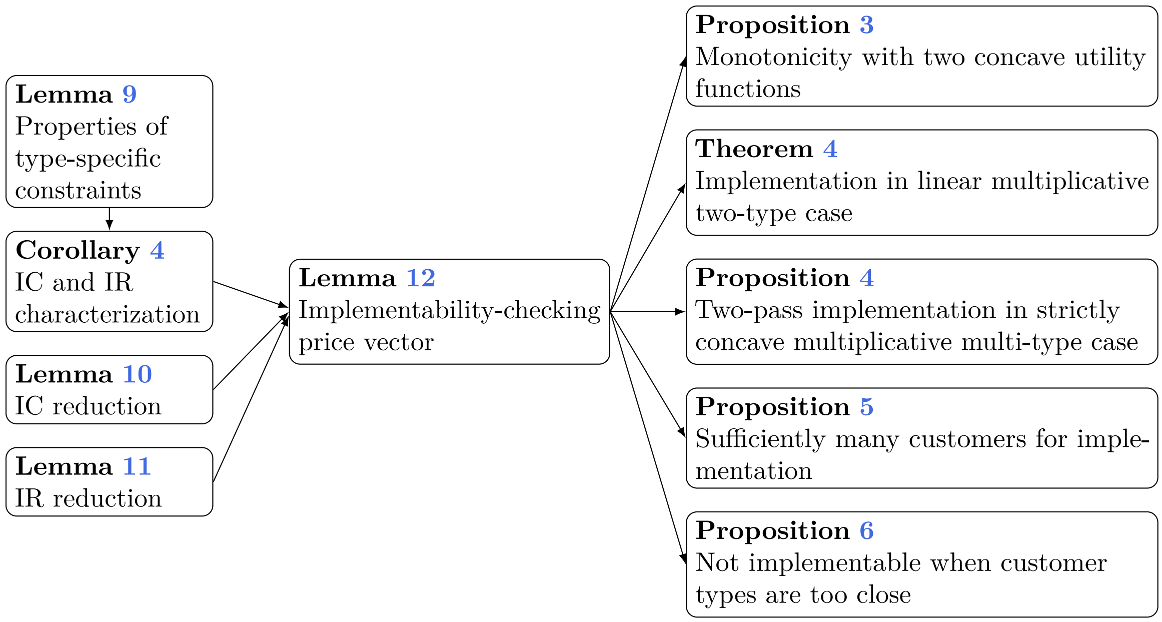

Figure 7: Roadmap of Appendix C.1.

Therefore, if (p, q) satisfies $\mathrm{IC}_{m,m+1}$ for every $m \in \{j, \ldots, k-1\}$, then

$$
\begin{aligned}
p_j - p_k &= \sum_{m=j}^{k-1} (p_m - p_{m+1}) \\
&\leq \sum_{m=j}^{k-1} \left[v^h(\theta_m) - v^h(\theta_{m+1}; \theta_m) \right] && \text{(by (49))} \\
&= v^h(\theta_j) - v^h(\theta_k) - \sum_{m=j}^{k-1} \left[v^h(\theta_{m+1}; \theta_m) - v^h(\theta_{m+1}) \right] \\
&\leq v^h(\theta_j) - v^h(\theta_k) - \left[v^h(\theta_k; \theta_{k-1}) - v^h(\theta_k) \right] && \text{(by Claim } 1'') \\
&= v^h(\theta_j) - v^h(\theta_k; \theta_j). && \text{(by Claim } 1'')
\end{aligned}
$$

Thus, (p, q) satisfies IC_{jk}.

Now assume $q^l_m > 0$ for every $m \in \{j, \ldots, k-1\}$, which by part (a) of Corollary 4 implies that IC_{jk} is equivalent to IC^l_{jk} and $\mathrm{IC}_{m,m+1}$ is equivalent to $\mathrm{IC}^l_{m,m+1}$. By part (a) of Lemma 1, (p, q) satisfies IC_{jk}. □

Lemma 11 (IR Reduction with two types)**.** *Fix $((N^h, N^l), K, (u^h, u^l))$ such that $K > 1$. Fix a scheme q and a price vector p. Assume $q^l_j = 0$ or $q^l_k > 0$ for some $j, k \in \{1, \ldots, K\}$ such that $j < k$. If (p, q) satisfies IC_{jk} and IR_k, then it satisfies IR_j.*

Proof. Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. Assume that (p, q) satisfies IC_{jk} and

IR_k. Assume $q_j^l = 0$, which implies that IC_{jk} is equivalent to IC_{jk}^h and IR_j is equivalent to IR_j^h. As (p,q) satisfies IR_k, by the definition of customer types, we have

$$v^h(\theta_k) - p_k = \max_{t\in\{h,l\}} v^t(\theta_k) - p_k \geq 0. \tag{50}$$

Therefore, we have

$$\begin{aligned} v^h(\theta_j) - p_j &\geq v^h(\theta_k;\theta_j) - p_k && (\text{by } \text{IC}_{jk}^h)\\ &\geq v^h(\theta_k) - p_k && (\text{by Claim } 1'')\\ &\geq 0. && (\text{by } (50)) \end{aligned}$$

Thus, (p,q) satisfies IR_j.

Now assume $q_k^l > 0$, which by part (d) of Corollary 4 implies that IR_k is equivalent to IR_k^l. For every $t \in \{h,l\}$ such that $q_j^t > 0$, we have

$$\begin{aligned} v^t(\theta_j) - p_j &\geq v^t(\theta_k;\theta_j) - p_k && (\text{by } \text{IC}_{jk}^t)\\ &\geq v^l(\theta_k;\theta_j) - p_k && (\text{by the definition of types})\\ &\geq v^l(\theta_k) - p_k && (\text{by Claim } 1'')\\ &\geq 0. && (\text{by } \text{IR}_k^l) \end{aligned}$$

Thus, (p,q) satisfies IR_j. □

The following result is analogous to Lemma 2 in the single-type case and provides a price vector to check implementability. It turns out that when there is more than one customer type, we no longer have the partial upward IC constraint reduction as in part (c) of Lemma 2.

Lemma 12 (Two-type implementation)**.** *Fix $((N^h,N^l),K,(u^h,u^l))$ and a scheme q such that $v^{\bar{t}_K}(\theta_K) \geq 0$. Assume that for every $j \in \{1,\ldots,K-1\}$, $q_j^l > 0$ implies $q_{j+1}^l > 0$. Let $p^* = (p_1^*,\ldots,p_K^*)$ be such that $p_K^* = v^{\bar{t}_K}(\theta_K)$ and $p_j^* - p_{j+1}^* = v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_{j+1};\theta_j)$ for every $j \in \{1,\ldots,K-1\}$. The following statements are equivalent:*

(a) q is implementable.

(b) p^* implements q.

(c) For any $j,k \in \{1,\ldots,K\}$ such that $j<k$, (p^*,q) satisfies IC_{kj}. If $q_0^h + q_0^l > 0$, then (p^*,q) satisfies IC_{0j} for every $j \in \{1,\ldots,K\}$.

Proof. The following result, which generalizes Claim 3 to the two-type case, is useful in showing that $p^* \in \mathbb{R}_+^K$. We omit its proof since it is analogous to the proof of Claim 3.

Claim 3″. *Fix* $((N^h, N^l), K, (u^h, u^l))$ *and a scheme* q. *For any* $j, k \in \{1, \ldots, K\}$ *such that* $j < k$ *and every* $t \in \{h, l\}$, *we have* $v^t(\theta_j) > v^t(\theta_k; \theta_j)$.

By Claim 3″, for every $j \in \{1, \ldots, K-1\}$, we have $p^*_j - p^*_{j+1} = v^{\bar{t}_j}(\theta_j) - v^{\bar{t}_j}(\theta_{j+1}; \theta_j) > 0$. Moreover, $p^*_K = v^{\bar{t}_K}(\theta_K) \geq 0$ by assumption. Therefore, p^* is a valid price vector.

It is clear that statement (b) implies statement (a). The proof is complete if we can show that statement (a) implies statement (c) and that statement (c) implies statement (b).

Proof for (a)⟹(c) Assume that q is implementable and let p be a price vector that implements q. Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. By part (b) of Corollary 4, (p, q) satisfies IC_{kj} if and only if $p_j - p_k \geq v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k)$. Note that

$$
\begin{aligned}
p_j - p_k &= \sum_{m=j}^{k-1} p_m - p_{m+1} \\
&\leq \sum_{m=j}^{k-1} \left[v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1}; \theta_m) \right] && (\text{by } \mathrm{IC}^{\bar{t}_m}_{m,m+1}) \\
&= \sum_{m=j}^{k-1} p^*_m - p^*_{m+1} && (\text{by the definition of } p^*) \\
&= p^*_j - p^*_k. && (51)
\end{aligned}
$$

Thus, we have $p^*_j - p^*_k \geq p_j - p_k \geq v^{\underline{t}_k}(\theta_j; \theta_k) - v^{\underline{t}_k}(\theta_k)$, that is, (p^*, q) satisfies IC_{kj}.

Assume $q^h_0 + q^l_0 > 0$. We can pick some $t \in \{h, l\}$ such that $q^t_0 > 0$. Fix $j \in \{1, \ldots, K\}$. As (p, q) satisfies IC_{0j}, it satisfies IC^t_{0j}, which is equivalent to

$$
v^t(\theta_j; \theta_0) - p_j \leq 0. \tag{52}
$$

In (51), if we set $k = K$, we have $p_j - p^*_j \leq p_K - p^*_K$. Moreover, as (p, q) satisfies IR_K, by the definition of p^*, we have $p_K \leq v^{\bar{t}_K}(\theta_K) = p^*_K$. Therefore, $p_j \leq p^*_j$ holds and by (52), we have $v^t(\theta_j; \theta_0) - p^*_j \leq 0$, that is, (p^*, q) satisfies IC^t_{0j}. As the choice of t is arbitrary as long as $q^t_0 > 0$, we have shown that (p^*, q) satisfies IC_{0j}.

We have shown that statement (a) implies statement (c).

Proof for (c)⟹(b) Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. By the definition of p^*, part (a) of Corollary 4 implies that (p^*, q) binds $\mathrm{IC}_{m,m+1}$ for every $m \in \{j, \ldots, k-1\}$. If $q^l_j = 0$, then the conditions of Lemma 10 hold. If $q^l_j > 0$, then by assumption $q^l_m > 0$ for every $m \in \{j, \ldots, k-1\}$, in which case the conditions of Lemma 10 again hold.

By Lemma 10, (p^*, q) satisfies IC_{jk}. Therefore, (p^*, q) satisfies every downward IC constraint.

By the definition of p^*, part (d) of Corollary 4 implies that (p^*, q) binds IR_K. Fix $k \in \{1, \ldots, K-1\}$. If $q_k^l > 0$, then by assumption $q_K^l > 0$. Thus, $q_k^l = 0$ or $q_K^l > 0$, that is, the conditions of Lemma 11 hold. As (p^*, q) satisfies IC_{kK} and IR_K, by Lemma 11, (p^*, q) satisfies IR_k. Therefore, (p^*, q) satisfies every IR constraint.

Assume that (p^*, q) satisfies IC_{kj} for any $j, k \in \{1, \ldots, K\}$ such that $j < k$. By the definition of schemes, $q_0^h + q_0^l \geq 0$ holds. If $q_0^h + q_0^l = 0$, then IC_{0j} is undefined for every $j \in \{1, \ldots, K\}$. In this case, (p^*, q) satisfies every constraint in the set of IC and IR constraints. For the case where $q_0^h + q_0^l > 0$, if additionally (p^*, q) satisfies IC_{0j} for every $j \in \{1, \ldots, K\}$, then again (p^*, q) satisfies every constraint in the set of IC and IR constraints. Overall, we have shown that p^* implements q.

We have shown that statement (c) implies statement (b). □

C.2 Proof of Proposition 3

Proof. Assume that a scheme q is implementable and let p be a price vector that implements q.

We first show that every customer is in at most two priority passes. Towards a contradiction, assume that there exists $t \in \{h, l\}$ and $j, k, m \in \{1, \ldots, K\}$ such that $j < k < m$, $q_j^t > 0$, $q_k^t > 0$, and $q_m^t > 0$. Note that (p, q) satisfies IC_{mj}^t if and only if

$$p_j - p_m \geq v^t(\theta_j; \theta_m) - v^t(\theta_m). \tag{53}$$

However, because u^t is concave, we have

$$\begin{aligned}
p_j - p_m &= p_j - p_k + p_k - p_m \\
&\leq v^t(\theta_j) - v^t(\theta_k; \theta_j) + v^t(\theta_k) - v^t(\theta_m; \theta_k) && (\text{by } \text{IC}_{jk}^t \text{ and } \text{IC}_{km}^t) \\
&< v^t(\theta_j) - v^t(\theta_m; \theta_j) && (\text{by Claim } 1'') \\
&\leq v^t(\theta_j; \theta_m) - v^t(\theta_m), && (\text{by Lemma } 3'')
\end{aligned}$$

which contradicts (53). Therefore, each customer type is in at most two priority passes.

We next show monotonicity. For this purpose, fix $j \in \{1, \ldots, K\}$ such that $q_j^l > 0$. We first show that $q_k^h = 0$ for every $k \in \{j+1, \ldots, K\}$. Towards a contradiction, assume that $q_k^h > 0$ for some $k \in \{j+1, \ldots, K\}$. As $q_j^l > 0$ and $q_k^h > 0$, both IC_{jk}^l and IC_{kj}^h are defined. Note that

$$v^l(\theta_j) - v^l(\theta_k; \theta_j) < v^h(\theta_j) - v^h(\theta_k; \theta_j) \qquad (\text{by the definition of types})$$

$$\leq v^h(\theta_j;\theta_k) - v^h(\theta_k). \qquad \text{(by Lemma 3'')} \tag{54}$$

However, that (p, q) satisfies both IC^l_{jk} and IC^h_{kj} implies that

$$v^h(\theta_j;\theta_k) - v^h(\theta_k) \leq p_j - p_k \leq v^l(\theta_j) - v^l(\theta_k;\theta_j),$$

which contradicts (54). Therefore, for every $j \in \{1, \ldots, K\}$, $q^l_j > 0$ implies $q^h_k = 0$ for every $k \in \{j+1, \ldots, K\}$.

We next show that $q^h_0 = 0$. Towards a contradiction, assume $q^h_0 > 0$. Fix $t \in \{h, l\}$ such that $q^t_{K-1} > 0$. That (p, q) satisfies $\text{IC}^t_{K-1,K}$ implies

$$v^t(\theta_{K-1}) - p_{K-1} \geq v^t(\theta_K;\theta_{K-1}) - p_K.$$

Because u^t is concave, by Lemma 3'',

$$v^t(\theta_{K-1};\theta_K) - v^t(\theta_{K-1}) \geq v^t(\theta_K) - v^t(\theta_K;\theta_{K-1}).$$

Adding up the two inequalities above, we obtain

$$v^t(\theta_{K-1};\theta_K) - p_{K-1} \geq v^t(\theta_K) - p_K. \tag{55}$$

If $j = K$, then $q^l_K > 0$ by the definition of j. If $j < K$, because $q^l_j > 0$, by our finding so far, the implementability of q implies $q^h_K = 0$ and hence $q^l_K > 0$. In both cases, IR^l_K is defined. Note that we have

$$\begin{aligned}
v^h(\theta_{K-1};\theta_0) - p_{K-1} &= v^h(\theta_{K-1};\theta_K) - p_{K-1} && \text{(by Claim 1'')} \\
&\geq v^t(\theta_{K-1};\theta_K) - p_{K-1} && \text{(by the definition of types)} \\
&\geq v^t(\theta_K) - p_K && \text{(by (55))} \\
&\geq v^l(\theta_K) - p_K && \text{(by the definition of types)} \\
&\geq 0. && \text{(by } \text{IR}^l_K\text{)} \qquad (56)
\end{aligned}$$

In (56), if $t = l$, then $v^h(\theta_{K-1};\theta_K) - p_{K-1} > v^t(\theta_{K-1};\theta_K) - p_{K-1}$; if $t = h$, then $v^t(\theta_K) - p_K > v^l(\theta_K) - p_K$. Thus, at least one of the inequalities in (56) must be strict, which means that if $q^h_0 > 0$, (p, q) would not satisfy $\text{IC}^h_{0,K-1}$, a contradiction. Therefore, $q^h_0 = 0$.

We have shown that q is monotone.

This completes the proof. □

C.3 Proof of Theorem 4

Since q is regular, the conditions of Lemma 12 hold. By the lemma, q is implementable if and only if p^* implements q. Fix $j, k \in \{1, \ldots, K\}$ such that $j < k$. That (p^*, q) satisfies IC_{kj} is equivalent to $v^{\underline{t}_k}(\theta_j; \theta_{j+1}) - v^{\underline{t}_k}(\theta_k) \le p_j^* - p_k^*$. We have

$$\begin{aligned} p_j^* - p_k^* &= \sum_{m=j}^{k-1} p_m^* - p_{m+1}^* \\ &= \sum_{m=j}^{k-1} v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1}; \theta_m) && \text{(by the definition of } p^*\text{)} \\ &= \sum_{m=j}^{k-1} \beta^{\bar{t}_m} \left[v(\theta_m) - v(\theta_{m+1}; \theta_m) \right]. && \text{(by the definition of multiplicative case)} \end{aligned}$$

Therefore, by the definition of the multiplicative case, (p^*, q) satisfies IC_{kj} if and only if

$$\beta^{\underline{t}_k} \left[v(\theta_j; \theta_{j+1}) - v(\theta_k) \right] \le \sum_{m=j}^{k-1} \beta^{\bar{t}_m} \left[v(\theta_m) - v(\theta_{m+1}; \theta_m) \right]. \tag{57}$$

If $j = k - 1$, then the regularity of q only admits the following two cases: $\bar{t}_j = \underline{t}_k$; $\bar{t}_j = h$ and $\underline{t}_k = l$. In the case where $\bar{t}_j = \underline{t}_k$, (57) is equivalent to

$$v(\theta_{k-1}; \theta_k) - v(\theta_k) \le v(\theta_{k-1}) - v(\theta_k; \theta_{k-1}), \tag{58}$$

which holds since by (16) and (17) in the proof of Theorem 2, the linearity of u implies that $v(\theta_{k-1}) - v(\theta_k; \theta_{k-1}) = v(\theta_{k-1}; \theta_k) - v(\theta_k)$. In the case where $\bar{t}_j = h$ and $\underline{t}_k = l$, (57) is equivalent to

$$v(\theta_{k-1}; \theta_k) - v(\theta_k) \le \beta \left[v(\theta_{k-1}) - v(\theta_k; \theta_{k-1}) \right],$$

which is implied by (58) because $v(\theta_{k-1}) - v(\theta_k; \theta_{k-1}) > 0$ holds by Claim $3''$ in the proof of Lemma 12 and $\beta \ge 1$ holds by definition. Therefore, if $j = k - 1$, then (p^*, q) satisfies IC_{kj}.

If $j < k - 1$ (which is possible since $K > 2$), by the regularity of q, $\bar{t}_j = h$ and $\underline{t}_k = l$. For this case, in (57), $\beta^{\underline{t}_k} = 1$ and $\beta^{\bar{t}_m} = \beta$ for at least one $m \in \{j, \ldots, k-1\}$. Therefore, in this case, IC_{kj} in (57) can be equivalently written in the form $\beta \ge \underline{\beta}_{kj}$ for some $\underline{\beta}_{kj}$ whose value is independent of β.

Let $\underline{\beta} = \max\{\underline{\beta}_{kj} : j, k \in \{1, \ldots, K\}, j < k-1\}$. We observe that $\underline{\beta}$ is independent of β. Note that (57) holds for any $j, k \in \{1, \ldots, K\}$ such that $j < k$ if and only if $\beta \ge \underline{\beta}$. Therefore, by Lemma 12, q is implementable if and only if $\beta \ge \underline{\beta}$.

To see that $\underline{\beta} > 1$, consider IC_{31} with respect to p^*, which is defined since $K > 2$. The regularity of q implies that $\bar{t}_1 = h$ and $\underline{t}_3 = l$. In this case, by (57), (p^*, q) satisfies IC_{31} if and only if

$$v(\theta_1;\theta_2) - v(\theta_3) \leq \beta\left[v(\theta_1) - v(\theta_2;\theta_1)\right] + \beta^{\bar{t}_2}\left[v(\theta_2) - v(\theta_3;\theta_2)\right], \tag{59}$$

which implies a lower bound on β. Because $\beta \geq \beta^{\bar{t}_2}$ and the right-hand side of (59) is increasing in both β and $\beta^{\bar{t}_2}$, if (59) holds for some $\beta \leq 1$, then (59) must hold for $\beta = \beta^{\bar{t}_2} = 1$. Note that (59) with $\beta = \beta^{\bar{t}_2} = 1$ is equivalent to

$$v(\theta_1) - v(\theta_1;\theta_2) \geq v(\theta_2;\theta_1) - v(\theta_2) + v(\theta_3;\theta_2) - v(\theta_3),$$

which does not hold because $v(\theta_1) - v(\theta_1;\theta_2) \leq v(\theta_2;\theta_1) - v(\theta_2)$ by Lemma $3''$ and $v(\theta_3;\theta_2) - v(\theta_3) > 0$ by Claim $1''$. Thus, (59) does not hold for $\beta \leq 1$. Therefore, the lower bound on β implied by (59) must be strictly larger than 1, and hence $\underline{\beta} > 1$.

C.4 Proof of Proposition 4

By the regularity of q and Lemma 12, we can check the implementability of q by p^* as defined in the lemma. If $K > 2$, then at least one customer type has customers in two different priority passes, which together with the assumption that $N > 2$ makes q not implementable by Theorem 1.

Assume instead $K = 2$. Because $q_0^h = q_0^l = 0$ by the regularity of q, if $0 < q_j^t < N^t$ for some $j \in \{1, 2\}$ and $t \in \{h, l\}$, then $q_k^t > 0$ for some $k \in \{1, 2\}$ such that $k \neq j$, which together with the assumption that $N > 2$ makes q not implementable by Theorem 1. Therefore, since $K = 2$, if q is implementable, then $q_j = N^h$ and $q_k = N^l$ for some $j, k \in \{1, 2\}$ such that $j \neq k$. Moreover, by Proposition 3, we have $q_1 = N^h$ and $q_2 = N^l$ if q is implementable.

Now assume $q_1 = N^h$ and $q_2 = N^l$. By the regularity of q and Lemma 12, q is implementable if and only if p^* satisfies IC_{21}, that is, $p_1^* - p_2^* \geq v^l(\theta_1;\theta_2) - v^l(\theta_2)$. By the definitions of multiplicative two-type case and p^*, we have $p_1^* - p_2^* = \beta\left[v(\theta_1) - v(\theta_2;\theta_1)\right]$ and $v^l(\theta_1;\theta_2) - v^l(\theta_2) = v(\theta_1;\theta_2) - v(\theta_2)$. Therefore, (p^*, q) satisfies IC_{21} if and only if

$$\beta\left[v(\theta_1) - v(\theta_2;\theta_1)\right] \geq v(\theta_1;\theta_2) - v(\theta_2),$$

which holds if and only if $\beta \geq \underline{\beta}$, where $\underline{\beta} = \frac{v(\theta_1;\theta_2) - v(\theta_2)}{v(\theta_1) - v(\theta_2;\theta_1)}$. Thus, q is implementable if and only if $K = 2$, $q_1 = N^h$, $q_2 = N^l$, and $\beta \geq \underline{\beta}$. Lastly, because $N > 2$, by Lemma $3''$, the strict concavity of u implies $\underline{\beta} > 1$.

C.5 Proof of Proposition 5

If $K = 1$, q is the unique regular scheme, and it is implemented by $p = v^l(\theta_1) \geq 0$.

Assume $K > 1$. By the regularity of q and Lemma 12, q is implementable if and only if p^* as defined in the lemma implements q. By the definition of p^*, (p^*, q) satisfies $\text{IC}_{j+1,j}$ for $j \in \{1, \dots, K-1\}$ if and only if

$$\beta^{\bar{t}_j}\left[v(\theta_j) - v(\theta_{j+1};\theta_j)\right] \geq \beta^{t_{j+1}}\left[v(\theta_j;\theta_{j+1}) - v(\theta_{j+1})\right], \tag{60}$$

where $\beta^h = \beta$ and $\beta^l = 1$. Because u is linear, by (16) and (17), $v(\theta_j) - v(\theta_{j+1};\theta_j) = v(\theta_j;\theta_{j+1}) - v(\theta_{j+1})$. Moreover, as $\beta^{\bar{t}_j} \geq \beta^{t_{j+1}}$ by the regularity of q, (60) holds, that is, (p^*, q) satisfies $\text{IC}_{j+1,j}$. Thus, q is implementable if $K = 2$.

Assume $K > 2$. Fix $k \in \{1, \dots, K-2\}$ and $j \in \{1, \dots, k\}$. It remains to show that (p^*, q) satisfies $\text{IC}_{k+2,j}$. Towards this end, note that the regularity of q implies that $q^h_{k+2} = 0$. Therefore, by part (b) of Corollary 4, (p^*, q) satisfies $\text{IC}_{k+2,j}$ if and only if $p^*_j - p^*_{k+2} \geq v(\theta_j;\theta_{k+2}) - v(\theta_{k+2})$, which by the definition of p^* is equivalent to

$$\sum_{m=j}^{k+1}\left[v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1};\theta_m)\right] - \left[v(\theta_j;\theta_{k+2}) - v(\theta_{k+2})\right] \geq 0. \tag{61}$$

Let $d = u_1 - u_2$. Because each base utility function is linear, we have

$$\begin{aligned} v^{\bar{t}_m}(\theta_m) - v^{\bar{t}_m}(\theta_{m+1};\theta_m) &= \frac{\beta^{\bar{t}_m}}{2}(u_{Q_{m-1}(q)+1} + u_{Q_m(q)}) - \frac{\beta^{\bar{t}_m}}{2}(u_{Q_m(q)} + u_{Q_{m+1}(q)}) \\ &= \frac{\beta^{\bar{t}_m} d}{2}(q_m + q_{m+1} - 1) \end{aligned}$$

for each $m \in \{j, \dots, k+1\}$, and

$$\begin{aligned} v(\theta_j;\theta_{k+2}) - v(\theta_{k+2}) &= \frac{1}{2}(u_{Q_{j-1}(q)+1} + u_{Q_j(q)+1}) - \frac{1}{2}(u_{Q_{k+1}(q)+1} + u_{Q_{k+2}(q)}) \\ &= \frac{d}{2}\left(q_j + q_{k+2} - 1 + 2\sum_{m=j+1}^{k+1} q_m\right). \end{aligned}$$

Therefore, (61) is equivalent to

$$\sum_{m=j}^{k+1}\frac{\beta^{\bar{t}_m}}{2}(q_m + q_{m+1} - 1) - \frac{1}{2}\left(q_j + q_{k+2} - 1 + 2\sum_{m=j+1}^{k+1} q_m\right) \geq 0. \tag{62}$$

By the regularity of q, $q^l_k = 0$, and hence $\beta^{\bar{t}_j} = \cdots = \beta^{\bar{t}_k} = \beta^h = \beta > 1$ and $\beta^{\bar{t}_{k+1}} \geq 1$, which together imply that the left-hand side of (62) is increasing in q_m for

$m \in \{j, \dots, k+2\}$. Thus, (62) is implied by the following inequality:

$$\frac{\beta}{2}(k+1-j)(2\underline{m}-1) + \frac{1}{2}(2\underline{m}-1) - \frac{1}{2}\left[2\underline{m}(k+2-j)-1\right] \geq 0,$$

where $\underline{m} = \min_{1 \leq k \leq K}\left(q_k^h + q_k^l\right)$. This inequality is equivalent to

$$\beta(k+1-j)(2\underline{m}-1) + (2\underline{m}-1) - \left[2\underline{m}(k+2-j)-1\right] \geq 0. \tag{63}$$

Now assume $q_k^h + q_k^l \geq \frac{\beta}{2(\beta-1)}$ for every $k \in \{1, \dots, K\}$, which is equivalent to $\underline{m} \geq \frac{\beta}{2(\beta-1)}$. We will show that q is implementable. Since $\underline{m} \geq \frac{\beta}{2(\beta-1)}$, the left-hand side of (63) is weakly decreasing in j. Therefore, the left-hand side of (63) weakly decreases if we set $j = k$ since $j \leq k$ by definition. Thus, to show that (63) holds, it suffices to show that it holds with $j = k$, that is, (63) is implied by:

$$\beta(2\underline{m}-1) + (2\underline{m}-1) - (4\underline{m}-1) \geq 0,$$

which holds for $\underline{m} \geq \frac{\beta}{2(\beta-1)}$. We have shown that (p^*, q) satisfies $\text{IC}_{k+2,j}$ for any $j, k \in \{1, \dots, K-2\}$ such that $j \leq k$. Therefore, (p^*, q) satisfies IC_{kj} for any $j, k \in \{1, \dots, K\}$ such that $j < k$. Thus, by Lemma 12, q is implementable.

Overall, for every $K \geq 1$, q is implementable. This completes the proof.

C.6 Proof of Proposition 6

Because $K > 2$, by the regularity of q, we can find $k \in \{1, \dots, K-2\}$ such that $q_k^l = q_{k+2}^h = 0$. By Lemma 12, q is implementable if and only if p^* as defined in the lemma implements q. By the definition of p^* and the assumption that $q_k^l = q_{k+2}^h = 0$, (p^*, q) does not satisfy $\text{IC}_{k+2,k}$ if and only if

$$\underbrace{\beta\left[v(\theta_k) - v(\theta_{k+1};\theta_k)\right]}_{p_k^* - p_{k+1}^*} + \underbrace{\beta^{\bar{t}_{k+1}}\left[v(\theta_{k+1}) - v(\theta_{k+2};\theta_{k+1})\right]}_{p_{k+1}^* - p_{k+2}^*} - \left[v(\theta_k;\theta_{k+2}) - v(\theta_{k+2})\right] < 0,$$

where we let $\beta^h = \beta$ and $\beta^l = 1$. By the linear multiplicative setup, the above inequality is equivalent to

$$\frac{\beta}{2}(q_k + q_{k+1} - 1) + \frac{\beta^{\bar{t}_{k+1}}}{2}(q_{k+1} + q_{k+2} - 1) - \frac{1}{2}(q_k + 2q_{k+1} + q_{k+2} - 1) < 0. \tag{64}$$

The left-hand side of (64) is increasing in $\beta^{\bar{t}_{k+1}}$, q_k, q_{k+1}, and q_{k+2}. Specifically, if we set q_k, q_{k+1}, and q_{k+2} to $\bar{M} := \max_{1 \leq k \leq K}\left(q_k^h + q_k^l\right)$ and $\beta^{\bar{t}_{k+1}}$ to β, the left-hand side of (64) weakly increases. Therefore, (64) is implied by

$$\frac{\beta}{2}(2\bar{M}-1)+\frac{\beta}{2}(2\bar{M}-1)-\frac{1}{2}(4\bar{M}-1)<0. \tag{65}$$

Now assume $q_k^h + q_k^l < \frac{\beta-\frac{1}{2}}{2(\beta-1)}$ for every $k \in \{1,\dots,K\}$, which is equivalent to $\bar{M} < \frac{\beta-\frac{1}{2}}{2(\beta-1)}$. Note that (65) is equivalent to $\bar{M} < \frac{\beta-\frac{1}{2}}{2(\beta-1)}$. Thus, (p^*, q) does not satisfy $\mathrm{IC}_{k+2,k}$. Therefore, by Lemma 12, q is not implementable.