Online Supplementary Appendix to: Fair Matching under Constraints: Theory and Applications

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Appendix A. Proofs

A.1. Proof of Theorem 3.

Proof. Suppose the constraint $\mathcal{F}_s$ of s is not a general upper-bound. Consider the following two cases.

(1) Suppose that $\emptyset$ is infeasible at s. Then, assume all students find s to be unacceptable. It is clear that there is no feasible and individually rational matching in this case.

(2) Suppose that $\emptyset$ is feasible at s. Then there exist non-empty sets $I'' \subsetneq I' \subseteq I$ such that I' is feasible at s but I'' is infeasible at s. Let s' be a school different from s (note that such a school exists by the assumption that $|S| \geq 2$). Fix student preferences as follows:

$$\succ_i : s, s', \text{for every } i \in I'',$$
$$\succ_i : s', s, \text{for every } i \in I' \setminus I'',$$

and every other student finds all schools unacceptable. In addition, assume that each school other than s has a capacity constraint with a capacity of $|I|$.

In this problem, both of the following matchings are fair as well as feasible and individually rational:

(1) every student in I' is matched to s and every other student is unmatched.
(2) every student in I' is matched to s' and every other student is unmatched.

Therefore, if there is an SOFM, then it should match every student in I'' to s, every student in $I' \setminus I''$ to s', and leave every other student unmatched. But such a matching is infeasible because $I'' \notin \mathcal{F}_s$. □

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A.2. **Proof of Proposition 2.**

Proof. Because μ is feasible in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ by definition of SOFM and $\mathcal{F}_s \subseteq \mathcal{F}'_s$ for every $s \in S$ by assumption, μ is feasible in $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$ as well. Moreover, because μ is individually rational and fair in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ by definition of SOFM, μ is individually rational and fair in $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$ as well. Therefore, since μ' is the SOFM in $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$, it follows by the definition of SOFM that $\mu'_i \succeq_i \mu_i$ for every $i \in I$, completing the first part of the proposition statement. This conclusion, together with individual rationality of μ and $\mu_i \in S$, imply $\mu'_i \succeq_i \mu_i \succ_i \emptyset$ and hence $\mu'_i \in S$, completing the second part of the proposition statement. $\square$

A.3. **Proof of Propositions 3 and 4.**

Proof. Proposition 3 is a corollary of Proposition 4, hence we only present the proof of the latter.

Part 1 of Proposition 4: Consider the maximizer μ' of (5.2) for the SOFM μ. For each s, $\mu'_s \setminus \mu_s$ is a candidate for the maximizer $\bar{I}$ in (5.3), so we must have $|\mu'_s \setminus \mu_s| \leq \hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S)$. Hence,

$$\sum_{s \in S} |\mu'_s \setminus \mu_s| \leq \sum_{s \in S} \hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) = \hat{W}(\succ_I, \succ_S, \mathcal{F}_S).$$

Since the left-most side is the waste of μ, the proof is complete.

Part 2 of Proposition 4: Fix $\mathcal{F}_S$ and let μ be the SOFM. In the remainder of the proof, we will show $\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) \leq \bar{w}_s(\mathcal{F}_s)$ for each $s \in S$. For this purpose, fix any $s \in S$ and define $I^0 := \{i \in I | s \succ_i \mu_i\}$ to be the set of all students who strictly prefer s to the match at μ. Consider the following cases.

(1) Suppose $I^0 = \emptyset$. Because $\{i \in I | s \succ_i \mu_i \text{ and } \mu_i = \emptyset\} \subseteq I^0$ in general, this implies $\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) = 0$. Since $\bar{w}_s(\mathcal{F}_s)$ is nonnegative in general, this implies $\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) \leq \bar{w}_s(\mathcal{F}_s)$.
(2) Suppose $I^0 \neq \emptyset$. Let $j \in I^0$ be the student who has the highest priority at s among students in I^0. Now, define a new matching μ'' by

$$\mu''_i = \begin{cases} s & \text{if } i = j, \\ \mu_i & \text{otherwise.} \end{cases}$$

Then, μ'' is individually rational and fair by construction. In addition, $\mu''_{s'} \in \mathcal{F}_{s'}$ for all $s' \neq s$ because $\mu''_{s'} = \mu_{s'} \in \mathcal{F}_{s'}$ for every $s' \neq s, \mu_j$ and, if $\mu_j \neq \emptyset$, μ''_{μ_j} is feasible since $\mathcal{F}_{\mu_j}$ is a general upper-bound. This implies $\mu_s \cup \{j\} = \mu''_s \notin \mathcal{F}_s$ because otherwise μ'' is an individually rational, feasible, and fair matching that is weakly more preferred to μ by every student $i \neq j$ and strictly by j, a contradiction to the assumption that μ is the SOFM.

Now, consider any $I'' \subseteq I$ such that $\mu_s \cup I'' \in \mathcal{F}_s$, and $\mu_i = \emptyset$ and $s \succ_i \emptyset$ for each $i \in I''$. Letting $\mu_s = I'$, the triple (j, I', I'') satisfies the conditions $I' \cup \{j\} \notin \mathcal{F}_s, I' \cup I'' \in \mathcal{F}_s$, and $I' \cap I'' = \emptyset$ as in the definition of $\bar{w}_s(\mathcal{F}_s)$. Hence $|I''| \leq \bar{w}_s(\mathcal{F}_s)$ by the definition of $\bar{w}_s(\mathcal{F}_s)$. Since μ is the SOFM, inspecting the definition of $\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S)$ (5.3) shows that

$$\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) = \max_{I'' \subseteq I}\{|I''| : \mu_s \cup I'' \in \mathcal{F}_s, \text{ and } \mu_i = \emptyset \text{ and } s \succ_i \emptyset \text{ for all } i \in I''\},$$

it follows that $\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) \leq \bar{w}_s(\mathcal{F}_s)$.

Therefore, $\hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) \leq \bar{w}_s(\mathcal{F}_s)$ for every $s \in S$, as desired. □

A.4. **Proof of Theorem 4.**

Proof. Suppose the constraint $\mathcal{F}_s$ of s is not a capacity constraint while being a general upper-bound. Let k be the largest nonnegative integer such that all sets of students with cardinality k or smaller are feasible at s (note that k may be 0. Also note that k is well-defined, for $\emptyset$ is feasible at s because $\mathcal{F}_s$ is a general upper-bound, and I is finite).

Claim 2. *There exist $I_1 \in \mathcal{F}_s$ and $I_2 \notin \mathcal{F}_s$ such that $|I_1 \cap I_2| = k$ and $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$.*

Proof of Claim 2. Let $\mathcal{I}_1 = \{I' \subseteq I | |I'| = k+1, I' \in \mathcal{F}_s\}$ and $\mathcal{I}_2 = \{I' \subseteq I | |I'| = k+1, I' \notin \mathcal{F}_s\}$. The former is nonempty because otherwise $\mathcal{F}_s$ would be a capacity constraint, and the latter is nonempty due to the definition of k. Let $l := \min\{|I'_1 \setminus I'_2| : I'_1 \in \mathcal{I}_1, I'_2 \in \mathcal{I}_2\}$ and assume for contradiction that $l > 1$; note that the minimum exists because $\mathcal{I}_1$ and $\mathcal{I}_2$ are nonempty finite sets. Fix arbitrarily $\bar{I}_1 \in \mathcal{I}_1$ and $\bar{I}_2 \in \mathcal{I}_2$ such that $|\bar{I}_1 \setminus \bar{I}_2| = l$. Then, fix $i_1 \in \bar{I}_1 \setminus \bar{I}_2$ and $i_2 \in \bar{I}_2 \setminus \bar{I}_1$ and define $\bar{I} := (\bar{I}_1 \setminus \{i_1\}) \cup \{i_2\}$. If $\bar{I} \in \mathcal{F}_s$, then $\bar{I} \in \mathcal{I}_1$ and $|\bar{I} \setminus \bar{I}_2| = l - 1 < l$, a contradiction to the minimality of l. If $\bar{I} \notin \mathcal{F}_s$, then $\bar{I} \in \mathcal{I}_2$ and $|\bar{I}_1 \setminus \bar{I}| = 1 < l$, again a contradiction to the minimality of l. □

In the remainder, we assume the condition in Claim 2 holds for I_1 and I_2. Denote by i_1 and i_2 the agents such that $\{i_1\} = I_1 \setminus I_2$ and $\{i_2\} = I_2 \setminus I_1$.

Now consider the following preference profile: every student in $I_1 \cup I_2$ finds only s acceptable; every other student finds all schools unacceptable. Also assume school s ranks all students in $I_1 \cap I_2$ first (in an arbitrary order), then the (unique) student $i_2 \in I_2 \setminus I_1$, then the (unique) student $i_1 \in I_1 \setminus I_2$, and then every other student (in an arbitrary order).

Suppose that μ is a stable matching. Now, note I_2 is infeasible at s by assumption, but because $k = |I_2| - 1$ is such that any set of doctors whose cardinality is at most k is feasible at s, non-wastefulness requires that at least k doctors are matched at s under μ. Because of the construction of $\succ_s$ and the requirement of fairness, all students in $I_1 \cap I_2$ should be matched at s. Because I_1 and I_2 satisfy the condition described by Claim 2, $(I_2 \cup \{i_1\}) \setminus \{i_2\} = I_1$ is feasible at s. By non-wastefulness, i_1 should be matched at s (this is because, if i_1 is not matched to s but some student in $I \setminus (I_1 \cup I_2)$ is, then the matching violates fairness), and this implies i_2 is not matched to s and hence unmatched. This is a contradiction to fairness because $i_2 \succ_s i_1$ and $s \succ_{i_2} \emptyset$. $\square$

A.5. **Proof of Theorems 5 and 6.**

Proof. Because Theorem 5 is a special case of Theorem 6, we only provide a proof for the latter result. Suppose the constraint $\mathcal{F}_s$ of s is not a capacity constraint while being a general upper-bound. Let k be the largest nonnegative integer such that all sets of students with cardinality k or smaller are feasible at s (note that k may be 0. Also note that k is well-defined, for $\emptyset$ is feasible at s because $\mathcal{F}_s$ is a general upper-bound and I is finite). Then, Claim 2 implies that there exist subsets of students, I_1 and I_2, such that I_1 is feasible at s while I_2 is not, $|I_1 \cap I_2| = k$, and there exist $i_1, i_2 \in I$ such that $\{i_1\} = I_1 \setminus I_2$ and $\{i_2\} = I_2 \setminus I_1$. Now, fix a school $s' \neq s$ and consider the following preference and priority profiles as well as constraints: s ranks all doctors in $I_1 \cap I_2$ as the highest (in an arbitrary order), then i_2, then i_1, and then all other students (in an arbitrary order). School s' ranks i_1 first and i_2 second (while the ranking over all other students are arbitrary) and is subject to the capacity constraint with capacity of 1. Each student in $I_1 \cup I_2$ prefers s first and s' second (while preferences on all other schools are arbitrary), and all other students find all schools unacceptable.

Fix a mechanism φ that satisfies feasibility, fairness, and unanimity in $(I, S, \mathcal{F}_S)$. Under φ, i_2 is not matched to s. To see this, assume for contradiction that i_2 is matched to s. Then, since s is the most preferred by every student in $I_1 \cap I_2$, fairness implies that every student in $I_1 \cap I_2$ is matched to s, so every student in $(I_1 \cap I_2) \cup \{i_2\} = I_2$ is matched to s. But this is a contradiction to feasibility because $I_2 \notin \mathcal{F}_s$ by assumption and $\mathcal{F}_s$ is a general upper-bound.

Because i_2 is not matched to s and i_2 has higher priority than i_1 at s, it follows that i_1 is not matched to s. Given that, it also follows that i_2 is not matched to s' because i_1 has higher priority than i_2 at s' and s' has the capacity of one.

If i_2 misreports and declares that only s' is acceptable to her, then because φ satisfies unanimity, it matches all students in I_1 to s and i_2 to s', while leaving all other students unmatched. Thus, i_2 benefits from a misreport, and hence φ is not strategy-proof. $\square$

A.6. **Proof of Proposition 5.**

Proof. Fix $\succ_I$ arbitrarily and let $\succ'_i$ be a truncation of $\succ_i$. Denote μ and μ' be SOFMs under $\succ_I$ and $\succ'_I := (\succ'_i, \succ_{I\setminus\{i\}})$, respectively (note that SOFMs exist for all preference profiles under general upper-bound). If suffices to show that $\mu_i \succeq_i \mu'_i$.

First, suppose $\mu'_i = \emptyset$. Because the SOFM is individually rational, we have $\mu_i \succeq_i \emptyset$. These two relations imply $\mu_i \succeq_i \mu'_i$ as desired.

Second, suppose $\mu'_i \neq \emptyset$. For any $j \neq i$, j does not have justified envy toward anyone at μ' under $\succ_j$ because $\succ_j = \succ'_j$ and μ' is fair under $\succ'_I$. Moreover, if $s \succ_i \mu'_i$, then $s \succ'_i \mu'_i$ (because $\mu'_i \neq \emptyset$ and $\succ'_i$ is a truncation of $\succ_i$), and hence $j \succ_s i$ for every $j \in \mu'_s$ (because μ' is fair under $\succ'_I$). This implies that i does not have justified envy toward anyone at μ' under $\succ_i$. Therefore, overall, μ' is fair under $\succ_I$. Because μ is the SOFM under $\succ_I$, we conclude $\mu_i \succeq_i \mu'_i$ as desired. $\square$

A.7. **Proof of Proposition 6.**

Proof. Let $\tilde{s}$ be i's match under $\succ_I$. Since the cutoff adjustment algorithm implements the SOFM mechanism when the constraints are general upper-bounds and the SOFM is individually rational by definition, $\tilde{s} \succeq_i \emptyset$ holds. Together with $s \succ_i \tilde{s}$, we have $s \in S$.

To show $p_s > p'_s$, recall the definition of demand:

$$D_{\bar{s}}(p) := \{i \in I | i \succeq_{\bar{s}} i^{(\bar{s}, p_s)} \text{ and } \bar{s} \succ_i \emptyset; i \succeq_{s'} i^{(s', p_{s'})} \Rightarrow \bar{s} \succeq_i s'\}.$$

First, suppose that $\tilde{s} = \emptyset$. Then, $i^{(s,p_s)} \succ_s i$ holds by the definition of $D_{\bar{s}}(\cdot)$ for all $\bar{s} \in S$, and $i \succeq_s i^{(s,p'_s)}$ holds by the definition of $D_s(\cdot)$. Thus, we have $i^{(s,p_s)} \succ_s i^{(s,p'_s)}$, and by the definition of $i^{(s,\cdot)}$, we obtain $p_s > p'_s$.

Second, suppose that $\tilde{s} \neq \emptyset$. Letting $\bar{s} = \tilde{s}$, we have $i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow \tilde{s} \succeq_i s'$ for any s'. Since $s \succ_i \tilde{s}$, we have $i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'$. Since we have already shown $s \succ_i \emptyset$, if $i \succeq_s i^{(s,p_s)}$ holds then we must have $i \in D_s(p)$ by letting $\bar{s} = s$. This contradicts $\tilde{s} \neq s$. Hence, $i^{(s,p_s)} \succ_s i$ must hold. Under $(\succ'_i, \succ_{I\setminus\{i\}})$, i is matched with s, hence $i \succeq_s i^{(s,p'_s)}$ must hold. Overall, we have $i^{(s,p_s)} \succ_s i^{(s,p'_s)}$, implying $p_s > p'_s$. $\square$

A.8. **Proof of Proposition 7.**

Proof. Here we prove the general result stated in Remark 7.

Observe that the outcome of the cumulative offer algorithm defined in Section 6.4 does not change even if students apply to schools sequentially one by one, in any order. In particular, for the problem Π^n, we can first let students in T_1^n apply to schools until there is no more rejection, then students in T_2^n apply to schools until there is no more rejection, and so forth. Given the definition of the cumulative offer algorithm and the tier structure, it is not possible for a student in T_k^n to be newly rejected when students in $T_{k'}^n$ apply for any $k' > k$.

This implies that the cumulative offer algorithm is equivalent to the following algorithm in which, for each Round k (consisting of possibly multiple steps), only students in T_k^n apply and their matching is finalized in that round.

- Round $k \geq 1$, Step $t \geq 1$: Each student in T_k^n applies to her first choice school among those that have never rejected anyone whose priority is weakly higher than her if it is acceptable, while making no application otherwise. For each school s, let $\{i_1, i_2, \ldots, i_l\}$ be the set of students who have ever applied to it, with $i_1 \succ_s i_2 \succ_s \ldots \succ_s i_l$. If $\{i_1, i_2, \ldots, i_l\} \in \mathcal{F}_s$, then let s temporarily keep $\{i_1, i_2, \ldots, i_l\} \cap T_k^n$; otherwise, let s temporarily keep the set of students of the form $\{i_1, i_2, \ldots, i_{l'}\} \cap T_k^n$ such that $\{i_1, i_2, \ldots, i_{l'}\} \in \mathcal{F}_s$ and $\{i_1, i_2, \ldots, i_{l'+1}\} \notin \mathcal{F}_s$. School s rejects all the remaining students in T_k^n who have ever applied to it, $\{i_{l'+1}, \ldots, i_l\} \cap T_k^n$.
  - If no student is rejected by a new school, finalize the matching for each student $i \in T_k^n$, and let μ_i^n denote the (unique) school which currently keeps (and thus is permanently matched to) student i if such a school exists, and $\mu_i^n = \emptyset$ otherwise.
    * If $k = K^n$, terminate the algorithm and define the outcome as the matching in which each student i is matched with μ_i^n (the procedure up to this point uniquely determines μ_i^n for each $i \in I^n$).
    * If $k < K^n$, then go to Round $k + 1$, Step 1.
  - Otherwise, go to Round k, Step $t + 1$.

Under this algorithm, by definition, for each school s, there is at most one tier whose students are rejected from s.[1] If there exists such a tier, denote the index of that tier by $k(s)$, that is, the tier is $T_{k(s)}^n$.

We proceed by making two observations. First, by the definition of the algorithm, no students in tier $T_{k'}^n$ such that $k' > k(s)$ have a reporting strategy such that they can match

[1]Note that each student applies to her first choice school among those *that have never rejected anyone whose priority is weakly higher than her* if it is acceptable, while making no application otherwise.

with s. Second, the matching of a student i in tiers $T^n_{k'}$ such that $k' < k(s)$ is weakly more preferred than s because i is not rejected by s by the definition of $T^n_{k(s)}$.

Hence, if a student i has an incentive to misreport their preferences and matches with s, then $i \in T^n_{k(s)}$. This implies that, if a student i has an incentive to misreport their preferences, then $i \in \bigcup_{s\in S^n} T^n_{k(s)}$. Therefore, $D(\Pi^n) \subseteq \bigcup_{s\in S^n} T^n_{k(s)}$.

Thus,

$$\frac{|D(\Pi^n)|}{|I^n|} \leq \frac{|\bigcup_{s\in S^n} T^n_{k(s)}|}{|I^n|} \leq \frac{L'(n)\cdot|S^n|}{|I^n|} \leq \frac{L'(n)\cdot L(n)}{|I^n|} \to 0 \quad \text{as} \quad n\to\infty.$$

Since $\frac{|D(\Pi^n)|}{|I^n|} \geq 0$, we have proved the desired result. $\square$

A.9. **Proof of Proposition 8.**

Proof. We use the same notations as in the proof of Theorem 2. Suppose that at steps $1,\dots,t$ of the algorithm, if s has rejected at least one student, let the highest-priority student s who has been rejected be $i^{(s,l')}$ and $p^t_s = l'+1$. Otherwise, let $p^t_s = 1$. Also, let $p^0_s = 1$. By definition of p^t_s, the cumulative offer algorithm is equivalent to the following algorithm:

- Step $t \geq 1$: Each student i applies to her first choice school in $\{s \in S | i \succeq_s i^{(s,p^{t-1}_s)}\}$, while making no application otherwise. Each school s keeps every student i such that $i \succeq_s i^{(s,p^t_s)}$ and rejects all the remaining students who have ever applied to it. If no student is rejected by a new school, then terminate the algorithm and define the outcome as the matching in which each school is matched to the set of students who it currently keeps. Otherwise, go to Step $t+1$.

For a profile $p = (p_s)_{s\in S}$, define $\tilde{T} : P \to P$ as follows:

$$\tilde{T}_s(p) = \begin{cases} \min\{p'_s | D_s(p'_s, p_{-s}) \in \mathcal{F}_s\} & \text{if } D_s(p) \notin \mathcal{F}_s \\ p_s & \text{if } D_s(p) \in \mathcal{F}_s \end{cases},$$

where the minimum exists because P is finite and $D_s(|I|+1, p_{-s}) = \emptyset \in \mathcal{F}_s$. By inspection, $p^t_s = \tilde{T}_s(p^{t-1})$ holds for each t. Hence, for each t, $p^t_s = \tilde{T}^t_s(p^0)$.

Observe that, by the definition of $D_s(\cdot)$ and the above algorithm, the set of students that each s keeps at the terminal step t is $D_s(p^t)$. Hence, it can be shown that the above algorithm produces the SOFM by a proof similar to those of Theorem 2 and Proposition 1. $\square$

A.10. **Proof of Proposition 9.**

Proof. (2) $\Rightarrow$ (1) is straightforward. To show (1) $\Rightarrow$ (2), define the set of services Σ as $\Sigma := 2^I \setminus \mathcal{F}_s$. For each $\sigma \in \Sigma$, suppose

$$\nu^i_\sigma = \begin{cases} 1 \text{ if } i \in \sigma, \\ 0 \text{ otherwise,} \end{cases}$$

and let $\kappa^s_\sigma = |\sigma| - 1/2$.[2] We will show that the conclusion of the proposition holds with respect to these parameters for multidimensional constraints.

To show the "if" direction, suppose $I' \in \mathcal{F}_s$. Then, because $\mathcal{F}_s$ is a general upper-bound, it follows that, for any $\sigma \in \Sigma$, $I' \not\supseteq \sigma$ and hence $\sigma \cap I' \leq |\sigma| - 1/2$. So, $\sum_{i \in I'} \nu^i_\sigma \leq |\sigma| - 1/2 = \kappa^s_\sigma$. Therefore I' is DKT-feasible. To show the "only if" direction, suppose $I' \notin \mathcal{F}_s$. Then, the service $\sigma = I'$ is in Σ, and thus $\sum_{i \in I'} \nu^i_\sigma = |I'| > |I'| - 1/2 = \kappa^s_\sigma$. Thus, I' is not DKT-feasible. $\square$

A.11. **Proof of Proposition 10.** As mentioned in footnote 12, we will show the following stronger result: *Given a problem with n students from each group as in Proposition 10, let $m \in \mathbb{N}$ be the minimum cardinality of the sets of services that describe the constraint.*[3] *Then $m = n$.*

Proof. We first show that in the problem of n students from each group, we need at least n services. To show this, suppose without loss of generality that each service capacity is normalized to 1.[4] Consider a partition of all students into pairs of students from different groups. More specifically, label the students from one group as $I_1 = \{i_1, i_2, \ldots, i_n\}$ and those from the other group as $I_2 = \{i'_1, i'_2, \ldots, i'_n\}$, and form n pairs by paring two students of the same index from the two groups, i.e., $I = \bigcup_{t=1}^n \{i_t, i'_t\}$. For each t, because the pair $\{i_t, i'_t\}$ is infeasible at s, there exists $\sigma \in \Sigma$ for which $\nu^{i_t}_\sigma + \nu^{i'_t}_\sigma > 1$. Choose such a service arbitrarily and denote it by σ_t. To prove our claim, it suffices to show $\sigma_t \neq \sigma_{t'}$ if $t \neq t'$. For this purpose, assume for contradiction that $\sigma_t = \sigma_{t'} =: \sigma$. Then $\nu^{i_t}_\sigma + \nu^{i'_t}_\sigma > 1$ and $\nu^{i_{t'}}_\sigma + \nu^{i'_{t'}}_\sigma > 1$, so $\nu^{i_t}_\sigma + \nu^{i'_t}_\sigma + \nu^{i_{t'}}_\sigma + \nu^{i'_{t'}}_\sigma > 2$. This implies $\nu^{i_t}_\sigma + \nu^{i_{t'}}_\sigma > 1$ or $\nu^{i'_t}_\sigma + \nu^{i'_{t'}}_\sigma > 1$. Hence $\{i_t, i_{t'}\} \notin \mathcal{F}_s$ or $\{i'_t, i'_{t'}\} \notin \mathcal{F}_s$ holds, a contradiction.

We next show that in the problem with n students from each group, there exist multi-dimensional constraints with n services that describe the given constraint. To do so, let

[2] Note that the requirement $\kappa^s_\sigma \in \mathbb{R}_{++}$ is satisfied because $\emptyset \in \mathcal{F}_s$ by the assumption that $\mathcal{F}_s$ is a general upper-bound and hence $\emptyset \notin \Sigma$.

[3] Note that the minimum cardinality exists in $\mathbb{N}$ because of the construction of the set of services in the proof of Proposition 9.

[4] This is without loss of generality because, given any nonzero service capacity and service demands, one can normalize that service capacity to one while changing service needs of each student for that service in the same proportion.

$\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and (as before) $I_1 = \{i_1, i_2, \ldots, i_n\}$. Set each service capacity as 1. For each $t \in \{1, \ldots, n\}$, the service needs for σ_t are given by

$$\nu^i_{\sigma_t} = \begin{cases} 1 \text{ if } i = i_t, \\ 0 \text{ if } i \in I_1 \setminus \{i_t\}, \\ \frac{1}{n} \text{ otherwise.} \end{cases}$$

Suppose $I' \in \mathcal{F}_s$. Then either $I' \subseteq I_1$ or $I' \subseteq I_2$. In the former case, for any service σ_t, $\sum_{i \in I'} \nu^i_{\sigma_t} \leq \sum_{i \in I_1} \nu^i_{\sigma_t} = 1 + (n-1) \times 0 = 1 = \kappa^s_{\sigma_t}$, so I' is DKT-feasible. In the latter case, for any service σ_t, $\sum_{i \in I'} \nu^i_{\sigma_t} \leq \sum_{i \in I_2} \nu^i_{\sigma_t} = \frac{1}{n} \times n = 1 = \kappa^s_{\sigma_t}$, so I' is DKT-feasible.

Next suppose $I' \notin \mathcal{F}_s$. Then $I_1 \cap I' \neq \emptyset$ and $I_2 \cap I' \neq \emptyset$. Let $i_t \in I_1 \cap I'$ and $i' \in I_2 \cap I'$. Then, $\sum_{i \in I'} \nu^i_{\sigma_t} \geq \nu^{i_t}_{\sigma_t} + \nu^{i'}_{\sigma_t} = 1 + \frac{1}{n} > 1 = \kappa^s_{\sigma_t}$, so I' is not DKT-feasible. □

Appendix B. Analysis of data from Bunkyo City

In this section, we report our simulations using data on daycare seat allocation from Bunkyo City, Japan. The numerical analysis we report here suggests that the main findings for Yamagata are robust to data features.

As explained in Section 5.3, Bunkyo City is one of the 23 special districts of Tokyo, with about 230,000 residents as of 2018. Bunkyo is much more urban than Yamagata. It has a population density about 30 times that of Yamagata, has a high concentration of educational institutions, and attracts many dual-income families investing heavily in education and demanding childcare which, as we will see below, seems to make its daycare allocation problem more pressing than Yamagata's.[5] Part of our interest in studying Bunkyo's data is to investigate whether our numerical findings are robust to demographic features of different municipalities.

Our data involve applicants (who are anonymized), usually parents, representing children who would begin attending the daycare in April of 2018. There were 2114 applicants aged between 0 to 5 as of April 1, 2018 on which they would begin attending the daycare. For each applicant, the data show her reported preferences over the daycare centers and priority ranking (the priorities are common across daycare centers). Regarding reported preferences, we note that the mechanism in Bunkyo is based on serial dictatorship but restricts applicants to list at most five daycare centers in their ranking.[6] Because of this

[5]Bunkyo, whose literal translation would be "Literature Capital," is home to many higher education institutions such as University of Tokyo as well as prestigious elementary and secondary schools.

[6]As in the case of Yamagata City, there are a few additional differences between Bunkyo's mechanism and serial dictatorship, i.e., there are a few special rules, mainly regarding children with siblings. In our

restriction, Bunkyo's mechanism is not strategy-proof, so care is warranted when interpreting the results. Note that, however, there is a certain sense in which this mechanism is "less manipulable" than other mechanisms such as the Boston mechanism with the same length restriction (Pathak and Sönmez, 2013).

The priority order is based on the applicant characteristics such as parents' job status and the number of adults available for care at home. In fact, Bunkyo City (2018) discloses the explicit formula that converts relevant characteristics of each family to the (common) priority ranking. There are 63 daycare centers in our dataset. For each daycare center, the data show how many seats are supplied for each age.

In our simulation, we made several modeling choices given data limitation. First, as for Yamagata's data, Bunkyo's data involve ties although the actual priority order is strict. This is because our data lack information on some characteristics used by Bunkyo to determine the strict order, such as whether the child is currently in an alternative form of childcare and whether the family has a member with disability. As in our analysis of Yamagata's data, we randomly break ties using a single tie-breaking rule (that is, the tie-broken priorities are common across daycare centers) according to the uniform distribution. For each mechanism that we consider, we conducted 250 runs of simulations using such a tie-breaking rule.

The second limitation involves constraints. As is the case for Yamagata's dataset, for daycare centers, Bunkyo's dataset does not show the entire collection of feasible sets of children or the number of teachers corresponding to the flexible constraints. Instead, it only shows the number of advertised seats at each daycare center for each age, which is exactly enough to specify the rigid constraints. To overcome this limitation, we define m_s for each s in the daycare constraints (Equation (5.1)) by $m_s := \sum_{t \in \mathcal{T}} r_t \cdot q^t$, where r_t and q^t are those in the data (recall that r_t is the teacher-child ratio under the national regulation, and q^t is the number of advertised seats for age t at daycare center s). That is, m_s is the smallest possible number of teachers such that the constraint implied by the number of advertised seats in data is a rigid constraint associated with our daycare

numerical analysis, however, this difference causes only a minor difference between the assignments from serial dictatorship (with limited length of preference lists) and the actual one.

From/To	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
rigid SOFM	—	1091.76 (51.64%) (SE= 12.75)	913.72 (43.22%) (SE= 13.95)	1156.84 (54.72%) (SE= 12.63)
flexible SOFM	0	—	287.62 (13.61%) (SE= 6.80)	102.84 (4.86%) (SE= 8.17)
actual allocation	36.05 (1.71%) (SE= 5.14)	534.94 (25.30%) (SE= 8.09)	—	595.42 (28.17%) (SE= 8.91)
flexible ETSD	0	0	266.95 (12.63%) (SE= 6.36)	—

Table 3. The number of applicants who are made strictly better off by a change of a mechanism. For each mechanism that we consider, we conducted 250 runs of simulations, with each run corresponding to a realization of the tie-broken priorities. The percentage is out of all the 2114 applicants. The "SE" stands for the standard error of the raw number.

constraint.[7] This method is identical to the one we used in our analysis of Yamagata's data.

We find that the effect of allowing flexibility in constraints is substantial in our data from Bunkyo, just as is the case of data from Yamagata: the average number of children who are matched with a strictly preferred daycare center in the flexible SOFM compared to the rigid SOFM is 1091.76, which amounts to 51.64% of all applicants (Table 3).[8] By contrast, no applicant is made worse off, as implied by Proposition 2. The number of children who are unallocated changes from 1710.87 to 875.74, a 48.81% decrease (Figure 3). The average numbers of children who are matched to their first choice, first two choices, and the first three choices increase by 517.11%, 159.77% and 104.91%, respectively (Figure 4).[9] Our analysis suggests that substantial efficiency gains from utilizing the flexible nature of the constraints may be present not only in Yamagata City but more broadly.

[7]We set m_s as the bare minimum that is consistent with the data on advertised seats so that we do not overstate our estimate of the gains from removing the rigid constraint. In a similar spirit, we allow for non-integral values of m_s although the number of teachers is an integer in practice. With an alternative specification setting m_s to be the integer rounded up from our present definition, for instance, our estimate of the gains from removing rigid constraints would be larger.

[8]This table as well as others also report simulations of other mechanisms we discuss below.

[9]If an applicant lists k daycare centers in her reported preferences and gets unassigned to any of them, then we list her as being assigned to her $(k+1)$st choice.

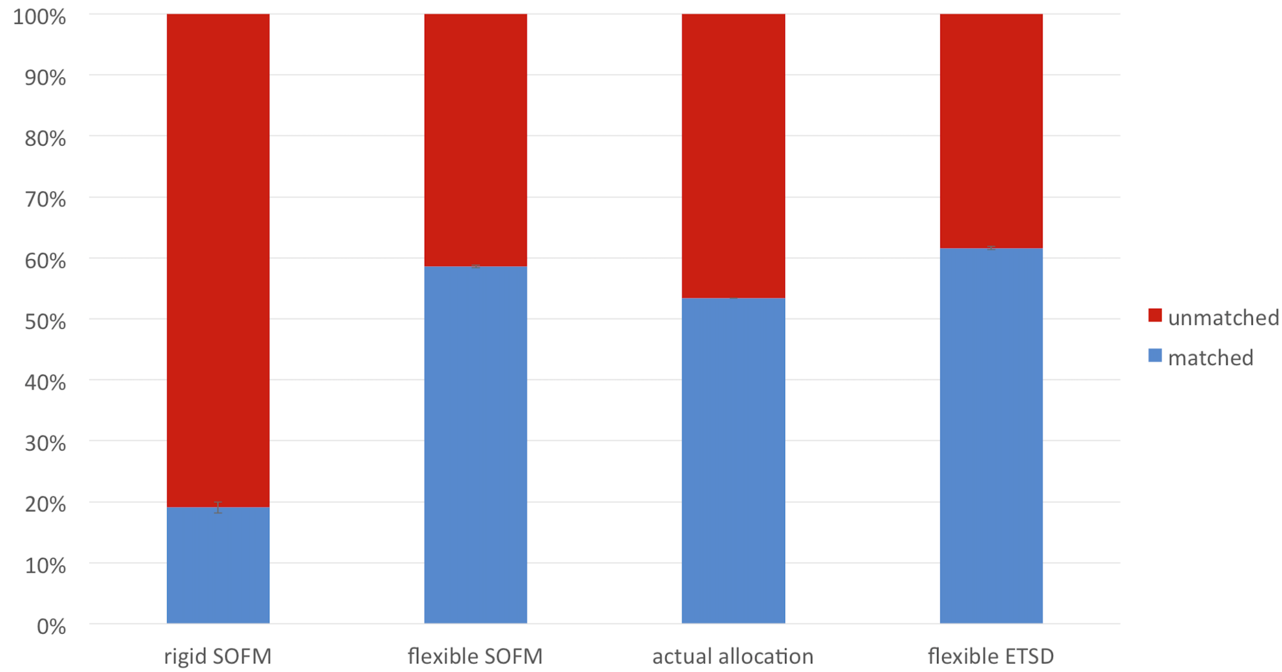

Figure 3. The fractions of matched applicants under different mechanisms. The error bars represent the standard errors.

Next, we compare the rigid and flexible SOFMs with Bunkyo's actual assignment. Bunkyo's mechanism is based on rigid (justified) envy-tolerating serial dictatorship (rigid ETSD), just as is the case with Yamagata (with the limitation on the length of preference list). This means that, among other things, there may remain justified envy between two children i and i' if they are of different ages, while by construction there is no justified envy between children of the same age. Bunkyo's assignment is expected to have some efficiency advantage over the rigid SOFM since justified envy is tolerated across different ages, while the comparison with the flexible SOFM is theoretically indeterminate because Bunkyo's assignment is based on the rigid constraint, which may or may not overwhelm the efficiency gains from tolerating justified envy across different ages.

We find that the flexible SOFM outperforms Bunkyo's assignment not only in terms of fairness but also in terms of efficiency. Regarding efficiency, all of our efficiency measures favor the flexible SOFM; the average fraction of unmatched children decreased by 11.18%, and 25.30% of children are matched with strictly preferred daycare under the flexible SOFM while only 13.61% are matched with strictly preferred daycare under the actual allocation. Turning our focus to fairness, Table 4 provides several measures of justified envy for Bunkyo's assignment (note that all measures of justified envy are zero for the rigid and flexible SOFM). There are 1622 pairs (i, s) such that i has a justified envy

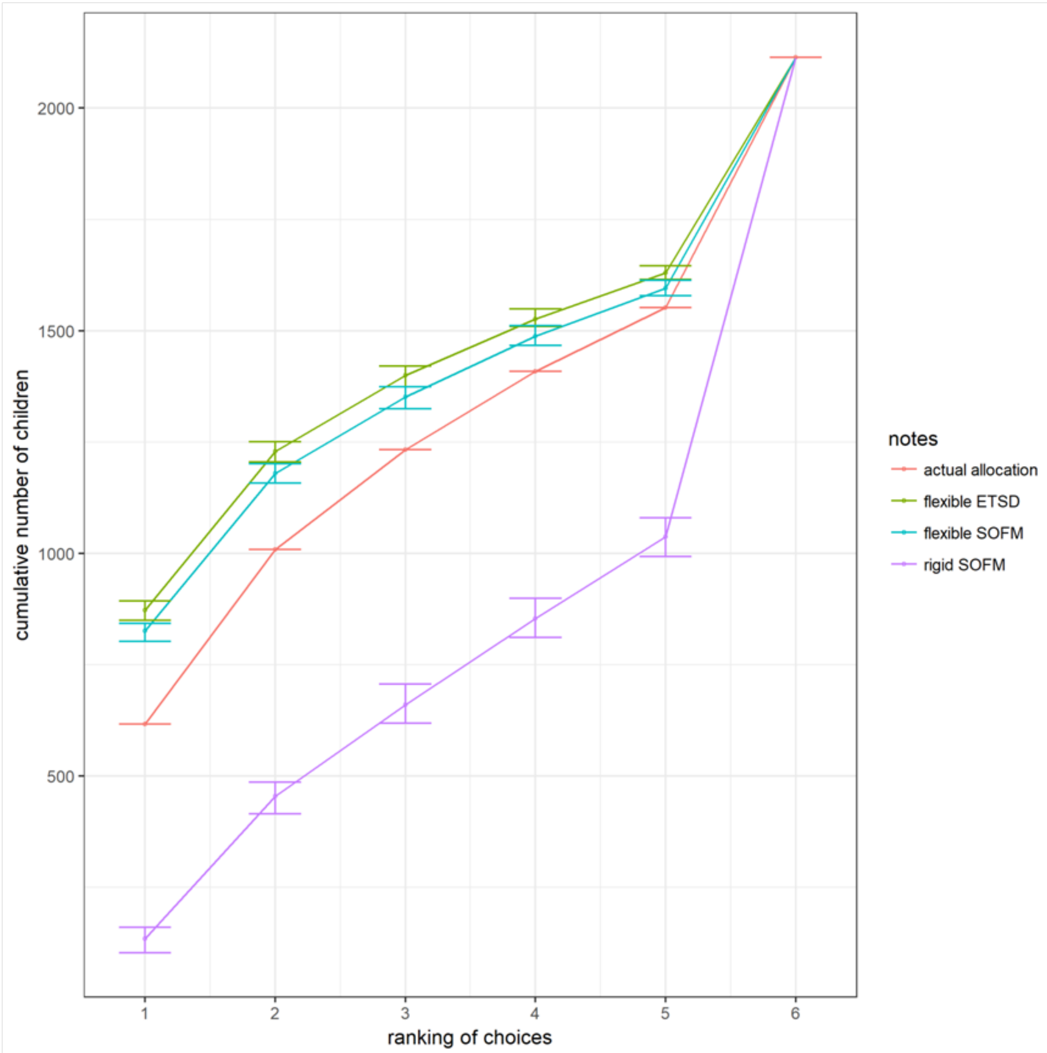

FIGURE 4. Rank distributions under different mechanisms: The graph reports the average cumulative number of children at each rank, as well as its range (shown by the error bars) across all 250 simulation runs.

toward some i' matched to s under the actual allocations. Also, students involved in at least one of such pairs and daycares involved are 40.82% and 96.83% of the respective total numbers. As for the analysis of Yamagata's data, the amount of justified envy for Bunkyo's actual assignment seems comparable to those in TTC on Boston and New Orleans data (Abdulkadiroglu et al., 2017).

We also study what happens in serial dictatorship if the rigid constraint is removed so that it is only subject to the daycare constraint. In the induced mechanism, flexible (justified) envy-tolerating serial dictatorship (flexible ETSD), some justified envy is tolerated while the constraint is flexible in this mechanism. Thus, its efficiency is expected to be

	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
pairs with justified envy	0	0	1622 (18.79%)	923.00 (10.70%) (SE= 114.76)
students with justified envy	0	0	863 (40.82%)	613.79 (29.03%) (SE= 50.03)
daycares with justified envy	0	0	61 (96.83%)	44.35 (70.40%) (SE= 2.94)

Table 4. Measures of justified envy under different mechanisms. The percentages for pairs with justified envy divide the numbers of pairs with justified envy by the numbers of pairs (i, s) such that s is acceptable to i. The "SE" stands for the standard error of the raw number.

even higher than both Bunkyo's actual assignment and SOFM under daycare constraint. Somewhat surprisingly, however, the magnitude of the improvement of this mechanism over the flexible SOFM seems rather small; the average number of unmatched children decreases only by 63.83 (7.29%), and the average number of children who become strictly better off under the flexible ETSD is 102.84 (4.86%). This difference is smaller than the improvement of the flexible SOFM over Bunkyo's assignment, whose corresponding numbers are 110.26 (11.18%) and 534.94 (25.30%), respectively. Meanwhile, the measures of justified envy show similar magnitudes to those for Bunkyo's assignment. These numbers may suggest that the flexible SOFM may be a potentially useful mechanism in daycare allocation.

Overall, the numerical analysis we report here suggests that the main findings for Yamagata are robust to data features.

Appendix C. Additional Discussions

C.1. **General Upper Bounds and Multidimensional Constraints.** In a recent work, Delacrétaz, Kominers and Teytelboym (2016) study a model of matching with multidimensional constraints. This subsection investigates the relationship between our model of general upper-bound and their model.

In the model with multidimensional constraints, there is a finite set of **services**, Σ. Each student i is associated with **service needs** $\nu^i = (\nu^i_\sigma)_{\sigma \in \Sigma} \in \mathbb{R}^{|\Sigma|}_+$, and each school

s is endowed with **service capacity profile** $\kappa^s = (\kappa^s_\sigma)_{\sigma\in\Sigma} \in \mathbb{R}^{|\Sigma|}_{++}$.[10] We say that a set of students I' is **DKT-feasible** at school s if $\sum_{i\in I'} \nu^i_\sigma \leq \kappa^s_\sigma$ for every $\sigma \in \Sigma$ and that matching μ is DKT-feasible if μ_s is DKT-feasible at every $s \in S$.

It is obvious that any constraint given as multidimensional constraints described above is a general upper-bound. The following proposition establishes that there is a specific sense in which these two classes of constraints are "equivalent" to each other, if one can specify any (possibly very large) set of services.

Proposition 9. *Fix I, s, and a constraint $\mathcal{F}_s$. The following two statements are equivalent.*

(1) *$\mathcal{F}_s$ is a general upper-bound.*
(2) *There exist a set of services Σ, a profile of students' service needs $(\nu^i)_{i\in I}$, and a service capacity profile κ^s such that a set of students I' is DKT-feasible at s if and only if $I' \in \mathcal{F}_s$.*

This proposition demonstrates that the class of constraints that can be described as general upper-bounds is the same as those that can be described by multidimensional constraints. This characterization exactly identifies what property is imposed on the types of constraints considered by Delacrétaz, Kominers and Teytelboym (2016) that use linear inequalities. Furthermore, this result is useful as it provides a potentially tractable "language" to code any general upper-bound using a number of linear inequalities. Related, the existence of an SOFM (our Theorem 2) can be obtained by exploiting the connection between these two models. More specifically, Proposition 6 of Delacrétaz, Kominers and Teytelboym (2016) shows the existence of an SOFM in the model with multidimensional constraints. This result and Proposition 9 provide an alternative proof for one of our results, i.e., the sufficiency of general upper-bound to guarantee existence of SOFM.

However, we also note that our "equivalence" result is subtle, and we need caution when interpreting this result. In order to establish that a given general upper-bound can be described by multidimensional constraints, the analyst needs to have the freedom to define the set of "services," as well as students' service needs and service capacities at each school. These services and related parameters defined in this attempt may not correspond to any physical services or other entities which one would regard as real services. In fact, in the

[10]Delacrétaz, Kominers and Teytelboym (2016) further assume that service needs and capacities are represented by integers. As they mention, none of the results in their paper or ours depends on this assumption. In a similar vein, they allow for zero service capacities but it does not affect any results in their paper or ours. Our assumptions are made only for convenience in proofs.

proof of the direction "(1) $\Rightarrow$ (2)" of Proposition 9, we define a "service" corresponding to every single infeasible set of students.

A related problem is that the number of services needed to describe a given general upper-bound may be unreasonably large even if the underlying constraint is simple and easily interpretable. To make this point in a simple setting, suppose that there is a school s, and the set of students I is partitioned into two groups, I_1 and I_2. Suppose $\mathcal{F}_s = \{I' | I' \subseteq I_1 \text{ or } I' \subseteq I_2\}$, that is, s can admit a set of students if and only if all of its members belong to a single group.[11] Now, suppose that each of I_1 and I_2 has n students. The following proposition demonstrates that even describing the above simple constraint requires an unboundedly large number of services as n grows.

Proposition 10. *Suppose that multidimensional constraints with the set of services Σ describe the above constraint for the problem with n students from each group. Then $|\Sigma| \geq n$.*[12]

This result calls for some caution when interpreting the "equivalence" result of Proposition 9. Although for any given general upper-bound one can find multidimensional constraints that describe it, the set of services–and hence the number of linear inequalities–needed to describe it may be large when there are many students. In such a case, the representation of a given general upper-bound by a system of linear inequalities may not be practical.

C.2. **The (Lack of) Connection with Hatfield and Milgrom (2005).** We sometimes receive comments that our results may be implied by Hatfield and Milgrom (2005). In this subsection, we illustrate a precise sense in which that is not the case. The discussion also clarifies that, although the present paper shares broad interest with the literature of matching with distributional constraints such as Kamada and Kojima (2015, 2018) and Kojima, Tamura and Yokoo (2018), the theoretical development in our present paper needs to be independent of those from such papers because the latter makes use of results from Hatfield and Milgrom (2005).

The argument for the "connection" between our analysis and theirs is based on defining each school's choice function which, faced with a set of students applying to the school, chooses the highest-ranked students until adding the next preferred student results in infeasibility. Formally, define a choice function $C_s : 2^I \to 2^I$ by

$$C_s(I') = \{i_1, i_2, \ldots, i_k\}, \tag{C.1}$$

[11] As detailed in Section 4.1, such a constraint is realistic in the context of refugee match.

[12] In Appendix A.11, we show that this bound is tight.

where $I' = \{i_1, i_2, \ldots, i_K\}$ is ordered by the school's priority so that $i_1 \succ_s i_2 \succ_s \ldots \succ_s i_K$, and k is the largest integer such that $\{i_1, i_2, \ldots, i_k\} \in \mathcal{F}_s$.[13] One can show that this choice function satisfies the substitutes condition. Then, it is argued that a result from Hatfield and Milgrom (2005) can be used to prove that a student-optimal stable matching exists, and that the equivalence can be shown between a student-optimal stable matching with respect to this choice function and a student-optimal fair matching in our setting.

However, in the setting of Hatfield and Milgrom (2005), the substitutes condition alone does not guarantee the existence of a stable matching, let alone a student-optimal stable matching. As pointed out by Aygün and Sönmez (2013), the existence is not guaranteed without another condition called the irrelevance of rejected contracts (IRC), and there is an example showing non-existence in the absence of that condition.

In fact, the choice function defined by (C.1) does not necessarily satisfy IRC. To see this point, let us first define IRC in our setting. A choice function C_s is said to satisfy IRC if for every subset of students I' and a student $i \notin I'$, $i \notin C_s(I' \cup \{i\})$ implies $C_s(I') = C_s(I' \cup \{i\})$. To see that the choice function defined by (C.1) does not necessarily satisfy IRC, consider the following example. The set of students is $\{i_1, i_2, i_3\}$, school s has a priority order $i_1 \succ_s i_2 \succ_s i_3$, and the feasibility constraint is $\mathcal{F}_s = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_3\}\}$. Then we have $C_s(\{i_1, i_2, i_3\}) = \{i_1\}$ and $C_s(\{i_1, i_3\}) = \{i_1, i_3\}$. Taking $I' = \{i_1, i_3\}$ and $i = i_2$, we see that $C_s(I') = \{i_1, i_3\} \neq \{i_1\} = C_s(I \cup \{i\})$. Hence, C_s violates IRC.

The above example has demonstrated that our existence theorem (Theorem 2) cannot be derived as a corollary of Hatfield and Milgrom (2005). In a similar vein, none of the results in our paper can be obtained as a corollary of Hatfield and Milgrom (2005).

It is worth noting that, to our knowledge, the IRC condition is always satisfied in the literature and plays a crucial role for the existence of a solution.[14] Our paper is an exception as illustrated above. This difference is due to the fact that we allow for general constraints. Our work presents an instance in which IRC is violated and investigates how to handle such a situation. We believe that this feature of the problem is of independent interest.

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[13]Hatfield and Milgrom (2005) consider a setting of matching with contracts. Here a choice function is defined over the family of subsets of students (the terms of contracts are fixed in our setting). An interpretation is that we are identifying a contract with the identity of the student involved in that contract.

[14]Hatfield and Kominers (2013) consider the technique of "completing" a choice function, by which the resulting choice function may satisfy IRC. This technique would not alter the current choice function, so the violation of IRC is not resolved by completion.

C.3. **Pareto-Undominated Fair Matchings under General Constraints.** The next example provides an environment in which schools' constraints are not general upper-bounds and there is no truncation-proof mechanism that always produces a Pareto-undominated fair matching.

Example 4. Suppose that there are $n \geq 2$ students $i_1 \ldots i_n$, and two schools, s_1 and s_2. Fix an integer m such that $1 < m < n$, and let preferences be as follows:

$$\succ_{i_k}: s_1, s_2 \qquad \text{for } k = 1, \ldots, m,$$

$$\succ_{i_k}: s_2, s_1 \qquad \text{for } k = m+1, \ldots, n.$$

The schools' priorities are arbitrary. The constraints are: $\mathcal{F}_{s_1} = \mathcal{F}_{s_2} = \{\emptyset, \{i_1, \ldots, i_n\}\}$. Note that these constraints are not general upper-bounds.

Now, consider the following two matchings μ^1 and μ^2: The matching μ^1 satisfies $\mu^1_{i_k} = s_1$ for all k, and the matching μ^2 satisfies $\mu^2_{i_k} = s_2$ for all k. It is straightforward to verify that, under $\succ_I$, μ^1 and μ^2 are the (only) Pareto-undominated fair matchings. Fix any mechanism φ that always produces a Pareto-undominated fair matching. We consider two (exhaustive) cases:

(1) Suppose that $\varphi^{\succ_S}(\succ_I) = \mu^1$. Consider $\succ'_{i_n}$ such that $s_2 \succ'_{i_n} \emptyset \succ'_{i_n} s_1$. Under the preference profile $(\succ'_{i_n}, \succ_{I \setminus \{i_n\}})$, μ^2 is a unique Pareto-undominated fair matching (note that μ^1 is no longer individually rational). This implies that $\varphi^{\succ_S}(\succ'_{i_n}, \succ_{I \setminus \{i_n\}}) = \mu^2$ because φ always produces a Pareto-undominated fair matching. Since $\mu^2_{i_n} \succ_{i_n} \mu^1_{i_n}$ and $\succ'_{i_n}$ is a truncation of $\succ_{i_n}$, φ is not truncation-proof.

(2) Suppose that $\varphi^{\succ_S}(\succ_I) = \mu^2$. Consider $\succ'_{i_1}$ such that $s_1 \succ'_{i_1} \emptyset \succ'_{i_1} s_2$. Under the preference profile $(\succ'_{i_1}, \succ_{I \setminus \{i_1\}})$, μ^1 is a unique Pareto-undominated fair matching (note that μ^2 is no longer individually rational). This implies that $\varphi^{\succ_S}(\succ'_{i_1}, \succ_{I \setminus \{i_1\}}) = \mu^1$ because φ always produces a Pareto-undominated fair matching. Since $\mu^1_{i_1} \succ_{i_1} \mu^2_{i_1}$ and $\succ'_{i_1}$ is a truncation of $\succ_{i_1}$, φ is not truncation-proof.

Overall, we have shown that φ cannot be truncation-proof, as desired. $\square$

The same example can be used to show that the cutoff adjustment algorithm does not always find a Pareto-undominated fair matching.

Example 4'. Consider the same environment as in Example 4. In the first step of the cutoff adjustment algorithm, the cutoff profile changes from $p^0 = (1,1)$ to $p^1 = T(p^0) = (2,2)$ because $\{i_1, \ldots, i_m\} \notin \mathcal{F}_{s_1}$ and $\{i_{m+1}, \ldots, i_n\} \notin \mathcal{F}_{s_2}$. This implies that $D_s(p^k) \neq I$ for all $s \in S$ and $k \geq 1$.

Since $\emptyset \in \mathcal{F}_s$ for each $s \in S$, the cutoff profiles are nondecreasing (recall that the definition of function T in Equation (3.1) implies $T_s(p) \not\geq p_s$ only when (i) $p_s = n+1$ and thus $D_s(p) = \emptyset$ as well as (ii) $D_s(p) \notin \mathcal{F}_s$). This means that p^k converges in a finite number of steps. Let p^* be the limit cutoff profile.

For any $k \geq 1$ such that $D_s(p^k) \neq \emptyset$ for some $s \in S$, because $D_s(p^k) \neq I$ as previously shown, it follows that $D_s(p^k) \notin \mathcal{F}_s$, and thus $T_s(p^k) = p_s^k + 1$. Therefore, p^* must satisfy $D_s(p^*) = \emptyset$ for each $s \in S$.[15] By the definition of the cutoff adjustment algorithm, the outcome μ^{p^*} thus satisfies $\mu_s^{p^*} = D_s(p^*) = \emptyset$ for each $s \in S$, i.e., the algorithm produces the empty matching.

Since the empty matching is Pareto-dominated by μ^1 (and μ^2 as well) that is fair, the cutoff adjustment algorithm does not produce a Pareto-undominated fair matching in this example. □

C.4. **Weak Fairness and Non-Existence: An Example.** Delacrétaz, Kominers and Teytelboym (2016) consider a slightly different setting from ours and find an example to show that their concept of stability may lead to non-existence. The following example, which is a slight variation of theirs, shows that there does not necessarily exist a matching that is feasible, individually rational, non-wasteful, and weakly fair.[16]

Example 5. Suppose that there are three students i_1, i_2, and i_3, and two schools, s_1 and s_2. Their preferences and priorities are as follows:

$$\succ_{i_1}: s_2, s_1 \qquad \succ_{s_1}: i_1, i_2, i_3$$

$$\succ_{i_2}: s_1, s_2 \qquad \succ_{s_2}: i_3, i_1, i_2$$

$$\succ_{i_3}: s_1, s_2$$

The feasibility constraints are $\mathcal{F}_{s_1} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_3\}\}$ and $\mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}\}$. Note that the constraint of school s_2 is a capacity constraint while the constraint of school s_1 is not, and both are general upper-bounds.

In this market, there is no matching satisfying feasibility, individual rationality, non-wastefulness, and weak fairness. To see this, consider the following (exhaustive) cases:

(1) Suppose i_1 is matched with s_2. Then i_3 should be matched with s_1 because otherwise i_3 is unmatched and hence has a feasible justified envy toward i_1. Then i_2 is unmatched, but this means i_2 has feasible justified envy toward i_3.

[15] In fact, one can show that $p^* = (n+1, n+1)$, although it is not necessary to show this for our conclusion.

[16] Adapted to our setting, their stability concept is slightly stronger than our requirements of feasibility, individual rationality, non-wasteful, and weak fairness.

(2) Suppose i_1 is matched with s_1. Then i_3 should be matched to s_1 because otherwise the allocation is wasteful (i_3 prefers s_1 most and $\{i_1, i_3\} \in \mathcal{F}_{s_1}$). This implies that i_2 is matched with s_2. But then i_1 has a feasible justified envy toward i_2.

(3) Suppose i_1 is unmatched. Then neither i_2 nor i_3 can be matched to s_1 as otherwise i_1 has a feasible justified envy toward the student who matches with s_1. But this is wasteful because, by letting μ denote the resulting matching, we have $s_1 \succ_{i_1} \emptyset = \mu_{i_1}$ and $\mu_{s_1} \cup \{i_1\} \in \mathcal{F}_{s_1}$. $\square$