Online Supplementary Appendix to: "Optimal Timing of Policy Announcements in Dynamic Election Campaigns"

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F Definition of Bayes Rule

Fix candidate i's history $h_i^t = \left(\left(t_i^k, X_i^k\right)_{k=1}^{k_i}, \left(t_j^l, X_j^l\right)_{l=1}^{l_j}, t, z_i \right)$ arbitrarily. If $t = T$, then candidate i believes that $h_j^t = (\emptyset, \emptyset, T, no)$ with probability one. Hence, we focus on $t < T$. Let $\left(t_i^l, X_i^l\right)_{l=1}^{l_i}$ be what candidate j can observe and is compatible with $\left(t_i^k, X_i^k\right)_{k=1}^{k_i}$: Let t_i^1 be the smallest time $t \in \{t_i^1, ..., t_i^{k_i}\}$ such that, for k with $t = t_i^k$, $X_i^k \neq X_i^0$ holds (that is, $-t_i^1$ is the first time for candidate i to change her policy set); given t_i^1, let t_i^2 be the smallest time $t \in \{t_i^1, ..., t_i^{k_i}\}$ such that $t > t_i^1$ and for k with $t = t_i^k$, $X_i^k \neq X_i^{k-1}$ holds (that is, $-t_i^2$ is the second time for candidate i to change her policy set), and so on. Fix $\left(t_i^l, X_i^l\right)_{l=1}^{l_i}$. Suppose that there exists $\left(t_j^k, X_j^k\right)_{k=1}^{k_j}$ with which $\left(t_j^l, X_j^l\right)_{l=1}^{l_j}$ is compatible, such that

$$h_j^t = \left(\left(t_i^l, X_i^l\right)_{l=1}^{l_i}, \left(t_j^k, X_j^k\right)_{k=1}^{k_j}, t, no \right)$$

happens with a positive probability by σ_j^* conditional on the realization of $\left(t_j^k\right)_{k=1}^{k_j}$, $\left(t_i^l, X_i^l\right)_{l=1}^{l_i}$, and t:[1] At each time t_j^k for $k = 1, ..., k_j$, given candidate j's history $h_j^{t_j^k} = \left(\left(t_i^l, X_i^l\right)_{l=1}^{l\left(t_j^k\right)}, \left(t_j^{k'}, X_j^{k'}\right)_{k'=1}^{k-1}, t_j^k, yes \right)$ with $l\left(t_j^k\right)$ being the largest l with $t_i^l < t_j^k$ (that is, $h_j^{t_j^k}$ is the history compatible with h_j^t), $\sigma_j^*(h_j^{t_j^k})\left(X_j^k\right) > 0$. Let $H_j^{\sigma_j^*}\left(h_i^t\right)$ be the set of candidate j's history satisfying this condition.

If $H_j^{\sigma_j^*}\left(h_i^t\right) \neq \emptyset$, then for each $h_j^t = \left(\left(t_i^l, X_i^l\right)_{l=1}^{l_i}, \left(t_j^k, X_j^k\right)_{k=1}^{k_j}, t, no \right) \in H_j^{\sigma_j^*}\left(h_i^t\right)$, we define the

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[1]Given candidate i's history, she believes that candidate j does not receive an opportunity at $-t$ when candidate i receives an opportunity. Hence, we require that the last element is *no*.

density f as follows:

$$f(h_j^t|h_i^t) = e^{-(T-t)\lambda} \frac{((T-t)\lambda)^{k_j}}{k_j!} \prod_{k=1}^{k_j} \sigma_j^*(h_j^{t_j^k}) \left(X_j^k\right) \tag{19}$$

for $k_j \neq 0$ and $f(h_j^t|h_i^t) = e^{-(T-t)\lambda}$ for $k_j = 0$. Note that $e^{-(T-t)\lambda}\frac{((T-t)\lambda)^{k_j}}{k_j!}$ is the probability that candidate j receives k_j opportunities between $-T$ and $-t$. Conditional on this event, every $\left(t_j^k\right)_{k=1}^{k_j}$, has the same density.

Using the density of h_j^t defined in (19),[2] we define

$$d\beta_i\left(h_j^t|h_i^t\right) = \frac{f(h_j^t|h_i^t)}{\int_{\tilde{h}_j^t \in H_j^{\sigma_j^*}(h_i^t)} f(\tilde{h}_j^t|h_i^t)d\tilde{h}_j^t} dh_j^t. \tag{20}$$

If $H_j^{\sigma_j^*}\left(h_i^t\right) = \emptyset$ (h_i^t cannot be explained without j's deviation), then $d\beta_i\left(h_j^t|h_i^t\right)$ is arbitrary, as long as $\int_{h_j^t \in H_j(h_i^t)} d\beta_i\left(h_j^t|h_i^t\right) = 1$.

Now, given a history h_i^t (note that this determines $\left(t_i^l, X_i^l\right)_{l=1}^{l_i}$ and $\left(t_j^l, X_j^l\right)_{l=1}^{l_j}$) and a set $\hat{H}_j\left(h_i^t\right) \subseteq H_j^{\sigma_j^*}\left(h_i^t\right)$, we can classify $h_j^t \in \hat{H}_j\left(h_i^t\right)$ into the following subsets: Given h_i^t and $\hat{H}_j\left(h_i^t\right)$, let KX_j be the set of k_j and $\left(X_j^k\right)_{k=1}^{k_j}$ such that there exists $(t_j^k)_{k=1}^{k_j}$ such that $\left(\left(t_i^l, X_i^l\right)_{l=1}^{l_i}, \left(t_j^k, X_j^k\right)_{k=1}^{k_j}, t, no\right) \in \hat{H}_j\left(h_i^t\right)$. Given h_i^t, $\hat{H}_j\left(h_i^t\right)$, and $(k_j, \left(X_j^k\right)_{k=1}^{k_j}) \in KX_j$, let T_j^1, $T_j^2\left(t_j^1\right)$, ..., $T_j^{k_j}(t_j^1, ..., t_j^{k_j-1})$ be, respectively, the set of t_j^1 such that there exists $(t_j^2, ..., t_j^{k_j})$ such that $\left(t_j^k, X_j^k\right)_{k=1}^{k_j}$ is compatible with $\left(t_j^l, X_j^l\right)_{l=1}^{l_j}$ and $\left(\left(t_i^l, X_i^l\right)_{l=1}^{l_i}, \left(t_j^k, X_j^k\right)_{k=1}^{k_j}, t, no\right) \in \hat{H}_j\left(h_i^t\right)$; the set of t_j^2 such that, given t_j^1, there exists $(t_j^3, ..., t_j^{k_j})$ such that $\left(t_j^k, X_j^k\right)_{k=1}^{k_j}$ is compatible with $\left(t_j^l, X_j^l\right)_{l=1}^{l_j}$ and $\left(\left(t_i^l, X_i^l\right)_{l=1}^{l_i}, \left(t_j^k, X_j^k\right)_{k=1}^{k_j}, t, no\right) \in \hat{H}_j\left(h_i^t\right)$; and so on, up to the set of $t_j^{k_j}$ such that, given $t_j^1, t_j^2, \ldots, t_j^{k-1}$, $\left(t_j^k, X_j^k\right)_{k=1}^{k_j}$ is compatible with $\left(t_j^l, X_j^l\right)_{l=1}^{l_j}$ and $\left(\left(t_i^l, X_i^l\right)_{l=1}^{l_i}, \left(t_j^k, X_j^k\right)_{k=1}^{k_j}, t, no\right) \in \hat{H}_j\left(h_i^t\right)$.[3] Given h_i^t and $H_j^{\sigma_j^*}\left(h_i^t\right)$, define KX_j^*, $T_j^{1,*}$, $T_j^{2,*}\left(t_j^1\right)$, ..., $T_j^{k_j,*}(t_j^1, ..., t_j^{k_j-1})$ in a similar manner. Then, given h_i^t and $\hat{H}_j\left(h_i^t\right)$, for

[2] Since the Poisson process has a density, we directly define the conditional probability using the ratio of the density functions. Since the Poisson process is right-continuous, we can micro-found this definition by a measure-theoretic definition as well. See Karazas and Shreve (1988, Proposition 1.13) for the details.

[3] For simple notation, we suppress the dependence of KX_j on h_i^t and $\hat{H}_j\left(h_i^t\right)$ and the dependence of T_j^1, $T_j^2\left(t_j^1\right)$, ..., $T_j^{k_j}(t_j^1, ..., t_j^{k_j-1})$ on h_i^t, $\hat{H}_j\left(h_i^t\right)$, and $(k_j, (X_j^k)_{k=1}^{k_j})$.

any $g\left(h_j^t\right)$, we define

$$
\begin{aligned}
&\int_{h_j^t \in \hat{H}_j\left(h_i^t\right)} g\left(h_j^t\right) d\beta_i\left(h_j^t | h_i^t\right) \\
:= &\frac{\sum_{\left(k_j,\left(X_j^k\right)_{k=1}^{k_j}\right)\in KX_j} \int_{t_j^1 \in T_j^1} \int_{t_j^2 \in T_j^2(t_j^1)} \cdots \int_{t_j^{k_j} \in T_j^{k_j}(t_j^1,\dots,t_j^{k_j-1})} \begin{pmatrix} g\left(h_j^t\right) e^{-(T-t)\lambda} \frac{((T-t)\lambda)^{k_j}}{k_j!} \\ \times \prod_{k=1}^{k_j} \sigma_j^*(h_j^{t_j^k}) \left(X_j^k\right) dt_j^{k_j} \cdots dt_j^1 \end{pmatrix}}{\sum_{\left(k_j,\left(X_j^k\right)_{k=1}^{k_j}\right)\in KX_j^*} \int_{t_j^1 \in T_j^{1,*}} \int_{t_j^2 \in T_j^{2,*}(t_j^1)} \cdots \int_{t_j^{k_j} \in T_j^{k_j,*}(t_j^1,\dots,t_j^{k_j-1})} \begin{pmatrix} g\left(h_j^t\right) e^{-(T-t)\lambda} \frac{((T-t)\lambda)^{k_j}}{k_j!} \\ \times \prod_{k=1}^{k_j} \sigma_j^*(h_j^{t_j^k}) \left(X_j^k\right) dt_j^{k_j} \cdots dt_j^1 \end{pmatrix}}.
\end{aligned}
$$

For example, given a fixed continuation strategy profile σ and h_i^t, the function $g\left(h_j^t\right)$ can be candidate i's continuation payoff $u_i\left(\sigma | h_i^t, h_j^t\right)$.

G Proofs Omitted in Appendix C

G.1 Proof of Proposition 9

Fix any $t \in (-\infty, 0]$, and suppose that no candidate enters at any $-\tau \in (-t, 0]$. On the one hand, if candidate i enters at $-t$, her payoff is $v_{i,t}(\text{enter})$. By Assumption 2, $v_{i,t}(\text{enter}) \leq v_{i,0}(\text{enter})$. Since $v_{i,0}(\text{enter}) = v_i\left(x_i^*, X\right)$ by definition and $v_i\left(x_i^*, X\right) < v_i\left(X, X\right)$ as we are in Case 1, we have $v_{i,t}(\text{enter}) < v_i\left(X, X\right)$. On the other hand, if she does not enter, then her payoff is $v_i\left(X, X\right)$. Hence, it is uniquely optimal not to enter at $-t$. Since the payoffs are continuous in time, there exists $\varepsilon > 0$ such that no candidate enters for any time in $(-t-\varepsilon, -t]$. Hence the continuous-time backward induction implies the desired result.

G.2 The Formal Definition of $\bar{v}_{i,t}(\text{not})$

Formally, $\bar{v}_{i,t}(\text{not})$ is defined by the following:

$$
\begin{aligned}
\bar{v}_{i,t}(\text{not}) \quad = \quad & e^{-(\lambda_i+\lambda_j)t} v_i(X,X) + e^{-\lambda_i t}(1-e^{-\lambda_j t}) v_i(X, x_j^*) + (1-e^{-\lambda_i t}) e^{-\lambda_j t} v_i(x_i^*, X) \\
& + \left(1 - e^{-\lambda_i t}\right)\left(1 - e^{-\lambda_j t}\right)\left(\frac{\lambda_i}{\lambda_i+\lambda_j} v_i^{BR_j} + \frac{\lambda_j}{\lambda_i+\lambda_j} \sup_{x_i} v_i(x_i, x_j^*)\right).
\end{aligned}
$$

G.3 Proof of Proposition 10

By the definition of t_0, there exists $\varepsilon > 0$ such that for all time in $(-t_0 - \varepsilon, -t_0]$, each candidate i enters under any PBE. Hence, if $t_i^* = t_j^* = \infty$, each candidate enters at all times in $(-\infty, -t_0]$. For the rest of the proof, we focus on the case in which at least one of t_A^* and t_B^* is less than ∞. Without loss, we assume $t_A^* \leq t_B^*$.

The following lemma shows that, for any PBE, candidate A does not enter at any time $-t < -t_A^*$:

Lemma 10 *Fix any σ_B such that (i) $\sigma_B(h_B^t) = x_B^*$ for any h_B^t with $\theta_B(h_B^t) = X$ for each $-t \in (-t_A^*, -t_0]$ and (ii) $\sigma_B(h_t) = BR_B(x_A)$ for any h_B^t with $\theta_A(h_B^t) = x_A$ for each $-t \in [-T, 0]$.*[4] *If σ_A is a best response to σ_B, then for any $h_t \in (X, X)$ with $-t < -t_A^*$, we have $\sigma_A(h_t)(X) = 1$.*

The proof of the lemma is complicated, so we first assume that the lemma holds and show the proposition, and then prove the lemma. If $t_A^* = t_B^*$, then Lemma 10 implies Proposition 10 with $t_i = t_i^*$ for each i. Hence, we assume $t_A^* < t_B^*$.

Fix a PBE and, for each $i = A, B$, let $v_{i,t}(\text{not})$ be candidate i's continuation payoff at time $-t$ when i does not enter. Given Lemma 10, for $t \in [t_A^*, t_B^{**}]$ with t_B^{**} defined below, we calculate $v_{i,t}(\text{not})$ assuming that only candidate B enters in the time interval $(-t, -t_A^*)$ and both candidates enter in the time interval $[-t_A^*, -t_0]$. For $\tau \geq t$, Lemma 10 implies that candidate A does not enter at times in $(-\tau, -t)$. Hence, we have $v_{B,\tau}(\text{not}) \geq v_{B,t}(\text{not})$ for $\tau \geq t$ because candidate B at $-\tau$ can receive $v_{B,t}(\text{not})$ by committing to a strategy in which he keeps skipping opportunities from $-\tau$ to $-t$. Let

$$t_B^{**} \equiv \inf \{t > t_0 : v_{B,t}(\text{not}) \geq v_{B,t}(\text{enter})\}.$$

There are the following two cases: $t_B^{**} < \infty$ or $t_B^{**} = \infty$. The following lemma is useful:

Lemma 11 *If $t_B^{**} < \infty$, then $v_2(x_B^*, X) > v_B^{BR_A}$.*

Proof. Suppose otherwise. Then, Assumption 2 implies $v_B(x_B^*, X) = v_B^{BR_A}$. Then, $v_{B,t}(\text{enter})$ is constant in $t \in [t_0, \infty)$. At time $-t_B^{**} < -t_A^*$, there are the following three cases:

1. Candidate A has the next opportunity at time $-t \in (-t_B^{**}, -t_A^*]$. Conditional on this event, candidate B obtains a payoff of $v_B^{BR_A} = v_B(x_B^*, X) = v_{B,t}(\text{enter})$ when he enters at $-t$

[4] Recall the definition of $\theta_i(\cdot)$ from Section 4.3.

and a payoff of $v_{B,t}(\text{not})$ when he does not. Since t_B^{**} is the infimum of $t > t_0$ such that $v_{B,t}(\text{not}) \geq v_{B,t}(\text{enter})$, candidate B prefers to enter in this event.

2. Candidate B has the next opportunity at time $-t \in (-t_B^{**}, -t_A^*]$. Conditional on this event, since candidate B receives $v_{B,t}(\text{enter})$ at any $-t$ upon entering, candidate B is indifferent between entering and not entering.

3. No candidate has an opportunity at any time $-\bar{t} \in (-t_B^{**}, -t_A^*]$. Conditional on this event, candidate B strictly prefers to enter since $v_{B,t_A^*}(\text{enter}) > v_{B,t_A^*}(X, X)$.

Hence, it is uniquely optimal to enter at $-t_B^{**}$, which is a contradiction. ■

Given this lemma, consider the following two cases:

1. $t_B^{**} < \infty$: In this case, we are left to prove Lemma 10. To see why, once we have shown Lemma 10, then for $t > t_B^{**}$, $v_{B,t}(\text{not}) \geq v_{B,t_B^{**}}(\text{not})$ since candidate B can skip opportunities until $-t_B^{**}$ without the opponent entering. Together with the fact that $v_{B,t}(\text{enter})$ is strictly decreasing in t by Lemma 11, we can conclude that candidate B does not enter at times in $(-\infty, -t_B^{**})$ in any PBE.

2. $t_B^{**} = \infty$: This means that candidate B enters at times in $(-\infty, 0]$ in any PBE given Lemma 10.

We now prove Lemma 10:

Proof of Lemma 10. Suppose now candidate A receives an opportunity at time $-\bar{t} < -t_A^*$ at a history in (X, X).

Fix candidate B's strategy arbitrarily. Once we fix his strategy, conditional on the event that candidate A does not enter at any time in $(-\infty, -t_A^*]$ and that candidate B has at least one opportunity in $(-\bar{t}, -t_A^*]$, we can define a random variable t that is the largest $\tau \in [t_A^*, \bar{t})$ such that candidate B enters at time $-\tau$.

From candidate A's perspective at time $-\bar{t}$, there are the following two possible events:

1. $t \leq t_A^*$ or candidate B does not have any opportunities in $(-\bar{t}, -t_A^*]$. Conditional on this event, not entering at times in $[-\bar{t}, -t_A^*)$ ensures candidate A the value of $v_{A,t_A^*}(\text{not})$. Since $v_{A,t_A^*}(\text{not}) = v_{A,t_A^*}(\text{enter})$ and $v_{A,t}(\text{enter})$ is weakly decreasing in t by Assumption 2, it is weakly better for candidate A not to enter at time $-\bar{t}$.

2. $t > t_A^*$. Conditional on this event, at time $-\bar{t}$, candidate A's continuation payoff from entering is weakly less than her continuation payoff from not entering if and only if

$$v_A^{BR_B} \leq (1 - e^{-\lambda_A t}) \left(\max_{X_A \in \mathcal{X}_A} v_A(X_A, x_B^*) \right) + e^{-\lambda_A t} v_A(X, x_B^*) =: \hat{v}_{A,t}.$$

To see why, note first that the left-hand side is the payoff from entering at time $-\bar{t}$, while the right-hand side is the payoff at time $-t$ when the current policy profile is (X, x_B^*). The payoff from not entering at time $-\bar{t}$ is a convex combination of the following two payoffs, where the weight on the latter payoff is strictly positive.

- The payoff under the event that candidate A receives at least one opportunity at which she enters in the time interval $(-\bar{t}, -t)$.
- The payoff under the event that candidate A does not receive any opportunity at which she enters in the time interval $(-\bar{t}, -t)$.

Note that the former payoff is equal to the left-hand side of the expression ($v_A^{BR_B}$), while the latter payoff is the same as the right-hand side of the expression. This implies the desired equivalence.

To compare the two values, it is instructive to examine why candidate A at $-t_A^*$ is indifferent between entering and not entering at histories in (X, X). Suppose now that candidate B has not entered at $-t_A^*$. There are following three events that can happen with positive probability until the deadline:

(a) Candidate A receives the next opportunity at $-\tau > -t_A^*$: In this case, candidate A receives $v_{A,t}(\text{enter})$ at $-\tau$ regardless of candidate A's choice at $-t_A^*$. Note that, even if candidate A has entered before $-\tau$, since we assume that candidate A enters at some policy in X_A^*, the situation is that candidate A enters at some policy in X_A^* and candidate B has not at $-\tau$ (note that all the policies in X_A^* give rise to the same payoff).

(b) Candidate B receives the next opportunity at $-\tau > -t_A^*$: Candidate A receives a payoff of $v_A^{BR_B}$ (candidate B best-responds to x_A^* at $-\tau$) if she enters before $-t_A^*$; and $\hat{v}_{A,\tau}$ (candidate B enters while candidate A has not at $-\tau$) if she does not enter before $-t_A^*$.

(c) No candidate receives any opportunity in the time interval $(-t_A^*, -t_0]$: Candidate A receives $v_{A,t_0}(\text{enter})$ if she enters at $-t_1^*$; and $v_{A,t_0}(\text{not})$ if she does not at $-t_A^*$. We have assumed that $v_{A,t_0}(\text{enter}) > v_{A,t_0}(\text{not})$.

Note that candidate A is indifferent between entering and not entering at $-t_A^*$ in case (2a) and strictly prefers entering in case (2c). Since case (2c) happens with positive probability, it must be the case that candidate A strictly prefers not entering to entering in case (2b), in order for her to be indifferent between entering and not entering at $-t_A^*$. This implies that there exists $-\tilde{t} \in (-t_A^*, t_0]$ such that $v_A^{BR_B} < \hat{v}_{A,\tilde{t}}$. Since candidate A can always skip opportunities between $-t$ and $-\tilde{t}$, we have $\hat{v}_{A,\tilde{t}} \leq \hat{v}_{A,t}$, implying $v_A^{BR_B} < \hat{v}_{A,t}$.

Hence, conditional on the event that candidate B has an opportunity and enters at $-t$, candidate A at $-\bar{t}$ strictly prefers not entering to entering.[5]

Now we prove that candidate A does not enter at any history in (X, X) at any time before $-t_A^*$ in any PBE. There are the following two cases:

1. If "$v_i^{BR_j} < \sup_{\{x_i\}\in\mathcal{X}_i} v_i(x_i, X)$ for each i" holds in Assumption 3, then $v_{A,t}(\text{enter})$ is strictly decreasing in t. Hence, conditional on the event that $t \leq t_A^*$ or candidate B does not have any opportunities in $(-\bar{t}, -t_A^*]$, candidate A at $-\bar{t}$ strictly prefers not entering to entering. Since it happens with a positive probability that candidate B does not receive any opportunity in $(-\bar{t}, -t_A^*]$, candidate A does not enter before $-t_A^*$ in any PBE.

2. If $t_A^* \neq t_B^*$ holds in Assumption 3, then $t_A^* < t_B^*$. By continuity of the continuation payoff in time, there exists $\varepsilon > 0$ such that candidate B enters for each $-t \in (-t_A^* - \varepsilon, -t_A^*]$. Hence, the event that candidate B has an opportunity and enters at some time in this time interval happens with a positive probability. Therefore, candidate A does not enter before $-t_A^*$ in any PBE.

■

[5] Here we are using Assumption 1 which implies that $v_A^{BR_B}$ must be independent of the time at which candidate A chooses x_A^*.

G.4 Proof of Part 1 of Proposition 12

By continuity of the continuation payoff in time, for times $-t < -\hat{t}_A$ sufficiently close to $-\hat{t}_A$, candidate B enters, and thus we focus on candidate A's incentive at those times. Let

$$\hat{v}_{A,t} := (1 - e^{-\lambda_A t}) \left(\max_{X_A \in \mathcal{X}_A} v_A(X_A, x_B^*) \right) + e^{-\lambda_A t} v_A(X, x_B^*)$$

be candidate A's payoff when she has not entered and candidate B has at time $-t$. The straightforward algebra shows that $\bar{v}^A_{A,t}(\text{not})$ satisfies

$$\begin{aligned}
\bar{v}^A_{A,t}(\text{not}) &= \int_0^t \lambda_B e^{-\lambda_B \tau} \hat{v}_{A,t-\tau} d\tau \\
&= \left(e^{-\lambda_A t} - e^{-(\lambda_1 + \lambda_2)t} \right) v_A\left(X, x_B^*\right) + \left(1 - e^{-2\lambda_B t} - 2e^{-\lambda_B t}\right) \max_{x_A} v_A\left(x_A, x_B^*\right).
\end{aligned}$$

In contrast, we have

$$v_{A,t}(\text{enter}) = e^{-\lambda_B t} v_A\left(x_A^*, X\right) + \left(1 - e^{-\lambda_B t}\right) v_A^{BR_B}.$$

Hence, $v_{A,t}(\text{enter})$ and $\bar{v}^A_{A,t}(\text{not})$ are differentiable in t. Since $\hat{t}_A$ is the infimum of t with $\bar{v}^A_{A,t}(\text{not}) \leq v_{A,t}(\text{enter})$, we have

$$\left. \frac{d}{dt} \bar{v}^A_{A,t}(\text{not}) \right|_{t=\hat{t}_A} < \left. \frac{d}{dt} v_{A,t}(\text{enter}) \right|_{t=\hat{t}_A}.$$

Consider candidate A's incentive at time $-\hat{t}_A$. For any $\varepsilon > 0$, there are the following three cases (assuming that candidate B enters as soon as she obtains an opportunity):

1. Candidate A has the next opportunity at time $-\bar{t} \in (-t, -(t-\varepsilon)]$. Conditional on this event, since we fix candidate B's strategy at histories in (X, X), candidate A is indifferent between entering and not entering.

2. Candidate B has the next opportunity at time $-\bar{t} \in (-t, -(t-\varepsilon)]$. Conditional on this event, candidate A obtains a payoff of $v_A^{BR_B}$ when she enters at $-t$ and a payoff of $\hat{v}_{A,\bar{t}}$ when she does not.

3. No candidate has an opportunity at any time $-\bar{t} \in (-t, -(t-\varepsilon)]$. Conditional on this event, candidate A obtains a payoff of $v_{A,t-\varepsilon}(\text{enter})$ when she enters at $-t$ and a payoff of $\bar{v}^1_{A,t-\varepsilon}(\text{not})$

when she does not.

Since candidate A is indifferent between entering and not entering at time $-\hat{t}_A$, for any $\varepsilon > 0$, we have

$$
\begin{aligned}
& \int_{\tau=0}^{\varepsilon} \underbrace{\lambda_B e^{-(\lambda_A+\lambda_B)\tau}}_{\text{Candidate } B \text{ has the next opportunity at time } -t+\tau} \left(v_A^{BR_B} - \hat{v}_{A,t-\tau}\right) d\tau \\
= \; & e^{-(\lambda_A+\lambda_B)\varepsilon} \left(\bar{v}^1_{A,t-\varepsilon}(\text{not}) - v_{A,t-\varepsilon}(\text{enter})\right).
\end{aligned}
$$

Dividing both sides by ε and taking the limit as $\varepsilon \downarrow 0$, we have

$$
v_A^{BR_B} - \hat{v}_{A,\hat{t}_A} = \left(\frac{d}{dt}\bar{v}^1_{A,t}(\text{not})\bigg|_{t=\hat{t}_A} - \frac{d}{dt}v_{A,t}(\text{enter})\bigg|_{t=\hat{t}_A} \right) < 0.
$$

By continuity of the continuation payoff in time, there exists $\bar{\eta} > 0$ such that, for each $\eta \in [0, \bar{\eta})$, we have

$$
v_A^{BR_B} - \hat{v}_{A,\hat{t}_A+\eta} < 0. \tag{21}
$$

We now consider candidate A's incentive at $-t < -\hat{t}_A$. There are again following three cases:

1. Candidate A has the next opportunity at time $-\bar{t} \in (-t, -\hat{t}_A]$. Conditional on this event, since we fix candidate B's strategy at histories in (X, X), candidate A is indifferent between entering and not entering.

2. Candidate B has the next opportunity at time $-\bar{t} \in (-t, -\hat{t}_A]$. Conditional on this event, candidate A obtains a payoff of $v_A^{BR_B}$ when she enters at $-t$ and a payoff of $\hat{v}_{A,\bar{t}}$ when she does not.

3. No candidate has an opportunity at any time $-\bar{t} \in (-t, -\hat{t}_A]$. Conditional on this event, candidate A is indifferent between entering and not entering.

Hence, (21) implies that, for $-t \in (-\hat{t}_A - \bar{\eta}, -\hat{t}_A)$, candidate A strictly prefers to enter in any PBE, as desired.

G.5 Proof of Part 2 of Proposition 12

Since candidate A does not enter for each $-t \in \left[-\hat{t}_B, 0\right]$, the fact that candidate B strictly prefers to enter at time 0 and becomes indifferent between entering and not entering at $-\hat{t}_B$ implies that he is strictly worse off if candidate A enters than if she does not enter, after candidate B enters:

Lemma 12 *$\hat{t}_B \leq \hat{t}_A$ implies $v_B\left(x_B^*, X\right) > v_B^{BR_A}$.*

Proof. Suppose otherwise. Then, by Assumption 2, we have $v_B\left(x_B^*, X\right) = v_B^{BR_A}$ and so we have

$$v_{B,t}(\text{enter}) = v_B\left(x_B^*, X\right).$$

At time $-\hat{t}_B$, consider the following three cases:

1. Candidate A has the next opportunity at time $-t \in (-\hat{t}_B, 0]$. Conditional on this event, candidate B obtains a payoff of $v_B^{BR_A} = v_B\left(x_B^*, X\right)$ when he enters at $-t$ and a payoff of $\bar{v}_{B,t}^1(\text{not})$ when he does not.

2. Candidate B has the next opportunity at time $-t \in (-\hat{t}_B, 0]$. Conditional on this event, since we fix candidate A's strategy at histories in (X, X), candidate B is indifferent between entering and not entering.

3. No candidate has an opportunity at any time $-\bar{t} \in (-\hat{t}_B, 0]$. Conditional on this event, candidate B strictly prefers to enter since $v_B\left(x_B^*, X\right) > v_B\left(X, X\right)$.

Hence, candidate B strictly prefers to enter at $-\hat{t}_B$, which is a contradiction. ■

Given this lemma, we are left to show that, at each $-t \in (-\infty, -\hat{t}_B]$, given that no candidate enters for $-\tau \in (-t, -\hat{t}_B)$, each candidate strictly prefers not to enter at $-t < -t_B$.

On the one hand, if candidate i enters at $-t$, her payoff is $v_{i,t}(\text{enter})$. By Lemma 12, $v_{i,t}(\text{enter}) < v_{i,\hat{t}_B}(\text{enter})$. On the other hand, if she does not enter, then her payoff is $\bar{v}_{i,\hat{t}_B}^1(\text{not})$. Hence, it is indeed uniquely optimal not to enter at $-t$.

H Example for Playing the Weakly Dominated Action

In Section D, we claimed that the conclusion of part 3 of Theorem 4 does not hold if we replace strictly dominant policy with weakly dominant policy. This appendix provides an example to illustrate this. Let $X = \{0, 1\}$ and define (v_A, v_B) by the payoff matrix as in Table 3.

	0	1
0	$0, 1$	$2, 1$
1	$1, 2$	$3, 0$

Table 3: Payoff matrix for an example with multiple PBE with weakly dominant policies

In addition, for each $i = A, B$, we define $v_i(X, a_j) = \sum_{a_i \in X} \frac{1}{2} v_i(a_i, a_j)$ for each $a_j \in X$; $v_i(a_i, X) = \sum_{a_j \in X} \frac{1}{2} v_i(a_i, a_j)$ for each $a_i \in X$; and $v_i(X, X) = \sum_{(a_i, a_j) \in X \times X} \frac{1}{4} v_i(a_i, a_j)$.

Notice that $(1, 0)$ is the weakly (but not strictly) dominant policy profile, meaning that the defining inequality for a strictly dominant policy is required to hold only weakly for all $X_i \in \mathcal{X}_i \backslash \{x_i^*\}$ except at least one $X_i \in \mathcal{X}_i \setminus \{x_i^*\}$ for which the inequality needs to hold strictly. However, if candidate B announces $\{1\}$ after candidate A announces $\{0\}$ — he rewards her by taking the weakly dominated strategy, then it is possible in a PBE that candidate A announces $\{0\}$ for some time interval.

Specifically, suppose $\lambda_A = \lambda_B$, let $t^* = -\frac{1}{\lambda} \ln 2$, and consider the strategy profile as follows: Candidate A chooses $\{1\}$ except when B's current policy set is X and the time is in the interval $[-T, -t^*)$, at which she takes $\{0\}$. Player B chooses $\{0\}$ except when candidate A has already chosen $\{0\}$, in which case he takes $\{1\}$. We show that this strategy profile is a PBE.

It is straightforward to check that candidate B is taking a best response. We check candidate A's incentive. Suppose first that $-t \in [-t^*, 0]$. The expected payoff from taking $\{1\}$ is

$$e^{-\lambda t} 2 + (1 - e^{-\lambda t}). \tag{22}$$

The expected payoff from taking $\{0\}$ is $e^{-\lambda t} 1 + (1 - e^{-\lambda t}) 2$, and this is no more than (22) if $-t \in [-t^*, 0]$. Next, the expected payoff from taking $\{0, 1\}$ is

$$\begin{aligned} &\int_0^t e^{-2\lambda(t-s)} 2\lambda \frac{\left(e^{-\lambda s} + (1 - e^{-\lambda s}) 2\right) + \left(e^{-\lambda s} 0.5 + (1 - e^{-\lambda s})\right)}{2} ds + e^{-2\lambda t} 1.5 \\ &= 3 - 3e^{-\lambda t} + 1.5 e^{-2\lambda t}. \end{aligned}$$

Given (22), the payoff from $\{1\}$ is larger than the payoff from $\{0, 1\}$ if and only if

$$e^{-\lambda t}2 + (1 - e^{-\lambda t}) > 3 - 3e^{-\lambda t} + 1.5e^{-2\lambda t} \iff 4e^{-\lambda t} > 2 + 1.5e^{-2\lambda t}.$$

This holds if and only if $-t \in (-t^*, 0]$.

Second, suppose that $-t \in (-\infty, -t^*)$. The expected payoff from taking $\{0\}$ and the one from taking $\{1\}$ have the same expressions as before, and the former is strictly greater than the latter if $-t \in (-\infty, -t^*)$. The expected payoff from taking $\{0, 1\}$ is at most a strict convex combination of (i) the expected payoff at time $-t^*$ from the continuation strategy profile that coincides with the specified strategy profile, (ii) the expected payoff from taking $\{0\}$ at time $-t$, and (iii) the expected payoff from the opponent taking $\{0\}$ at time $-t$. Since we have shown that (i) is less than (ii) for any $-t = -t^*$ and (ii) is increasing in t, (i) is less than (ii) for any $-t \in (-\infty, -t^*)$. Hence, it suffices to show that (iii) is no more than (ii), which is equivalent to

$$e^{-\lambda t} + (1 - e^{-\lambda t})2 > e^{-\lambda t}0.5 + (1 - e^{-\lambda t}),$$

and this holds for any $t \geq 0$.

Overall, we have shown that candidate A is taking a best response conditional on any history.

I A Proof and Additional Discussions for Section 3.1

This section provides discussions of the valence election campaign model. First, Section I.1 provides a proof of Proposition 2. Next, Section I.2 derives empirical implications of our model. Although we see these findings as only suggestive, they are consistent with the empirical findings such as those presented in Campbell (1983). Then, Section I.3 conducts a welfare analysis, comparing our model with that of Aragonès and Palfrey (2002).

The dynamic model we have analyzed in Section 3.1 was kept as simple as possible to highlight the complexity added by the fact that candidates face dynamic incentive problems in the presence of valence. In Appendix I.4, we extend and modify this model to examine robustness of our prediction that candidates use ambiguous language at the early stages of the campaign.

I.1 Proof of Proposition 2

Note that Assumptions 1 and 2 in Section 4.1 are satisfied given $X_i^* = \{1\}$. Moreover, we have $v_i^{BR_j} < \sup_{x_i \in \mathcal{X}_i} v_i(x_i, X)$ (Assumption 3) and first-mover disadvantage is satisfied for $i = W$.

Fix a PBE σ arbitrarily. Given candidate i's history $h_i^t = \left(\left(t_i^k, X_i^k\right)_{k=1}^{k_i}, \left(t_j^l, X_j^l\right)_{l=1}^{l_j}, t, z_i \right)$ at $-t$, let $w_t^i\left(\sigma, h_i^t\right)$ be candidate i's continuation payoff at time $-t$ given σ and h_i^t. In addition, let $\theta(h_i^t) = (X_i^{k_i}, X_j^{l_j})$ be the profile of policy sets that are chosen most recently, where we always write S's current policy set first in this proof. Since the most recently chosen policy sets are observable, we have $\theta(h_S^t) = \theta(h_W^t)$. For simple notation, we write $\theta(h_S^t) = \theta(h_W^t) = \theta\left(h^t\right)$. By Theorem 3, there exists $v_{i,t}\left(\theta\left(h^t\right)\right)$ such that $w_t^i\left(\sigma, h_i^t\right) = v_{i,t}\left(\theta\left(h^t\right)\right)$ in any PBE σ.

From Lemma 1, the following statements are true:

- If $\theta(h^t) = (\{x\}, \{0,1\})$ with $x \in \{0,1\}$ and if W can move, then W is indifferent between entering at $x' \in \{0,1\}$ with $x' \neq x$ and announcing $\{0,1\}$. S wins if and only if the median voter is located at x.

- If $\theta(h^t) = (\{0,1\}, \{x\})$ with $x \in \{0,1\}$ and if S can move, then S enters at x and wins.

Hence, we have

$$
\begin{aligned}
v_{S,t}\left(\theta\left(h^t\right)\right) &= 1 - (1-p)\, e^{-\lambda t} \text{ if } \theta(h^t) = (\{0,1\}, \{1\}) \\
v_{W,t}\left(\theta\left(h^t\right)\right) &= (1-p)\, e^{-\lambda t} \text{ if } \theta(h^t) = (\{0,1\}, \{1\})\,.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
v_{S,t}\left(\theta\left(h^t\right)\right) &= 1 - p \text{ if } \theta\left(h^t\right) = (\{1\}, \{0,1\}) \\
v_{W,t}\left(\theta\left(h^t\right)\right) &= p \text{ if } \theta\left(h^t\right) = (\{1\}, \{0,1\})\,.
\end{aligned}
$$

When $-t$ is sufficiently close to the deadline 0, then at any h^t with $\theta(h^t) = (\{0,1\}, \{0,1\})$, the following are true:

- If W can move, then W enters at 1. Note that, since $-t$ is sufficiently close to zero, with a probability close to 1, there is no more opportunity to announce a policy. Hence, $\{1\}$ gives

W the payoff close to $1-p$, $\{0\}$ gives W the payoff close to p, and $\{0,1\}$ gives W the payoff close to zero. S wins if and only if the median voter is located at 0.

- If S can move, then S does not enter. Note that, since $-t$ is sufficiently close to zero, with a probability close to 1, there is no more opportunity to announce a policy. Hence, $\{1\}$ gives S the payoff close to $1-p$, $\{0\}$ gives S the payoff close to p, and $\{0,1\}$ gives S the payoff close to 1.

Hence, we are in Case 3 for Theorem 1 (with candidate A being S), and using the notation of Section 4.1, we have

$$\begin{aligned} \bar{v}_{S,t}^{S}(\text{not}) &= 1-(1-p)\,\lambda t e^{-\lambda t}; \\ v_{S,t}(\text{enter}) &= 1-p; \end{aligned}$$

and

$$\begin{aligned} \bar{v}_{W,t}^{S}(\text{not}) &= (1-p)\,\lambda t e^{-\lambda t}; \\ v_{W,t}(\text{enter}) &= (1-p)\,\lambda t e^{-\lambda t}. \end{aligned}$$

Hence, $\hat{t}_S$ and $\hat{t}_W$, whose notation is introduced in Section 4.1, are characterized, respectively, by

$$1-(1-p)\,\lambda \hat{t}_S e^{-\lambda \hat{t}_S} = 1-p \Leftrightarrow 1 > \frac{p}{1-p} = \lambda \hat{t}_S e^{-\lambda \hat{t}_S} \tag{23}$$

and

$$(1-p)\,\lambda t e^{-\lambda \hat{t}_W} = (1-p)\,e^{-\lambda \hat{t}_W} \Leftrightarrow \hat{t}_W = \frac{1}{\lambda}. \tag{24}$$

To fully characterize the candidates' strategies, we examine the following three possible cases.

Case (1): $\frac{p}{1-p} > e^{-1}$. In this case, we have $\frac{1}{\lambda} = \hat{t}_W < \hat{t}_S$. Hence, Proposition 12 ensures that both S and W announce $\{0,1\}$ for all time in $(-t,-t^*)$ with $t^* := \hat{t}_W$. By Lemma 5, we have shown the claims.

Case (2): $\frac{p}{1-p} < e^{-1}$**.** In this case, we have $\frac{1}{\lambda} = \hat{t}_W > \hat{t}_S$. Moreover, by the implicit function theorem, we have

$$\frac{d\hat{t}_S}{dp} = -\frac{\frac{d\lambda \hat{t}_S e^{-\lambda \hat{t}_S}}{d\hat{t}_S}}{\frac{d\left(\frac{p}{1-p}\right)}{dp}} = -\left(1-p\right)^2 \lambda e^{-\lambda \hat{t}_S}\left(1 - \lambda \hat{t}_S\right) < 0. \tag{25}$$

Recall that the definition of $-\hat{t}_S$ implies that, at time $-\hat{t}_S$, S becomes indifferent between entering at 1 and announcing $\{0,1\}$ given the continuation play in which S does not enter and W enters at times in $(-\hat{t}_S, 0]$. The definition implies that this indifference holds in any PBE. By part 1 of Proposition 12, there exists $\bar{\varepsilon} > 0$ such that both S and W strictly prefer entering at 1 for each $-t \in [-\hat{t}_S - \bar{\varepsilon}, \hat{t}_S)$. Therefore, we are in Case 2 for Theorem 1 with $t_0 = -\hat{t}_S - \bar{\varepsilon}$.

We will show that candidate S always enters at 1 for $-t < -\hat{t}_S$. Suppose S always enters at 1 for all time in $(-t, -\hat{t}_S)$. If S announces $\{0,1\}$ at $-t$, there are following three subcases to consider.

1. If W can move next by $-\hat{t}_S$, then one strategy that W can take is to announce $\{0,1\}$. The following two cases are possible: If S enters at $\{1\}$ by $-\hat{t}_S$, W gets p. If S does not enter by $-\hat{t}_S$, by the definition of $-\hat{t}_S$ (that is, S is indifferent between $\{1\}$ and $\{0,1\}$ at $-\hat{t}_S$), S gets $1-p$ and W gets p. In both cases, W gets at least p. Furthermore, if W can get the first revision opportunity sufficiently close to $-\hat{t}_S$, W gets strictly more than p since W strictly prefers entering at 1 to announcing $\{0,1\}$. Overall, W gets strictly more than p, which means S gets strictly less than $1-p$.

2. If S can move next by $-\hat{t}_S$, S enters and gets $1-p$.

3. If no candidate can move by $-\hat{t}_S$, then by definition, S gets $1-p$.

Therefore, the payoff from announcing $\{0,1\}$ is strictly less than $1-p$. This implies that it is uniquely optimal for S to enter at 1, as desired. Hence, $t_S = \infty$ in Proposition 10.

We will now examine candidate W's incentives. Since first-mover disadvantage for W holds, there exists

$$t_W^* > \hat{t}_S \tag{26}$$

such that it is uniquely optimal for W not to enter at $-t < -t_W^*$ and uniquely optimal for W to enter at $-t \in (-t_W^*, 0]$.[6]

[6]This notation of t_W^* is introduced in Section 4.1.

Moreover, $t^*_W \equiv \inf\{t > t_0 : \bar{v}_{W,t}(\text{not}) \geq v_{W,t}(\text{enter})\}$ implies

$$(1-p)\, e^{-\lambda t} = \int_0^{t-\hat{t}_S} e^{-2\lambda\tau} \lambda (1-p)\, e^{-\lambda(t-\tau)} d\tau + p \left(1 - \int_0^{t-\hat{t}_S} \lambda e^{-2\lambda\tau} d\tau\right)$$

$\Leftrightarrow$

$$e^{-\lambda\left(2t^*_W - \hat{t}_S\right)} = \frac{p}{1-p}\frac{1}{2}\left(1 + e^{-2\lambda\left(t^*_W - \hat{t}_S\right)}\right).$$

Since $\frac{p}{1-p} = \lambda \hat{t}_S e^{-\lambda \hat{t}_S}$ by the definition of $\hat{t}_S$, this inequality is equivalent to

$$e^{-\lambda\left(2t^*_W - \hat{t}_S\right)} = \lambda \hat{t}_S e^{-\lambda \hat{t}_S} \frac{1}{2}\left(1 + e^{-2\lambda\left(t^*_W - \hat{t}_S\right)}\right) \Leftrightarrow e^{-2\lambda t^*_W} = \frac{\frac{1}{2}\lambda \hat{t}_S}{1 - \frac{1}{2}\lambda \hat{t}_S} e^{-2\lambda \hat{t}_S}.$$

Taking the log of both sides and rearranging, we obtain

$$t^*_W = \hat{t}_S - \frac{1}{2\lambda} \log\left(\frac{\frac{1}{2}\lambda \hat{t}_S}{1 - \frac{1}{2}\lambda \hat{t}_S}\right).$$

Hence, we have

$$\frac{dt_W}{dp} = \frac{dt^*_W}{d\hat{t}_S}\frac{d\hat{t}_S}{dp} = \left(1 - \frac{1}{\lambda \hat{t}_S \left(2 - \lambda \hat{t}_S\right)}\right)\frac{d\hat{t}_S}{dp}.$$

Recalling that $\lambda \hat{t}_S \in (0,1)$, we have

$$\sqrt{\lambda \hat{t}_S \left(2 - \lambda \hat{t}_S\right)} < \frac{1}{2}\left(\lambda \hat{t}_S + \left(2 - \lambda \hat{t}_S\right)\right) = 1,$$

and so

$$\frac{1}{\lambda \hat{t}_S \left(2 - \lambda \hat{t}_S\right)} > 1.$$

Therefore, together with (25), we have

$$\text{sign}\frac{dt_W}{dp} = \text{sign}\left(1 - \frac{1}{\lambda \hat{t}_S \left(2 - \lambda \hat{t}_S\right)}\right)\text{sign}\frac{d\hat{t}_S}{dp} = 1. \tag{27}$$

The inequalities (25), (26), and (27) prove part 2(c) of Proposition 2.

Case (3): $\frac{p}{1-p} = e^{-1}$. At time $-t^* = \frac{1}{\lambda}$, for each h^{t^*} with $\theta(h^{t^*}) = (\{0,1\}, \{0,1\})$, S is indifferent between "announcing $\{1\}$ and thereby ensuring $1-p$," and "announcing $\{0,1\}$." At the same time,

W is indifferent between announcing $\{1\}$ and $\{0, 1\}$.

For $-t < -t^*$, on the one hand, when W can move, his payoff from not entering is at least p since he gets p if S enters at 1 by $-t^*$. If S does not enter by $-t^*$, by the definition of $-t^*$, S gets $1-p$ and W gets p. On the other hand, entering at 1 gives W a payoff of $1-p$ times the probability of S not having any future revision opportunity, which is equal to $(1-p)e^{-\lambda t} < (1-p)e^{-\lambda t^*} = p$. Therefore, W strictly prefers not entering.

Given this, S is always indifferent between "announcing $\{1\}$ and thereby ensuring $1-p$," and "announcing $\{0, 1\}$."

I.2 Empirical Implications

In this section, we derive empirical implications of the results from the model of valence election campaign. We see these implications as only suggestive, but as will be seen in Appendix I.4, it is possible to enrich the model by incorporating various features (such as heterogenous arrival rates and general utilities from the outcomes). This suggests that, if one wants to conduct empirical research, then it will be possible to extend the model to incorporate more characteristics and to derive testable implications from such a general model, as we do here for the base model.

First, we show that ambiguity is likely when the probability distribution of the median voter's position is close to uniform, that is, when p is close to $\frac{1}{2}$. Specifically, fix a horizon length $T \in (\frac{1}{\lambda}, \infty)$. Let p^W be the p such that $t_W = T$.[7] By definition, $p^W < \frac{1}{1+e}$. Proposition 2 implies the following:

1. For $p \in (0, \frac{1}{2}) \setminus \{\frac{1}{1+e}\}$, the probabilities of W and S announcing the ambiguous policy are both nondecreasing in p.

2. For $p \in (0, p^W)$, the probability of W announcing the ambiguous policy is constant in p, and that of S announcing the ambiguous policy is strictly increasing in p.

3. For $p \in (p^W, \frac{1}{1+e})$, the probabilities of W and S announcing the ambiguous policy are both strictly increasing in p.

4. For $p \in (\frac{1}{1+e}, \frac{1}{2})$, the probabilities of W and S announcing the ambiguous policy are constant in p.

[7] Such p^W exists and is unique due to Proposition 2 2(c) and $t^* = \frac{1}{\lambda}$.

Hence, roughly, as the position of the median voter becomes more unpredictable, the probability of ambiguous policy announcement at the election date increases. This is consistent with Campbell (1983) who suggests that opinion dispersion has a strong positive effect on the ambiguity in candidates' language.[8]

Next, suppose that there are two candidates A and B, and outside researchers know $p > \frac{1}{1+e}$ but do not know which candidate is strong and which candidate is weak. They have a prior that assigns a positive probability to both candidate A's being strong and candidate B's being strong. If the researchers can observe the campaign phase, the first entrant can be inferred to be weak (and if there is no entrance, then the posterior about valence is the same as the prior). In contrast, if they cannot observe the campaign phase but only the final policy choices by the candidates, then if only one candidate enters, such a candidate can be inferred to be weak. Otherwise, the posterior about valence is the same as the prior.

I.3 Welfare Comparison with the Static Model

As mentioned in Remark 3, conducting a welfare analysis necessitates us to impose some specific assumption about the voter distribution. Here, we assume that there is a single voter. It is then necessary that this voter's ideal policy is 0 with probability p and 1 with probability $1-p$. we focus on the case in which $p > \frac{1}{1+e}$. Normalize the voter's payoff so that $u(0) - u(1) = 1$. With this normalization, if a candidate $i \in \{S, W\}$ with the ideal policy $y \in \{0, 1\}$ wins and implements a policy $x \in \{0, 1\}$, the voter's payoff can be written as $\mathbb{I}_{x=y} + \delta \cdot \mathbb{I}_{i=S}$.

Aragonès and Palfrey (2002) consider the one-shot game where each of candidates S and W simultaneously chooses a policy. Here we consider a version of their model adopted to our environment in which the policy space is $\{0, 1\}$. That is, each candidate chooses either 0 or 1, and there is no choice of $\{0, 1\}$.

Since their expected payoffs are represented by the following payoff matrix, the unique mixed-strategy Nash equilibrium is that S takes 0 and 1 with probabilities p and $1-p$, respectively; and W takes 0 and 1 with probabilities $1-p$ and p, respectively:

[8] As discussed in footnote 33 of the main text, we have in mind a situation where n voters are independently distributed over $\{0, 1\}$ where the probability on the policy 0 is $q < \frac{1}{2}$. A higher q suggests more option dispersion (a higher standard deviation of the preferred policies among the voters. Campbell (1983) also considers standard deviation), and corresponds to a higher p.

S's payoff \ W's payoff	0	1
0	$1, 0$	$p, 1-p$
1	$1-p, p$	$1, 0$

Given this equilibrium strategy, the expected welfare of the voter is

$$\begin{aligned} & \underbrace{p}_{y=0}\left(\underbrace{p}_{x_S=0}(1+\delta)+\underbrace{(1-p)^2}_{x_S=1 \text{ and } x_W=0}+\underbrace{p(1-p)}_{x_S=1 \text{ and } x_W=1}\delta\right) \\ & +\underbrace{(1-p)}_{y=1}\left(\underbrace{(1-p)}_{x_S=1}(1+\delta)+\underbrace{p^2}_{x_S=0 \text{ and } x_W=1}+\underbrace{(1-p)\,p}_{x_S=0 \text{ and } x_W=0}\delta\right) \\ = \quad & (1+\delta)\left(1-p+p^2\right), \end{aligned}$$

where x_i for $i = S, W$ denotes the realized policy choice by candidate i. This expected payoff converges to $W(p) := 1 - p + p^2$ in the limit as δ goes to 0.

Next, consider our model of valence election campaign. Since $p > \frac{1}{1+e}$, given Proposition 2, in any PBE, W does not enter for each $-t < -t_W = -\frac{1}{\lambda}$, and enters at $x = 1$ for each $-t > -\frac{1}{\lambda}$, while S never enters unless W enters. Hence, (i) with probability $e^{-\lambda \cdot \frac{1}{\lambda}} = e^{-1}$, no candidate enters; (ii) with probability $\int_0^{\frac{1}{\lambda}} \lambda e^{-\lambda s} e^{-\lambda\left(\frac{1}{\lambda}-s\right)} ds = e^{-1}$, W enters at policy 1 but S does not enter; and (iii) with probability $1 - 2e^{-1}$, both candidate enter at policy 1. In the respective cases, (i) if no candidate enters, then the voter's expected payoff is $\frac{1}{2} + \delta$ (recall that we assume that a candidate without specifying her policy takes each policy with probability $\frac{1}{2}$); (ii) if W enters at 1 while S does not enter, then the expected payoff is $1 - p + p(1 + \delta)$; and (iii) if both candidates enter at 1, then the expected payoff is $1 - p + \delta$. In total, the expected payoff is

$$\begin{aligned} & e^{-1}\left(\frac{1}{2}+\delta\right)+e^{-1}(1-p+p(1+\delta))+\left(1-2e^{-1}\right)(1-p+\delta) \\ & =1-\frac{1}{2}e^{-1}-p+2pe^{-1}+\delta-(1-p)\,\delta e^{-1}. \end{aligned}$$

This expected payoff converges to $V(p) := 1 - \frac{1}{2}e^{-1} - \left(1 - 2e^{-1}\right)p$ in the limit as δ goes to 0.

Finally, we compare the two expected payoffs.

$$W(p) > V(p) \iff 1 - p + p^2 > 1 - \frac{1}{2}e^{-1} - \left(1 - 2e^{-1}\right) p$$
$$\iff p^2 + 2pe^{-1} + \frac{1}{2}e^{-1} > 0,$$

which holds for any p. Hence, in particular, we obtain $W(p) > V(p)$ for $p > \frac{1}{1+e}$.

Hence, the voter's expected payoff in our model is smaller than under a unique mixed Nash equilibrium model in which each candidate chooses between 0 and 1 as in Aragonès and Palfrey (2002) when $p > \frac{1}{1+e}$, $\delta > 0$ is sufficiently small, and T is sufficiently large.

I.4 A Generalized Model with Valence Candidates

I.4.1 Heterogeneous Arrival Rates

This section discusses the effect of heterogeneous arrival rates. Let the arrival rate for candidate i be $\lambda_i > 0$, and allow for the possibility that $\lambda_S \neq \lambda_W$. We define $r = \frac{\lambda_S}{\lambda_W}$ as the relative frequency of the opportunities to enter between the candidates.

First, it is straightforward to show that the basic structure of the equilibrium does not change even if $\lambda_S \neq \lambda_W$: The equilibrium behaviors after some candidate has already entered are the same as before. When both candidates are announcing the ambiguous policy, there exist p^* and t^* such that if $p > p^*$, then W enters if $-t < -t^*$, he does not if $-t > -t^*$, and S never enters in any PBE. If $p < p^*$, then W enters after some cutoff and S enters as soon as possible until another cutoff. The former cutoff for W to start entering precedes in time the latter for S to stop entering.

When $r \neq 1$, the cutoff p^* can be calculated as $p^* = r^{\frac{r}{1-r}}/(1 + r^{\frac{r}{1-r}})$, and the expected payoff profile for S and W when $p > p^*$ is $\left(1 - r^{\frac{r}{1-r}}, r^{\frac{r}{1-r}}\right)$. Note that these values converge to the ones in the base model as $r \to 1$.

Since $r^{\frac{r}{1-r}}$ is decreasing in $r = \frac{\lambda_S}{\lambda_W}$, it follows that p^* is decreasing in r and S's payoff is increasing in r. Thus, having a relatively higher arrival rate makes the candidate better off. This is intuitive. With W's strategy being fixed, if S has a higher arrival rate, she has a greater chance to copy W's position. In contrast, with S's strategy being fixed, if W has a higher arrival rate, then he can wait longer at the policy profile $\{0, 1\}$ to reduce the probability of being copied afterward. Of course W's strategy is not constant in the former case and S's is not in the latter, so determination of the equilibrium strategy profile is more complicated, but these are the main driving forces of the comparative statics.

Note that Calcagno et al. (2014) show that having a higher arrival rate makes the player worse off in their analysis of battle-of-the-sexes games. This follows because having a higher arrival rate decreases his/her commitment power. The difference from our result is due to the nature of the stage game being analyzed. In a battle of the sexes, player i's ability to commit to an action a_i can help induce his or her opponent to take a_j such that (a_i, a_j) constitutes player i's favorite Nash equilibrium. In contrast, in the valence election campaign, the game is a constant-sum game, so being unable to change an action over a longer time means that the player can react to the opponent less quickly and suffers a low payoff with a larger probability.

I.4.2 Model with General Payoff Functions

Model

The simple model of "valence election campaign" presented in Section 3.1.1 was intended to provide a basic intuition for the dynamic incentive problems faced by candidates. This section extends this base model to more general cases. The policy space is X, and available policy sets are $\mathcal{X} = \{X\} \cup \left(\bigcup_{x \in X} \{\{x\}\}\right)$ for each i. The two candidates, S and W, correspond to the strong and the weak candidates, respectively. The candidates are purely office-motivated, so $v_S(X_S, X_W) + v_W(X_W, X_S) = 1$ for any $(X_S, X_W) \in \mathcal{X}_S \times \mathcal{X}_W$. The strong candidate's payoff when only the weak candidate enters is $\min_{x \in X} v_S(X, \{x\}) =: \alpha$.[9] We assume that the policy to which the strong candidate enters does not depend on the time of the entry. Formally, we assume: $\arg\max_{x \in X} v_S(\{x\}, X) \cap \arg\max_{x \in X} \min_{y \in X} v_S(\{x\}, \{y\}) \neq \emptyset$, and let an (arbitrary) element of this intersection be x^*. With this assumption, we let the weak candidate's payoff when only the strong candidate enters be $v_W(X, \{x^*\}) =: \beta$. Also, the weak candidate's payoff when the strong candidate enters and then the weak candidate enters is $\max_{y \in \mathcal{X}} v_W(\{y\}, \{x^*\}) =: \gamma$. Finally, the strong candidate wins for sure if the two candidates announce the same policy set, so

[9] As will be seen, we assume that W loses for sure if S enters after W enters. Hence, when W chooses his policy to enter, he maximizes his payoff from his entry, that is, he minimizes S's payoff, conditional on the event that S will not enter afterward.

$v_S(X_S, X_W) = 1$ if $X_S = X_W$. To summarize, the payoffs are represented as follows:

$$(S\text{'s payoff}, W\text{'s payoff}) = \begin{cases} (\alpha, 1-\alpha) & \text{if only } W \text{ enters;} \\ (1-\beta, \beta) & \text{if only } S \text{ enters;} \\ (1-\gamma, \gamma) & \text{if } S \text{ enters and then } W \text{ enters;} \\ (1, 0) & \text{if } W \text{ enters and then } S \text{ enters, or if neither enters.} \end{cases}$$

We assume $\alpha \in [0, 1)$ and $\beta, \gamma \in [0, 1]$.[10,11] We let S's arrival rate and W's arrival rate be $\lambda_S > 0$ and $\lambda_W > 0$, respectively. Call this model the *generalized valence election campaign*. It is characterized by a tuple $(\alpha, \beta, \gamma, \lambda_S, \lambda_W)$.

Note that the crucial assumptions that we make here are (i) the payoff from the game is determined solely by the policy sets at the election, (ii) S wins for sure if S and W choose the same policy, and (iii) the position in the policy space that S enters does not depend on the timing of entry.[12] These are the only restrictions that we impose. These assumptions are satisfied in our base model, with $\lambda_S = \lambda_W = \lambda$ and $\alpha = \beta = \gamma = p$.

Moreover, the specification fits other cases as well. For example, this general model can be applied to the case of a continuous policy space, the model that the literature on elections often considers. Specifically, $\mathcal{X}_i = \{x\}_{x \in [0,1]} \cup [0, 1]$ for each $i = A, B$, i.e., we allow the candidates to announce either a specific policy $x \in [0, 1]$ or an ambiguous policy $[0, 1]$. Analogous to the base model, the policy set at time $-T$ is $[0, 1]$. If candidate i wins the election and implements policy $x \in [0, 1]$, then the voter's utility with position $y \in [0, 1]$ is defined as $u(x, y) + \delta \cdot \mathbb{I}_{i=S}$, where the utility function u is strictly concave with respect to x (i.e., the voters are risk-averse). If a candidate with the ambiguous policy $[0, 1]$ wins, then the voter believes that the candidate will implement the policies in $[0, 1]$ according to the uniform distribution. Hence, the expected payoff

[10] We assume $\alpha \neq 1$ because otherwise W obtains a payoff of 0 in any equilibrium.

[11] It is not crucial that S receives the payoff exactly equal to 1 when W enters and then S enters. Specifically, with all other parameters fixed, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$, all the cutoffs characterizing the equilibrium behavior change continuously if the payoff profile from W entering and then S entering is $(1 - \varepsilon, \varepsilon)$ and if all those cutoffs are distinct from each other at such a payoff profile.

[12] Note that (i) and (ii) imply that the position that W enters is also independent of the timing of his entry. This is because since W loses if S enters afterward by (ii), when W chooses his policy to enter, he can condition on the event that S will not enter afterward. Under such an event, by (i), W's payoff is determined solely by his policy announcement. Hence, the position that W enters is independent of his entry time.

is $\int_0^1 u(x,y)dx + \delta \cdot \mathbb{I}_{i=S}$.[13] The probability distribution of the median voter is uniform over the policy space $[0,1]$. Again, we assume that the valence term is $\delta > 0$, but is sufficiently small so that W at policy $\frac{1}{2}$ beats S with the ambiguous policy.[14,15]

In this model with the continuous policy space, if S enters before W does, she enters at policy $\frac{1}{2}$ regardless of the timing of her entry. This is because (i) this policy uniquely maximizes her payoff if W enters afterward, and (ii) it guarantees a payoff of 1 if W does not enter. If W enters before S does, he enters at a policy around $\frac{1}{2}$ regardless of the timing of her entry. This is because (i) if S enters afterward then S copies W's policy so W loses for sure, and (ii) if S does not enter afterward, policies around $\frac{1}{2}$ guarantee a payoff of 1 since voters are risk-averse.

Since the payoffs are constant-sum, Theorem 3 implies that the model with private monitoring is outcome-equivalent to the one with public monitoring. For simple notation, for the rest of this section, we assume public monitoring. That is, we assume that $h^t = \left(\left(t_S^k, x_S^k\right)_{k=1}^{k_S}, \left(t_W^k, x_W^k\right)_{k=1}^{k_W}, t \right)$ is public and analyze SPE.

Analysis and Equilibrium Dynamics

To state our result, we define three pieces of notation. First, write $Q_t = (E, N)$ if in all SPE, (i) S enters if she receives an opportunity at time $-t$ when W has not entered, and (ii) W does not enter if he receives an opportunity at $-t$ when S has not entered. That is, the first element Q_t denotes S's action at time $-t$ and the second element denotes W's action at the same time. The symbol E stands for "entering" and the symbol N stands for "not entering." Define $Q_t = (E, E)$, $Q_t = (N, E)$, and $Q_t = (N, N)$ analogously.

Second, we define functions

$$f_S(t) \quad : \quad = \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{\gamma - \beta, 0\}}{1-\alpha} & \text{if } r \neq 1; \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{\gamma - \beta, 0\}}{1-\alpha} & \text{if } r = 1, \end{cases}$$

$$f_W(t) \quad : \quad = \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1; \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1, \end{cases}$$

[13] The integral is well defined because u is concave and thus it is measurable.

[14] Specifically, $\int_0^1 u(x,y)dx + \delta < u(\frac{1}{2}, y)$ for all y. Note that such a $\delta > 0$ exists by the strict concavity of u.

[15] As we mentioned in the literature review, if we assume convexity, ambiguity does not need valence: If candidates are symmetric, it is optimal for a candidate to announce $[0,1]$ when the opponent is announcing $\{\frac{1}{2}\}$.

where $r = \frac{\lambda_S}{\lambda_W}$.[16]

Finally, let t_S be the smallest positive solution for $f_S(t) = 0$ (if there is no solution, then define $t_S = \infty$); and let t_W be the smallest positive solution for $f_W(t) = 0$ (since $f_W(t)$ is negative for sufficiently small $t > 0$, positive for sufficiently large t, and continuous, there always exists a positive solution).[17]

The equilibrium behavior is characterized as follows:

Proposition 14 *For the generalized valence election campaign with $(\alpha, \beta, \gamma, \lambda_S, \lambda_W)$, in any SPE, S enters at the same position as W once W has entered but S has not. In addition, the following hold.*

1. *If $\beta \geq \gamma$, then the following are true.*
   (a) *If $-t_S < -t_W$, then $Q_t = (N, E)$ for all $-t \in (-t_W, 0]$; and $Q_t = (N, N)$ for all $-t \in (\infty, -t_W)$.*
   (b) *If $-t_S > -t_W$, then there exists $t_W^* \in (t_S, \infty)$ such that $Q_t = (N, E)$ for all $-t \in (-t_S, 0]$; $Q_t = (E, E)$ for all $-t \in (-t_W^*, -t_S)$; and $Q_t = (E, N)$ for all $-t \in (-\infty, -t_W^*)$.*[18]

2. *If $\beta < \gamma$, then the following are true.*
   (a) *If $-t_S < -t_W$, then $Q_t = (N, E)$ for all $-t \in (-t_W, 0]$; and $Q_t = (N, N)$ for all $-t \in (\infty, -t_W)$.*
   (b) *If $-t_S > -t_W$, then there exists $\varepsilon > 0$ such that $Q_t = (N, E)$ for all $-t \in (-t_S, 0]$; $Q_t = (E, E)$ for $-t \in (-t_S - \varepsilon, -t_S)$. The equilibrium behavior for $-t < -t_S$ depends on the details of the parameters, but the following properties hold:*
      i. *There exists $t_W^{**} \in (t_S, \infty)$ such that W does not enter for all $-t \in (-\infty, -t_W^{**})$; and*
      ii. *There exists $\bar{r} \leq 1$ such that $r \geq \bar{r}$ if and only if there exists $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S)$.*

3. *All the time-cutoffs described above can be taken independent of T.*

[16]One can show that $f_S(t)$ and $f_W(t)$ are continuous in r at $r = 1$.

[17]The smallest positive solutions always exist because f_S and f_W are both continuous. We note that the notations t_S and t_W defined here are different from the ones we introduced in Section 4.1.

[18]t_W^* in the statement is the same as t_W^* in Section 4.1.

This means that, for a sufficiently long election campaign phase, W uses ambiguous language (and for many cases S uses such language as well) for a long time during the early stages of the election campaign, but the candidates' incentive to do so changes as the election date approaches. This basic pattern is common across a wide range of parameter specifications, although the exact way the incentives change varies across different specifications. Notice that in the base model, the parameters satisfy $\beta = \gamma$. In this case, if p is sufficiently small, then S enters as soon as possible. Thus, Proposition 14 claims that, if S expects even the slightest cost of W entering after her own entry (i.e., $\beta < \gamma$), then she will not enter when the election date is far away.[19]

Recall that the model includes the case of a continuous policy space with a concave payoff function. Thus, the proposition implies that the essence of our result is orthogonal to the convexity of payoff functions. This is in contrast to the models of Shepsle (1972) and Aragonès and Postlewaite (2002) in which the convexity of payoff functions is essential to the ambiguous policy announcement.

We now offer comparative statics of the cutoff times with respect to the parameter values:

Proposition 15 *In the generalized valence election campaign with* $(\alpha, \beta, \gamma, \lambda_S, \lambda_W)$*, the following comparative statics hold:*

1. *For each* (α, β, γ)*, there exists* $r^* \in (0, \infty)$ *such that* $-t_S < -t_W$ *if and only if* $r^* < r$.
2. *For each* $(\beta, \gamma, \lambda_S, \lambda_W)$*, there exists* $\alpha^* \in [0, 1)$ *such that* $-t_S < -t_W$ *if and only if* $\alpha^* < \alpha$.
3. *For each* $(\alpha, \gamma, \lambda_S, \lambda_W)$*, there exists* $\beta^* \in [0, 1)$ *such that* $-t_S < -t_W$ *if and only if* $\beta^* < \beta$.
4. *For each* $(\alpha, \beta, \lambda_S, \lambda_W)$*, there exists* $\gamma^* \in [0, 1]$ *such that* $-t_S < -t_W$ *if and only if* $\gamma^* < \gamma$.
5. *For each* $(\alpha, \beta, \lambda_S, \lambda_W)$*, there exists* $\bar{\gamma} \in [0, 1)$ *such that, for each* $\bar{\gamma} < \gamma$*, there exists* $-\bar{t}$ *such that* S *does not enter at all* $-t < -\bar{t}$.

Part 1 of this proposition implies that, for sufficiently large r, Case 1(a) or 2(a) in Proposition 14 applies. Intuitively, since S can move quickly compared to W, W enters only if the deadline is very close ($-t_W$ is close to 0).

Parts 2 and 3 imply that for sufficiently large α or β, Case 1(a) or 2(a) in Proposition 14 applies. To see the intuition, notice that high α implies that S gets a high payoff when only W enters, and

[19] Note that $\beta < \gamma$ implies that S's payoff when she is the only one who enters, $1 - \beta$, is strictly greater than her payoff when W enters afterward, which is $1 - \gamma$.

high β implies that S gets a low payoff when only S enters. Hence, in these situations, S has only a small incentive to enter.

If $\beta \geq \gamma$, since W never enters after S enters, the value of γ does not affect the cutoff times. On the other hand, if $\beta < \gamma$, Part 4 implies that for sufficiently large γ, Case 1(a) or 2(a) in Proposition 14 applies. Intuitively, high γ implies that S gets a small payoff when W enters after S's entry. In such a situation, S has only a small incentive to enter.

Part 5 implies that, if γ is sufficiently large, then S does not enter if the election is sufficiently far away. To see this, consider the extreme case with $\gamma = 1$. In this case, S's payoff is zero if S enters first and then W enters afterward. Hence, if S enters when the election is far away, then with a high probability W will enter and S's payoff is close to zero. Therefore, in equilibrium, S does not enter when the election is far away.

Remark 7 (Sufficient condition for $-t_S < -t_W$) The numbers t_S and t_W that appear in Proposition 14 are only implicitly defined as the smallest solutions of $f_S(t) = 0$ and $f_W(t) = 0$, respectively. There is a sufficient condition to ensure that $-t_S < -t_W$. The sufficient condition is that $\phi < 0$, where[20]

$$\phi := \begin{cases} -\frac{\gamma}{1-\alpha} & \text{if } \gamma > \beta \text{ and } r < 1 - \frac{1-\alpha}{\gamma-\beta} \\ e^{\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}-1} - \frac{\max\{\beta,\gamma\}}{1-\alpha} & \text{if } r = 1 \\ \left(\frac{1}{r} - \frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}\right)^{\frac{r}{r-1}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} & \text{otherwise} \end{cases}.$$

In fact, we use the condition in Remark 7 to show that r^* is finite and α^* and β^* are strictly less than 1 in Proposition 15. Moreover, Part 5 of Proposition 15 ensures the existence of $\bar{\gamma}$ such that $\gamma > \bar{\gamma}$ implies S does not enter if the deadline is far. In total, if at least one of these parameters is sufficiently high, then there is a long period of no entry by any candidate.

Recall that in the base model, $r = 1$ and $\alpha = \beta = \gamma = p$. Proposition 2 implies that for sufficiently large p, there is a cutoff time $-t^*$ such that no candidate enters for all $-t < -t^*$. The specification of the base model implies that the three parameters α, β, and γ move simultaneously as p varies, so it is not possible in the base model to examine the effects of individual parameters.

[20] One can show that ϕ is continuous in r at $r = 1$.

Proposition 15 ensures that if at least one of these parameters is sufficiently high, then no candidate enters when the deadline is far, as in the case of high p's. In addition, in Section I.4.1, we define p^* to be a cutoff of p such that $p > p^*$ implies the existence of t^* with which, in any equilibrium, (i) no candidate enters for all $-t < -t^*$ and (ii) W enters and S does not for all $-t > -t^*$. Part 1 of Proposition 15 generalizes the claim that p^* is decreasing in r (and it converges to 0 as $r \to \infty$). Overall, the insight from the base model carries over to the general setting.

J Proofs for Section 3.2

J.1 Proof of Proposition 3

Part 1: Policy $x^*(X, \mu)$ is a Condorcet winner. To see why, we have $v_i(\{x^*(X, \mu)\}, X) = 1$ by assumption. Moreover, given the definition of $\mathcal{M}$, Theorem 7.2 of Roemer (2001) implies that $x^*(X, \mu)$ is a best response to $x^*(X, \mu)$, and for each $x_i' \neq x^*(X, \mu)$, $v_i(x^*(X, \mu), x_j) > v_i(x_i', x_j')$ for each $x_j \in BR_j(x_i)$ and each $x_j' \in BR_j(x_i')$.

Since the game is symmetric and constant-sum, Theorem 2 implies that, in any PBE, each candidate enters at $x_i \in X_i^*$ as soon as possible.

Part 2: There exists a function $y : X \to X$ such that $P_i(x, y(x)) < \frac{1}{2}$ for each $x \in X$ for each $i = A, B$. If candidate i has not entered and j has already entered at x, then it is optimal for i to enter at $y(x)$, which gives the highest feasible payoff. If a candidate enters while the other candidate has not yet entered, then she is indifferent among any policy x with $v_i(\{x\}, X) = 1$ (which exists by assumption) since once the other candidate enters later, she will lose for sure. Therefore, Assumptions 1-3 and first-mover disadvantage for i in Section 4.1 are satisfied for each $i \in \{A, B\}$. Moreover, each candidate has a strict incentive to enter at $t = 0$. Hence, we have Case 2 with $t_0 = 0$ for Theorem 1. Hence, Theorem 1 implies that, for each i, there exists t_i such that candidate i enters at all times $-t \in (-t_i, 0]$ and does not enter at all times $-t \in (-\infty, -t_i)$.

In addition, t_i^* in Section 4.1 is calculated as follows: On the one hand, i's expected payoff of entering is the probability that the other candidate will not have an opportunity to enter. That is, $v_{i,t}(\text{enter}) = e^{-\lambda_j t}$. On the other hand, supposing that each player enters at every time $-\tau \in (-t, 0]$, we have

$$\bar{v}_{i,t}\,(\text{not}) = \left(\int_0^t \underbrace{e^{-(\lambda_i+\lambda_j)\tau}}_{\text{no one moves until time } -(t-\tau)} \left(\underbrace{\lambda_i d\tau}_{i \text{ moves at time } -(t-\tau)} \right) \underbrace{e^{-\lambda_j(t-\tau)}}_{j \text{ does not move in } (-(t-\tau),0]} \right)$$

$$+ \left(\int_0^t \underbrace{e^{-(\lambda_i+\lambda_j)\tau}}_{\text{no one moves until time } -(t-\tau)} \left(\underbrace{\lambda_j d\tau}_{j \text{ moves at time } -(t-\tau)} \right) \left(\underbrace{1 - e^{-\lambda_i(t-\tau)}}_{i \text{ can move in } (-(t-\tau),0]} \right) \right) + \underbrace{e^{-(\lambda_i+\lambda_j)t}}_{\text{no one moves until time } 0} \frac{1}{2}.$$

Hence, t_i^* is characterized by $f_i\left(t_i^*\right) = 0$ with

$$f_i(t) := -e^{-\lambda_i t} + \frac{\lambda_j}{\lambda_i + \lambda_j}(1 - e^{-(\lambda_i+\lambda_j)t}) + e^{-(\lambda_i+\lambda_j)t}\frac{1}{2}. \tag{28}$$

Differentiating $f_i(t)$, we get

$$f_i'(t) = \lambda_i(e^{-\lambda_i t} - e^{-(\lambda_i+\lambda_j)t}\frac{1}{2}) + \lambda_j e^{-(\lambda_i+\lambda_j)t}\frac{1}{2} > 0.$$

Since $f_i(t)$ is $-\frac{1}{2}$ at $t = 0$, continuous and strictly increasing in t, and approaches $\frac{\lambda_j}{\lambda_i+\lambda_j} > 0$ as $t \to \infty$, there exists a unique t such that $f_i(t) = 0$. The cutoff t_i^* is such t.

J.2 Proof of Proposition 4

Equation (28) implies that $e^{-\lambda_A t_B^*}$ is strictly more than $1/2$. The reason is that, letting $i = A$, the sum of the second and the third terms is a strict convex combination of $\frac{\lambda_B}{\lambda_A+\lambda_B} < \frac{1}{2}$ and $\frac{1}{2}$. Hence, $f_B\left(t_B^*\right) = 0$ implies that $e^{-\lambda_A t_B} < \frac{1}{2}$. Since this is B's continuation payoff from entering at time $-t_B^*$ and B is indifferent between entering and not entering at time $-t_B^*$, B's continuation payoff from not entering at time $-t_B^*$ is also strictly less than $1/2$. Hence, A's continuation payoff from not entering at time $-t_B^*$ is strictly greater than $1/2$. One strategy A can take is not to enter until time $-t_B^*$ and then enter for all the times in $(-t_B^*, 0]$. This gives a lower bound of A's PBE payoff that is strictly greater than $1/2$ because B does not enter for times in $(-\infty, -t_B^*)$ in any PBE. This implies that A's payoff is strictly greater than B's.

K A Proof and Additional Discussions for Section 3.3

K.1 Proof of Proposition 5

First, we compute a lower bound of the probability of candidate i winning conditional on her being able to move at time $-t$. To calculate such a bound, suppose candidate i does not enter for each time in the time interval $(-t, -\tau)$, and then enters for each time in the time interval $[-\tau, 0]$. A lower bound of the probability of winning when i uses this strategy, denoted by $\bar{p}_\tau$, is given by the following consideration: Since the second entrant can win for sure, the minimum winning probability is given by the assumption that the opponent will not enter until a candidate enters. The bound can be computed as follows:

$$\bar{p}_\tau = \int_0^\tau \underbrace{\lambda_i e^{-\lambda_i s}}_{i \text{ enters at } -(\tau-s)} \times \underbrace{e^{-\lambda_j(\tau-s)}}_{j \text{ cannot enter after } i \text{ enters}} ds = \begin{cases} \frac{\lambda_i\left[e^{-\lambda_i\tau}-e^{-\lambda_j\tau}\right]}{\lambda_j-\lambda_i} > 0 & \text{if } \lambda_i \neq \lambda_i \\ \lambda_i \tau e^{-\lambda_i \tau} & \text{if } \lambda_i = \lambda_j \end{cases}.$$

Another lower bound can be calculated by assuming that i enters at time $-t$, and it is given by $e^{-\lambda_j t}$. Hence, in total, we obtain a bound of $\max\{e^{-\lambda_j t}, \max_{\tau\in[0,t]} \bar{p}_\tau\}$. This implies that, if we take $\varepsilon < \frac{\min_{t\in[0,\infty)} \max\{e^{-\lambda_j t}, \max_{\tau\in[0,t]} \bar{p}_\tau\}}{\max_{x,y\in X}|u_i(x)-u_i(y)|}$, then at every time $-t$, there exists a strictly better strategy for candidate i than entering at a policy with which i will lose for sure.

These bounds can be used to derive an explicit expression of $\bar{\varepsilon}$:

$$\bar{\varepsilon} = \min\left\{1, \min_{i\in\{L,R\}} \frac{\min_{t\in[0,\infty)} \max\{e^{-\lambda_j t}, \max_{\tau\in[0,t]} \bar{p}_\tau\}}{\max_{x,y\in X}\left|u_i(x) - u_i(y)\right|}\right\}. \tag{29}$$

Given this definition of $\bar{\varepsilon}$, $\varepsilon < \bar{\varepsilon}$ ensures that it is a dominated strategy for candidate i to enter at a policy x such that i loses at a policy set profile $(\{x\}, X)$.

We next derive the set of policies with which candidate i can win given that candidate j has entered at x, which we denote by $X(i, x)$. If candidate j's policy is $x \in X$, candidate i can win if and only if her policy is x' (including the case where she picks X and her ideal policy is x') satisfying one of the following three conditions:

1. $x_1 \le x_1'$ and $x_2 \le x_2'$ (voters at $(1,0)$ and $(0,1)$ vote for her);

2. $x_1 \le x_1'$ and $x_1' + x_2' \le x_1 + x_2$ (voters at $(1,0)$ and $(0,0)$ vote for her); or

3. $x_2 \le x_2'$ and $x_1' + x_2' \le x_1 + x_2$ (voters at $(0,1)$ and $(0,0)$ vote for her).

Second, we derive the set of policies with which candidate L can win if candidate R does not enter. Since the voters believe that candidate R implements $\left(\frac{1}{2},\frac{1}{2}\right)$ if she does not enter, the set is the same as $X\left(L,\left(\frac{1}{2},\frac{1}{2}\right)\right)$. Similarly, candidate R can win with policies in $X\left(R,(0,0)\right)$ if candidate L does not enter.

We now consider each candidate's best response to the opponent's entry to x. First, suppose that R has entered at x. Candidate L enters at (x_1, x_2') with $x_2' \le x_1$ if $x_1 \le x_2$, and (x_1', x_2) with $x_1' \le x_2$ if $x_1 \ge x_2$. Given the tie breaking rule, we conclude that candidate L enters at $(\min\{x_1,x_2\}, \min\{x_1,x_2\})$.

Second, suppose that L has entered at x. Given this, suppose that R's entry to x' is a best response.

1. If $x_1 \le x_1'$ and $x_2 \le x_2'$, then the following hold.

   (a) If $x_1 \le \frac{1}{2}$ and $x_2 \le \frac{1}{2}$, then $x' = \left(\frac{1}{2},\frac{1}{2}\right)$. In this case, she receives $u_R(x') = \frac{1}{2}$.

   (b) Otherwise, given the tie breaking rule, x' is on the line segment connecting $(0,1)$ and $(1,0)$. In particular, $x_1' = x_1$ and $x_2' = 1 - x_1$ if $x_1 > \frac{1}{2}$; and $x_1' = 1 - x_2$ and $x_2' = x_2$ if $x_2 > \frac{1}{2}$. In this case, she receives $u_R(x') = 1 - \max\{x_1,x_2\}$.

2. If $x_1' + x_2' \le x_1 + x_2$, then $x' = \left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2}\right)$ and she receives $u_R(x') = \frac{x_1+x_2}{2}$.

Hence, for $x \in X\left(L,\left(\frac{1}{2},\frac{1}{2}\right)\right)$ (L never enters outside of $X\left(L,\left(\frac{1}{2},\frac{1}{2}\right)\right)$), R enters at $\left(\frac{1}{2},\frac{1}{2}\right)$ if $x = \left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right)$; she enters at x' with $x_1 \le x_1'$ and $x_2 \le x_2'$ if x satisfies $x_1 \le \frac{1}{2}$ and $x_2 \le \frac{1}{2}$ or x satisfies

$$\frac{x_1+x_2}{2} \le 1 - \max\{x_1,x_2\}; \tag{30}$$

and she enters at $\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2}\right)$ if x satisfies $\frac{x_1+x_2}{2} \ge 1 - \max\{x_1,x_2\}$ and $x \ne \left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right)$. Since $x_1 + x_2 = \max\{x_1,x_2\} + \min\{x_1,x_2\}$, (30) is equivalent to

$$\min\{x_1,x_2\} \le 2 - 3\max\{x_1,x_2\}.$$

Given this response of the other candidate, the following property holds for L. To formalize, let $X_t^L \subseteq X$ be the set of policies such that $x \in X_t^L$ if and only if L's continuation payoff is maximized if he enters at x at time $-t$ conditional on the event that R has not entered and L enters at $-t$. Let t_L^1 be the solution for

$$e^{-\lambda_R t} = \frac{1}{2}. \tag{31}$$

Lemma 13 $X_t^L = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ *for* $-t \in (-t_L^1, 0]$*, and* $X_t^L = \left\{\left(\frac{2}{3}, 0\right), \left(0, \frac{2}{3}\right)\right\}$ *for* $-t \in \left(-\infty, -t_L^1\right)$*.*

Proof. First, note that candidate L does not enter at a policy $x \notin X\left(L, \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ since R's best response against such x guarantees L to get payoff $\varepsilon u_L(\frac{1}{2}, \frac{1}{2})$, which is dominated by a payoff from a strategy of entering at $(\frac{1}{2}, \frac{1}{2}) \in X\left(L, \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. Second, $X\left(L, \left(\frac{1}{2}, \frac{1}{2}\right)\right) = \{x \in X | \max\{x_1, x_2\} \geq \frac{1}{2}\}$ holds. Third, we consider the following three exhaustive cases depending on which policy among $X\left(L, \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ candidate L enters at:

1. If L enters at $\left(\frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}\right)$, then R will win and implement $\left(\frac{1}{2}, \frac{1}{2}\right)$ if she enters afterward. Hence, L's payoff is

$$\underbrace{e^{-\lambda_R t}}_{\substack{\text{Probability of } R \text{ not} \\ \text{receiving an opportunity}}} + \varepsilon \underbrace{\left(-\frac{1}{2}\right)}_{\text{Utility from the policy is } -\frac{1}{2} \text{ anyway}}.$$

2. If L enters at x with $\min\{x_1, x_2\} \leq 2 - 3\max\{x_1, x_2\}$, then R, if she enters afterward, will win and implement (x_1', x_2') such that $x_1' = x_1$ and $x_2' = 1 - x_1$ if $x_1 > \frac{1}{2}$, and $x_1' = 1 - x_2$ and $x_2' = x_2$ if $x_2 > \frac{1}{2}$. Hence, L's payoff is

$$\underbrace{e^{-\lambda_R t}}_{\substack{\text{Probability of } R \text{ not} \\ \text{receiving an opportunity}}} - \varepsilon \underbrace{\left(-\max\{x_1, x_2\}\right)}_{\substack{\text{Utility from the policy is } -\max\{x_1,x_2\} \text{ anyway} \\ \text{since } \max\{x_1,x_2\}=\max\left\{x_1',x_2'\right\}}}.$$

Thus, among all x's in this case, L's payoff is maximized if and only if he enters at $\left(\frac{1}{2}, 0\right)$, $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, or any convex combination of them, and his payoff is then

$$e^{-\lambda_R t} + \varepsilon\left(-\frac{1}{2}\right).$$

3. If L enters at x with $\min\{x_1, x_2\} \geq 2 - 3\max\{x_1, x_2\}$, then R will win and implement

$\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right)$. Hence, L's payoff is

$$e^{-\lambda_R t} - \varepsilon e^{-\lambda_R t}\left(\max\{x_1, x_2\}\right) - \varepsilon\left(1 - e^{-\lambda_R t}\right)\left(-\frac{x_1 + x_2}{2}\right).$$

If (x_1, x_2) is the optimal policy for L under this case, then the constraint $\min\{x_1, x_2\} \geq 2 - 3\max\{x_1, x_2\}$ has to bind, since otherwise L wants to reduce $\max\{x_1, x_2\}$. The set of x's satisfying $\min\{x_1, x_2\} = 2 - 3\max\{x_1, x_2\}$ is expressed as

$$\left\{\left(\frac{2}{3} - \theta, 3\theta\right) \cup \left(3\theta, \frac{2}{3} - \theta\right) : \text{there exists } \theta \geq 0 \text{ and } \frac{2}{3} - \theta \geq 3\theta\right\}.$$

Given θ, L's payoff is equal to

$$\begin{aligned} & e^{-\lambda_R t} - \varepsilon e^{-\lambda_R t}\left(\frac{2}{3} - \theta\right) - \varepsilon\left(1 - e^{-\lambda_R t}\right)\left(\frac{\frac{2}{3} - \theta + 3\theta}{2}\right) \\ &= e^{-\lambda_R t} - \varepsilon e^{-\lambda_R t}\left(\frac{2}{3} - \theta\right) - \varepsilon\left(1 - e^{-\lambda_R t}\right)\left(\frac{1}{3} + \theta\right). \end{aligned}$$

Hence, if $e^{-\lambda_R t} \geq \frac{1}{2}$, then it is the best for L to enter at $\left(\frac{1}{2}, \frac{1}{2}\right)$; and if $e^{-\lambda_R t} \leq \frac{1}{2}$, then it is the best for him to enter at $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$.

In total, for $-t \in (-t_L^1, 0]$, candidate L enters at $\left(\frac{1}{2}, 0\right)$, $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, or any convex combination of them, and obtains a payoff of $e^{-\lambda_R t} - \varepsilon\frac{1}{2}$. Again, by the tie breaking rule, L enters at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

■

In addition, the following property holds for R:

Lemma 14 *For all $-t \in (-\infty, 0]$, if L enters at $\left(\frac{1}{2}, \frac{1}{2}\right)$ for all times in $(-t, 0]$, then R's unique best response at $-t$ is not to enter.*

Proof. Let σ_R^* be the strategy of R such that R does not enter unless L enters, and best-responds to L's policy once L enters. Consider the following two cases:

1. Conditional on the event under which L will have an opportunity at some $-\tau \in (-t, 0]$, (i) if R enters at $-t$, her payoff will be at most $\varepsilon\frac{1}{2}$, but (ii) σ_R^* gives her a payoff strictly greater than $\varepsilon\frac{1}{2}$ (since L enters at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and R can win if she can enter after L enters).

2. Conditional on the event under which L will not enter, both entering at $\left(\frac{1}{2},\frac{1}{2}\right)$ and σ_R^* are optimal for R.

Since the first event happens with strictly positive probability, the proof is complete. ■

We now pin down the candidates' strategies at $-t$ sufficiently close to 0. Let t_L^2 be the unique t satisfying the following.

$$\begin{cases} \frac{\lambda_R e^{-\lambda_R t} - \lambda_L e^{-\lambda_L t}}{\lambda_L - \lambda_R} = 0 & \text{if } \lambda_L \neq \lambda_R, \\ t = \frac{1}{\lambda} & \text{if } \lambda_L = \lambda_R = \lambda. \end{cases} \tag{32}$$

For each $t < \min\{t_L^1, t_L^2\}$, suppose that candidates take the following continuation play for each $-\tau \in (-t, 0]$: R does not enter unless L enters (and takes a static best-response once L enters) and L enters at $\left(\frac{1}{2},\frac{1}{2}\right)$. Then, we show that, at time $-t$, it is optimal for R not to enter at $-t$ and for L to enter at $\left(\frac{1}{2},\frac{1}{2}\right)$.

Given this continuation play, Lemma 14 ensures that R has a strict incentive not to enter at $-t$. Hence, we consider L's incentive. L's payoff when he does not enter at time $-t$ is

$$\int_0^t \lambda_L e^{-\lambda_L \tau}\left(e^{-\lambda_R(t-\tau)} - \varepsilon\frac{1}{2}\right) d\tau - e^{-\lambda_L t}\left(-\varepsilon\frac{1}{2}\right) = \begin{cases} \lambda_L \frac{e^{-\lambda_R t} - e^{-\lambda_L t}}{\lambda_L - \lambda_R} - \varepsilon\frac{1}{2} & \text{if } \lambda_L \neq \lambda_R \\ e^{-\lambda t}\lambda t - \varepsilon\frac{1}{2} & \text{if } \lambda_L = \lambda_R = \lambda \end{cases}.$$

Hence, L strictly prefers to enter at $\left(\frac{1}{2},\frac{1}{2}\right)$ at time $-t$ if the following holds: $t < t_L^1$ and

$$\begin{cases} \lambda_L \frac{e^{-\lambda_R t} - e^{-\lambda_L t}}{\lambda_L - \lambda_R} - \varepsilon\frac{1}{2} > e^{-\lambda_R t} - \varepsilon\frac{1}{2} & \text{if } \lambda_L \neq \lambda_R, \\ \lambda t > 1 & \text{if } \lambda_L = \lambda_R = \lambda \end{cases} \Leftrightarrow t < t_L^2.$$

Moreover, if $t_L^2 \leq t_L^1$, then L is indifferent between entering and not entering at time $-t_L^2$.

Therefore, by continuity of the continuation payoffs in time and the continuous-time backward induction, for each $t < \min\{t_L^1, t_L^2\}$, at time $-t$, it is uniquely optimal for R not to enter and for L to enter at $\left(\frac{1}{2},\frac{1}{2}\right)$. In what follows, we consider candidates' incentives at time $-t$ with $t > \min\{t_L^1, t_L^2\}$.

If time $-t_L^2$ is after the time at which L's optimal entering policy switches from $\left(\frac{1}{2},\frac{1}{2}\right)$ to $\left(0,\frac{2}{3}\right)$, that is, if $t_L^2 < t_L^1$, then neither L nor R enters for $-t < -t_L^2$. To see why, suppose this claim holds for $-\tau \in [-t, -t_L^2)$. Note that, on the one hand, L's payoff from entering at time $-t$ is strictly decreasing in t since the probability of candidate R entering afterward increases. On the other hand, given that R does not enter for each $-\tau$ with $\tau \leq t$, L can secure a payoff of

$$\begin{cases} \lambda_L \frac{e^{-\lambda_R t_L^2} - e^{-\lambda_L t_L^2}}{\lambda_L - \lambda_R} - \varepsilon \frac{1}{2} & \text{if } \lambda_L \neq \lambda_R \\ e^{-\lambda t_L^2} \lambda t_L^2 - \varepsilon \frac{1}{2} & \text{if } \lambda_L = \lambda_R = \lambda \end{cases}$$

by not entering in the time interval $[-t, -t_L^2)$. Since candidate L is indifferent between entering and not entering at $-t = -t_L^2$, he strictly prefers not entering for each $-t < -t_L^2$. With the same reasoning, one can show that R strictly prefers not entering for each $-t < -t_L^2$. Hence, by continuity of the continuation payoffs in time and the continuous-time backward induction, neither L nor R enters at any $-t < -t_L^2$ in any PBE.

Hence, we are left to consider the case in which $t_L^2 > t_L^1$. By continuity of the continuation payoff in time, there exists $\varepsilon > 0$ such that candidate L enters at $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$ for each $-t \in \left(-t_L^1 - \varepsilon, -t_L^1\right)$. Given this behavior of candidate L, candidate R faces the following trade-off:

1. Conditional on the event under which L will enter after R, either entering at $\left(\frac{1}{2}, \frac{1}{2}\right)$ or not entering (or both) is optimal for R. After R's entry to $\left(\frac{1}{2}, \frac{1}{2}\right)$, L's unique best response is to enter at $\left(\frac{1}{2}, \frac{1}{2}\right)$, and in particular entering at $\left(\frac{2}{3}, 0\right)$ and entering at $\left(0, \frac{2}{3}\right)$ are both suboptimal.

2. Conditional on the event under which L will not enter, both entering at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and not entering are the best for R.

Note that the advantage for R to enter at $\left(\frac{1}{2}, \frac{1}{2}\right)$ is to change L's policy from $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$ to $\left(\frac{1}{2}, \frac{1}{2}\right)$ (R's ideal policy). However, such an advantage is only valid when L enters after R enters. Since L will win for sure in such a case, we will prove that, for sufficiently small policy preference $\varepsilon > 0$, it is uniquely optimal for R not to enter:

Lemma 15 *Suppose $t_L^1 < t_L^2$. Fix $-t < -t_L^1$ and L's strategy such that he enters at $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$ for all times in $\left(-t, -t_L^1\right)$, and enters at $\left(\frac{1}{2}, \frac{1}{2}\right)$ for all times in $(-t_L^1, 0]$. Then, conditional on any history at time $-t$ at which no candidate has entered and R receives an opportunity, not entering is R's unique best response.*

Proof. Fix time $-t < -t_L^1$. Since R not entering at all in the time interval $[-t, -t_L^1)$ is one of the feasible continuation strategies, it suffices to show that, for each $-t$, this strategy is strictly better for R than her entering at $-t$. Consider the following two cases:

1. L obtains an opportunity in the time interval $(-t, -t_L^1)$. Conditional on this event, if R enters at $x \in X$ at time $-t$, then L enters at $y(L, x)$ and wins for sure. Hence, assuming that R enters, the optimal policy for her to enter is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and it gives R a payoff of $\varepsilon \frac{1}{2}$. Meanwhile, if R does not enter until $-t_L^1$, then R obtains

$$\underbrace{\left(1 - e^{-\lambda_R t_L^1}\right)}_{R \text{ can enter by the deadline after } t_L^1} \cdot \left(1 + \varepsilon \frac{1}{3}\right) + e^{-\lambda_R t_L^1} \varepsilon \cdot 0 \geq 1 - e^{-\lambda_R t_L^1}.$$

   Since (31) implies that $t_L^1 = \frac{\ln \frac{1}{2}}{-\lambda_R} = \frac{\ln 2}{\lambda_R}$ and (29) implies $\varepsilon < 1$, straightforward algebra shows that not entering is uniquely optimal for R at $-t$.

2. L does not obtain an opportunity in the time interval $(-t, -t_L^1)$. Conditional on this event, since R's unique best response is not to enter at time $-t_L^1$ (note that, conditional on the event that L does not obtain an opportunity in $(-t, -t_L^1)$, R wants to enter at $-t$ if and only if she wants to enter at $-t_L^1$), it is uniquely optimal for R not to enter at $-t$.

Therefore, conditional on both events, it is uniquely optimal for R not to enter at time $-t$. ■

Let $\bar{\sigma}_L$ be candidate L's strategy such that, if R has not entered, L enters at $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$ for each $-t \in \left(-\infty, -t_L^1\right)$ and at $\left(\frac{1}{2}, \frac{1}{2}\right)$ for each $-t \in (-t_L^1, 0]$ (and L chooses a static best response once R enters); and let $\bar{\sigma}_R$ be candidate R's strategy such that R never enters if L has not entered (and R chooses a static best response once L enters). By $t_1^L < t_2^L$ and Lemma 15, there exists $\varepsilon > 0$ such that $\bar{\sigma}_i$ is optimal for each $-t \geq -t_L^1 - \varepsilon$ and $i \in \{L, R\}$.

For $t < t_L^1$, suppose that the candidates take $\bar{\sigma}$ for each time $-\tau$ with $\tau < t$. Given that R never enters after $-t$, given $t < t_L^1$, we must have $X_t^L = \left\{\left(\frac{2}{3}, 0\right), \left(0, \frac{2}{3}\right)\right\}$. Note that the probability that L wins by entering at $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$—equivalently, the probability that R cannot enter after L enters—is decreasing in t and converges to 0 as $t \to \infty$. Hence, the payoff of entering converges to $-\varepsilon \frac{1}{3}$. By (29), for sufficiently large t, there exists $\tau' \in [0, t]$ such that L can obtain a payoff greater than $-\varepsilon \frac{1}{3}$ by instead not entering until $-\tau'$. Hence, given the bound of ε we imposed and continuity of the continuation payoff in time given $\bar{\sigma}$, there exists the smallest t such that L is indifferent between entering and not entering at $-t$. Let t_L^3 be such t.

By continuity of the continuation payoff in t and the continuous-time backward induction, $\bar{\sigma}_i$ is optimal for any $-t > -t_L^3$ in any PBE. Hence, we are left to show that no candidate enters at $-t < -t_L^3$. Let σ^* be a pair of strategies such that neither L nor R enters at $-t < -t_L^3$ and both of them take $\bar{\sigma}$ for any $-t > -t_L^3$. One can show that R chooses a best response in the same way as in Lemma 15 given the continuation play σ^*. L's incentive can be checked as follows: Let v_L^3 be L's payoff of entering at time $-t_L^3$ given the continuation play $\bar{\sigma}$. Entering at $-t < -t_L^3$ gives him a payoff strictly lower than v_L^3 since the probability that R can enter after L enters increases monotonically in t. Not entering until $-t_L^3$ guarantees a payoff of v_L^3 since L is indifferent between entering and not entering at $-t_L^3$ given the continuation play σ^*. Hence, by continuity of the continuation payoffs in time and the continuous-time backward induction, both candidates take σ^* in any PBE.

Finally, we examine the conditions under which we have $t_L^1 < t_L^2$ and $t_L^1 > t_L^2$, respectively. Note that (31) implies that $t_L^1 = \frac{\ln \frac{1}{2}}{-\lambda_R} = \frac{\ln 2}{\lambda_R}$. Since the left-hand side of (32) is negative for $t \in \left(0, t_L^2\right)$, and positive for $t > t_L^2$, we have $t_L^1 < t_L^2$ if and only if the left-hand side of (32) is negative for $t = t_L^1$. Substituting $t = t_L^1 = \frac{\ln 2}{\lambda_R}$, the left-hand side of (32) is equal to

$$\frac{\lambda_R e^{-\lambda_R \frac{\ln 2}{\lambda_R}} - \lambda_L e^{-\lambda_L \frac{\ln 2}{\lambda_R}}}{\lambda_L - \lambda_R} = \frac{\frac{1}{2} - \frac{\lambda_L}{\lambda_R}\left(\frac{1}{2}\right)^{\frac{\lambda_L}{\lambda_R}}}{\frac{\lambda_L}{\lambda_R} - 1}.$$

Letting $l = \frac{\lambda_L}{\lambda_R}$, this is equal to $\frac{\frac{1}{2} - l\left(\frac{1}{2}\right)^l}{l-1}$. Taking the derivative of the numerator with respect to l yields

$$-\left(\frac{1}{2}\right)^l + l\left(\frac{1}{2}\right)^l \ln 2 = \left(\frac{1}{2}\right)^l (1 - l \ln 2).$$

Hence, the numerator is decreasing for $l \leq \frac{1}{\ln 2}$ and increasing for $l \geq \frac{1}{\ln 2}$.

Note that the numerator is zero at $l = 1 < \frac{1}{\ln 2}$ and at $l = 2 > \frac{1}{\ln 2}$. Hence, $\frac{1}{2} - l\left(\frac{1}{2}\right)^l$ is positive for $l < 1$, 0 for $l = 1$, negative for $l \in (1, 2)$, 0 for $l = 2$, and positive for $l > 2$. Together with the denominator (and using l'Hopital rule at $l = 1$), we have

$$\frac{\frac{1}{2} - l\left(\frac{1}{2}\right)^l}{l-1} \begin{cases} < 0 & \text{for } l \in (0,2) \\ = 0 & \text{for } l = 2 \\ > 0 & \text{for } l > 2 \end{cases}.$$

Therefore, $t_L^1 < t_L^2$ if and only if $\frac{\lambda_L}{\lambda_R} < 2$. In a similar vein, one can show that $t_L^1 > t_L^2$ if and only if $\frac{\lambda_L}{\lambda_R} > 2$.

K.2 Persuasion-Cost Election Campaign

In the policy-motivated election campaign in Section 3.3, L enters at suboptimal policies $\left(\frac{2}{3}, 0\right)$ or $\left(0, \frac{2}{3}\right)$ since, when R enters after L, this suboptimal policy will lead R to enter at a more favorable policy for L. Such a consideration does not occur if L does not care about what policy R picks when R wins. In such a case, the equilibrium dynamics are simpler than in the model in Section 3.3, while we can still conduct comparative statics with respect to the distribution of voters and the ideal points of the candidates more easily, keeping the advantage of the policy-motivated model over the purely office-motivated model as in Section 3.2.

Let X be an arbitrary policy space that is a full-dimensional compact subset of $\mathbb{R}^n$ for some n, and recall that $|\cdot|$ denotes the Euclidian distance. A unit mass of voters are distributed over X according to the distribution $\mu(x)$ over X. The voter located at x has utility of $-|x-y|$ from policy y.

There are two candidates L and R, and we let $\mathcal{X}_i = \{X\} \cup \left(\bigcup_{x \in X} \{\{x\}\}\right)$ for each $i = L, R$. Given a profile of policies $(x_L, x_R) \in X \times X$, we define candidate i's vote share $S_i(x_L, x_R)$ and probability of i's winning $P_i(x_L, x_R)$ as in Section 3.2. The definition of $P_i(X_i, X_j)$ when $X_i = X$ or $X_j = X$ holds is given later. We assume that $(X, \mu) \notin \mathcal{M}$.

The ideal policies of candidates L and R are x_L^* and x_R^*, respectively. The ideal policies are common knowledge among voters and candidates. The utility for candidate i is equal to

$$\begin{cases} \mathbb{I}_{i \text{ wins}} - \varepsilon |x_i^* - x| & \text{if } X_i = \{x\} \subseteq X \\ \mathbb{I}_{i \text{ wins}} & \text{if } X_i = X \end{cases},$$

where $\varepsilon > 0$. That is, the candidate incurs a cost $|x_i^* - x|$ associated with the policy to which she commits, regardless of whether she wins the election. For example, if the voters believe that x_i^* is i's ideal policy, committing to x far from x_i^* requires the cost of persuading the voters. Without specifying the policy—with $X_i = X$—, in contrast, she does not have to pay such a cost. We assume that

$$\varepsilon < \frac{1}{\max_{i \in \{R,L\}, x \in X} |x_i^* - x|}. \tag{33}$$

This condition implies that, the minimum (with respect to $x \in X$) of the payoffs from entering at some x and winning exceeds the payoff from not entering and losing. The denominator of the right-hand side of (33) is strictly positive because X is a full-dimensional subset of $\mathbb{R}^n$ and it is finite because X is compact.

Suppose that the voters believe that the candidates will implement their ideal policies once they get elected without specifying a policy. That is, we assume $S_i(X, x_j) = S_i(x_i^*, x_j)$, $S_i(x_i, X) = S_i(x_i, x_j^*)$, $S_i(X, X) = S_i(x_i^*, x_j^*)$, and the probability of winning P_i is accordingly defined when X is chosen by at least one candidate. They vote for the candidate whose policy implementation gives them the higher expected payoff. The candidate who attracts more votes will win the election. Given this, we assume that $P_R(x_R^*, x_L^*) = 1$, that is, R will win if neither candidate specifies their policies.[21] The payoff function v_i for each $i = L, R$ is specified accordingly. As in the policy-motivated election campaign in Section 3.3, we assume that the tie is broken in favor of the last candidate to specify the policy if the candidates enter at different times.[22]

Call this game a *persuasion-cost election campaign*. It is characterized by a tuple $(X, \mu, \varepsilon, T, \lambda_L, \lambda_R)$.

Let X^* be the set of policies with which L attracts weakly more votes than R if R does not specify a policy:

$$X^* = \left\{ \hat{x} : \int_x 1_{\{|x - x_R^*| \geq |x - \hat{x}|\}} \mu(x)\, dx \geq \frac{1}{2} \right\}.$$

In addition, given $x \in X$, let $X^*(x)$ be the set of policies such that R attracts weakly more votes than L given that L enters at x:

$$X^*(x) = \left\{ \hat{x} : \int_{\tilde{x}} 1_{\{|\tilde{x} - x| \leq |\tilde{x} - \hat{x}|\}} \mu(\tilde{x})\, d\tilde{x} \geq \frac{1}{2} \right\}.$$

[21] The case in which $P_R(x_L^*, x_R^*) = 0$ can be analyzed in a symmetric manner, so its analysis is omitted.

[22] We assume such a tie-breaking rule because $(X, \mu) \notin \mathcal{M}$ and thus there is no best response once the opponent enters. As in footnote 39 of the main text, the assumption corresponds to taking a limit of unique PBEs in the models with discrete policy spaces.

Given X^* and $X^*(x)$, we can characterize PBE:

Proposition 16 *The persuasion-cost election campaign with $(X, \mu, \varepsilon, T, \lambda_L, \lambda_R)$ has a PBE. Moreover, there exists $t_L^* < \infty$ such that for any PBE, the following hold:*

1. *L enters at $x \in \arg\min_{x \in X^*} |x_L^* - x|$ for $-t > -t_L^*$, while he does not enter for $-t < -t_L^*$.*

2. *R never enters unless L enters. Once L enters at x, R enters as soon as possible at $x' \in \arg\min_{x' \in X^*(x)} |x_R^* - x'|$.*

Candidate R does not have an incentive to enter before L enters since (i) R can win without entering if L cannot obtain an opportunity and (ii) R will lose by entering if L can obtain an opportunity afterward. Given this strategy of R, since L cannot win without entering, he enters if the deadline is near. If the deadline is far, then the probability that R can enter afterward is very large. Hence, entering gives L the payoff close to 0 (or negative if he pays the persuasion cost). Therefore, L does not enter when the deadline is far.

Once we specify x_R^*, x_L^*, and μ, it is straightforward to derive the distribution of the announced policies at the deadline. Thus, we can conduct the comparative statics about observable variables.[23]

K.2.1 Proof of Proposition 16

Consider a PBE. Given (33), there exists $\bar{t} > 0$ such that for all time $-t \in (\bar{t}, 0]$, L enters at some policy with which he can win. In addition, for each $-t$, if R has already entered, L takes a static best response.

Since R can win without incurring the persuasion cost if L does not enter, we can show that, for each $-t$, R does not enter:

Lemma 16 *Fix candidate L's strategy in which he takes a static best response after R enters. Then, conditional on any history at time $-t$ at which no candidate has entered and R receives an opportunity, not entering is R's unique best response.*

Proof. Since R's not entering until L enters is one of the feasible continuation strategies, it suffices to show that this strategy, denoted by $\bar{\sigma}^R$, is strictly better for R than her entering at $-t$ for each $t \geq 0$.

[23] The policy to which candidates enter is generically unique in Proposition 16.

Fix time $-t$ and a history at time $-t$ such that no candidate has entered. Consider the following two cases:

1. L obtains an opportunity in the time interval $(-t, 0]$. Fix time $\bar{t} > 0$ such that L enters for each $[-\bar{t}, 0]$ if no candidate enters. Conditional on this event, let p be the probability that L obtains an opportunity at some $-\tilde{t} \in [-\bar{t}, 0)$, and then R has an opportunity in some $-\hat{t} \in (-\tilde{t}, 0]$.

   Conditional on this event, entering at x gives R a payoff of $-\varepsilon |x - x_R| \leq 0$ while $\bar{\sigma}^R$ gives R a payoff no less than $p(1 - \max_{x \in X} |x_R^* - x|) > 0$ (strict inequality follows from (33)) since (i) if L has an opportunity at $-\tilde{t} \in [-\bar{t}, 0)$, then either L will have entered by $-\tilde{t}$ or he enters at $-\tilde{t}$, and (ii) if R has an opportunity at some $-\hat{t} \in (-\tilde{t}, 0]$, then she wins for sure by $\bar{\sigma}^R$.

2. L does not obtain an opportunity in the time interval $(-t, 0]$. Conditional on this event, $\bar{\sigma}^R$ gives R a payoff of 1, which is her largest feasible payoff.

Since $\bar{\sigma}^R$ is optimal conditional on each of these two events and the incentive is strict in the first case, it is uniquely optimal for R not to enter given the conditions in the statement of the lemma. ■

After L's entry, candidate R enters at the policy x' with which R can win with the lowest persuasion cost:

$$x' \in \arg \min_{x' \in X^*(x)} \left| x_R - x' \right|.$$

Given this reaction of R, L's payoff of entering at x at time $-t$ is $e^{-\lambda_R t} - \varepsilon |x_L - x|$. Hence, if he enters, then he enters at the policy with which L can win with the lowest persuasion cost assuming that R will not enter. His payoff of entering at $-t$ is, therefore,

$$e^{-\lambda_R t} - \min_{x \in X^*} \varepsilon |x_L - x|.$$

In contrast, his payoff of not entering at $-t$, given that he will enter as soon as possible in the interval $(-t, 0]$, is

$$\begin{aligned}&\int_0^t \lambda_L e^{-\lambda_L \tau}\left(e^{-\lambda_R (t-\tau)} - \min_{x\in X^*} \varepsilon\,|x_L - x|\right) d\tau \\ = \quad & \frac{e^{-\lambda_L t} - \lambda_L e^{-\lambda_R t}}{\lambda_L - \lambda_R} - (1 - e^{-\lambda_L t}) \min_{x\in X^*} \varepsilon\,|x_L - x|\,.\end{aligned}$$

Let

$$t_L^* = \frac{\log \frac{\lambda_L - (\lambda_L - \lambda_R)\min_{x\in X^*} \varepsilon|x_L - x|}{\lambda_R}}{\lambda_L - \lambda_R} \in (0, \infty)$$

be the smallest t such that L is indifferent between entering and not entering. By the continuous-time backward induction, for $(-t_L^*, 0]$, L enters at $x \in \arg\min_{x\in X^*} \varepsilon\,|x_L - x|$. We are left to show that L does not enter at any time $-t < -t_L^*$. Let σ_L^* be L's strategy such that, at any time $-t$, if R has not entered before $-t$, (i) L does not enter if $t > t_L^*$ and (ii) he enters at some $x \in \arg\min_{x\in X^*} \varepsilon\,|x_L - x|$ if $t < t_L^*$.

Consider the following two cases:

1. R obtains an opportunity in the time interval $(-t, -t_L^*)$. Conditional on this event, if L enters at x at time $-t$, then L's payoff is $-\varepsilon\,|x_L - x|$, while σ_L^* gives him a payoff of $e^{-\lambda_R t_L} - \min_{x\in X^*} \varepsilon\,|x_L - x|$ since no candidate will enter before $-t_L^*$ and L is indifferent between entering and not entering at $-t_L^*$. Hence it is uniquely optimal not to enter at $-t$.

2. R does not obtain an opportunity in the time interval $(-t, -t_L^*)$. Conditional on this event, L is indifferent between entering and not entering since he is indifferent between entering and not entering at $-t_L^*$.

Hence, it is uniquely optimal not to enter at $-t$ with $t > t_L^*$.

Overall, we have identified the equilibrium dynamics described in the statement of the proposition.

L Proofs for Section 3.4

Our convention throughout the paper is that, when we write $f_i(x, y)$ for a function f_i where $i \in \{A, B\}$, x is associated with candidate i and y is associated with candidate j. However, in this section of the appendix, we have that x is associated with candidate A and y is associated with candidate B. We use this alternative convention to avoid confusion about which candidate spends how much money to the campaign.

L.1 Proof of Part 1 of Proposition 6

Fix a PBE σ and an associated belief β. For any history h_i at time $-t$ at which i does not have an opportunity at $-t$, let $v_{i,t}(h_i)$ be i's continuation payoff given strategy profile σ and belief β. Let $H_{i,t}(x_A, x_B)$ be the set of i's histories such that the current time is $-t$ and the minimum spending amounts that are currently available are x_A for candidate A and x_B for candidate B. Since candidates A and B are symmetric, it is sufficient to consider the incentives in the histories in $H_{A,t}(0,0)$, $H_{A,t}(0,L)$, $H_{B,t}(0,L)$, $H_{A,t}(0,H)$, $H_{B,t}(0,H)$, $H_{A,t}(L,L)$, $H_{A,t}(L,H)$, and $H_{B,t}(L,H)$ (there is no choice to be made in histories in $H_i(H,H)$ for each $i = A, B$).

The simplest case is that a candidate has already spent H. In this case, the opponent spends H as soon as possible.

Lemma 17 *For any t and $h_A \in H_{A,t}(0,H) \cup H_{A,t}(L,H)$, candidate A spends H as soon as possible under σ, and the following hold.*

$$v_{A,t}(h_A) = \begin{cases} v_{A,t}(H,H) := \frac{\alpha}{2} & \textit{if } h_A \in H_{A,t}(H,H) \\ v_{A,t}(L,H) := e^{-\lambda t}\left[\alpha \frac{L}{H+L} + (1-\alpha)(H-L)\right] + \left(1-e^{-\lambda t}\right)\frac{\alpha}{2} & \textit{if } h_A \in H_{A,t}(L,H) \\ v_{A,t}(0,H) := e^{-\lambda t}(1-\alpha)H + \left(1-e^{-\lambda t}\right)\frac{\alpha}{2} & \textit{if } h_A \in H_{A,t}(0,H) \end{cases} .$$

$$v_{B,t}(h_B) = \begin{cases} v_{B,t}(H,H) := \frac{\alpha}{2} & \textit{if } h_B \in H_{B,t}(H,H) \\ v_{B,t}(L,H) := e^{-\lambda t}\alpha \frac{H}{H+L} + \left(1-e^{-\lambda t}\right)\frac{\alpha}{2} & \textit{if } h_B \in H_{B,t}(L,H) \\ v_{B,t}(0,H) := e^{-\lambda t}\alpha + \left(1-e^{-\lambda t}\right)\frac{\alpha}{2} & \textit{if } h_B \in H_{B,t}(0,H) \end{cases} .$$

Proof. This follows from the fact that $\{H\}$ is a unique static best response against $\{H\}$. ■

The next simple case is that both candidates have spent L, that is, the current profile of policy sets (X_A, X_B) satisfies $X_A = X_B = \{L, H\}$. Each candidate spends H as soon as possible in this case as well:

Lemma 18 *For any time $-t$ and $h_i \in H_{i,t}(L,L)$, each candidate i spends H as soon as possible under σ. Moreover, for each $i = A, B$, we have*

$$v_{i,t}(h_i) = v_{i,t}(L,L) := \frac{\alpha}{2} + e^{-\lambda t}\left(1 - \alpha\frac{1}{2(H+L)}\right)(H-L)$$

if $h_i \in H_{i,t}(L,L)$.

Proof. At histories in $H_{i,t}(L,L)$ for each $i = A, B$, the available policy sets for each candidate are $\{L,H\}$ and $\{H\}$. The payoff matrix of the relevant policy sets (that is, the payoffs of taking these policy sets at the deadline) is as follows.

	$\{L,H\}$	$\{H\}$
$\{L,H\}$	$\alpha\frac{1}{2} + (1-\alpha)(H-L), \alpha\frac{1}{2} + (1-\alpha)(H-L)$	$\alpha\frac{L}{H+L} + (1-\alpha)(H-L), \alpha\frac{H}{H+L}$
$\{H\}$	$\alpha\frac{H}{H+L}, \alpha\frac{L}{H+L} + (1-\alpha)(H-L)$	$\alpha\frac{1}{2}, \alpha\frac{1}{2}$

Note that H is a strictly dominant policy in this normal-form game. Hence, by part 3 of Theorem 4, spending H is uniquely optimal.

Given Lemma 17, for each $h_A \in H_{A,t}(L,L)$, we have

$$v_{A,t}(h_A) = \int_0^t 2\lambda e^{-2\lambda t}\frac{v_{A,t}(H,L) + v_{A,t}(L,H)}{2} + \underbrace{e^{-2\lambda t}}_{\text{No candidate has an opportunity until the deadline}} \times \left(\frac{\alpha}{2} + \alpha(H-L)\right).$$

Straightforward algebra shows that

$$v_{A,t}(h_A) = \frac{\alpha}{2} + e^{-\lambda t}\left(1 - \alpha\frac{1}{2(H+L)}\right)(H-L).$$

By symmetry, we have

$$v_{B,t}(h_B) = \frac{\alpha}{2} + e^{-\lambda t}\left(1 - \alpha\frac{1}{2(H+L)}\right)(H-L).$$

■

We are left to analyze histories in $H_{A,t}(0,L)$, $H_{B,t}(0,L)$, $H_{A,t}(0,0)$, and $H_{B,t}(0,0)$. Suppose now that $H_{A,t}(0,L)$ is reached. Note that $\{H\}$ strictly dominates $\{L,H\}$ for each $h_A \in H_{A,t}(0,L)$ for candidate A since $v_{A,t}(L,L) < v_{A,t}(H,L)$ for each t given Lemma 18. Hence, the remaining questions are (i) given $h_A \in H_{A,t}(0,L)$, whether candidate A wants to spend 0 or H; and (ii) given $h_B \in H_{B,t}(0,L)$, whether candidate B wants to stay at L or spend H.

At the deadline, the payoff matrix for the relevant policy sets is as follows.

	$\{L,H\}$	$\{H\}$
$\{0,L,H\}$	$(1-\alpha)H, \alpha+(1-\alpha)(H-L)$	$(1-\alpha)H, \alpha$
$\{H\}$	$\alpha\frac{H}{H+L}, \alpha\frac{L}{H+L}+(1-\alpha)(H-L)$	$\frac{\alpha}{2}, \frac{\alpha}{2}$

Note that candidate A's $\{H\}$ is a unique best response to candidate B's $\{L,H\}$, and candidate B's $\{L,H\}$ is a unique best response to candidate A's $\{0,L,H\}$. Hence, by continuity of the continuation payoffs in time, there exists $\bar{t}>0$ such that for all $t \in [0,\bar{t})$, in any PBE, candidate A takes $\{H\}$ while candidate B stays at $\{L,H\}$ at histories $h_A \in H_{A,t}(0,L)$ and $h_B \in H_{B,t}(0,L)$, respectively.

Let σ' be a strategy profile such that candidate A takes $\{H\}$ while candidate B stays at $\{L,H\}$ at histories $h_A \in H_{A,t}(0,L)$ and $h_B \in H_{B,t}(0,L)$ for any $t \geq 0$, respectively, and each candidate i follows σ_i if $h_i \notin H_{i,t}(0,L)$. Given the original β, let $\bar{v}_{i,t}(h_i)$ be candidate i's payoff at history h_i under σ' and β:

$$\begin{aligned}
\bar{v}_{A,t}(h_A) = v_{A,t}(0,L) &:= \int_0^t \lambda e^{-\lambda t} v_{A,\tau}(H,L)\,d\tau + e^{-\lambda t}(1-\alpha)H \\
&= e^{-\lambda t}\left(H + \alpha\left(e^{\lambda t}\frac{1}{2} - \frac{1}{2} - H + \frac{H-L}{2(H+L)}\lambda t\right)\right), \\
\bar{v}_{B,t}(h_B) = v_{B,t}(0,L) &:= \int_0^t \lambda e^{-\lambda t} v_{B,\tau}(H,L)\,d\tau + e^{-\lambda t}\left[\alpha + (1-\alpha)(H-L)\right] \\
&= e^{-\lambda t}\left((H-L)(1+\lambda t) + \alpha\left(\begin{array}{c} \frac{H}{2(H+L)}\left(1+e^{\lambda t}-\lambda t\right) - \frac{H^2}{(H+L)}(1+\lambda t) \\ +\frac{L}{2(H+L)}\left(e^{\lambda t} + (1+2L)(1+\lambda t)\right) \end{array}\right)\right).
\end{aligned} \tag{34}$$

Since there exists $\bar{t}>0$ such that σ and σ' coincide for all $t \in [0,\bar{t})$, candidate A takes $\{H\}$ and candidate B takes $\{L,H\}$ at time $-t$ under σ if

$$v_{A,t}(0,L) < v_{A,t}(H,L) \text{ and } v_{B,t}(0,L) > v_{B,t}(0,H).$$

By straightforward algebra, we have

$$v_{A,t}(0,L) \leq v_{A,t}(H,L) \Leftrightarrow t \leq t^* := \frac{1}{\lambda} \tag{35}$$

and

$$v_{B,t}(0,L) \geq v_{B,t}(0,H) \Leftrightarrow t \leq t^{**} := \begin{cases} \frac{1}{\frac{\alpha}{2(1-\alpha)(H+L)}-1}\frac{1}{\lambda} & \text{if } \frac{\alpha}{2(1-\alpha)(H+L)} - 1 > 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{36}$$

In both (35) and (36), the first inequality holds with equality if and only if the second holds with equality. Intuitively, near the deadline, since $\{H\}$ is a static best response against $\{L, H\}$, candidate A chooses $\{H\}$ as soon as possible. Since $\{H\}$ is a static best response to $\{H\}$ and candidate A enters at H regardless of candidate B staying at $\{L, H\}$ or choosing $\{H\}$, candidate B wants to spend H if the deadline is far ($-t < -t^{**}$) and hence it is sufficiently likely that candidate A enters at H.

By (2), we have $t^{**} < t^*$. Hence, for $t \leq t^{**}$, we have $v_{A,t}(h_A) = \bar{v}_{A,t}(h_A)$ and $v_{B,t}(h_B) = \bar{v}_{B,t}(h_B)$, and under σ, candidate A takes $\{H\}$ as a unique best response at any history $h_A \in H_{A,t}(0,L)$ for all time $-t \leq -t^{**}$ while candidate B stays at $\{L, H\}$ as a unique best response at any history $h_B \in H_{B,t}(0,L)$ for all time $-t < -t^{**}$. By continuity of the continuation payoff in time, the following result holds:

Lemma 19 *Under σ, there exists $\varepsilon > 0$ such that, for each $-t \in (-(t^{**} + \varepsilon), 0]$, candidate A takes $\{H\}$ at any $h_A \in H_{A,t}(0,L)$.*

Proof. The result follows from (35), $t^{**} < t^*$, and the continuity of the continuation payoff in time. ■

Before analyzing the candidates' incentives at $h_A \in H_{A,t}(0,L)$ and $h_B \in H_{B,t}(0,L)$ such that $-t < -t^{**}$, we consider their incentives when no candidate has spent anything during $-t > -t^{**}$.

Lemma 20 *Under σ, for any $-t \in (-t^{**}, 0]$, candidates A and B take $\{L, H\}$ at any $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$, respectively, and we have*

$$v_{A,t}(h_A) = v_{A,t}(0,0) := \frac{1}{2}\alpha + (1-\alpha)(H + (H-L)\lambda t)e^{-\lambda t},$$
$$v_{B,t}(h_B) = v_{B,t}(0,0) := \frac{1}{2}\alpha + (1-\alpha)(H + (H-L)\lambda t)e^{-\lambda t}.$$

Proof. At the deadline, taking $\{L, H\}$ is a static best response against $\{0, L, H\}$. Hence, by continuity of the continuation payoffs in time, there exists $\bar{t} > 0$ such that for all $t \in [0, \bar{t})$, candidates A and B take $\{L, H\}$ at histories $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$, respectively.

Let σ'' be a strategy profile such that candidates A and B take $\{L, H\}$ at histories $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$, respectively, and follow σ if $h_i \notin H_{i,t}(0,0)$. Given the original β, let $\hat{v}_{i,t}(h_i)$ be candidate i's payoff at history h_i under σ'' and β:

$$\begin{aligned} \hat{v}_{i,t}(h_i) &= \int_0^t 2\lambda e^{-2\lambda\tau} \frac{v_{A,\tau}(L,0) + v_{A,\tau}(0,L)}{2} d\tau + e^{-2\lambda t}\left(\frac{\alpha}{2} + (1-\alpha) H\right) \\ &= \frac{1}{2}\alpha + (1-\alpha)(H + (H-L)\lambda t) e^{-\lambda t}. \end{aligned}$$

Since there exists $\bar{t} > 0$ such that σ and σ'' coincide for $t \in [0, \bar{t})$, both candidates A and B take $\{L, H\}$ at histories $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$ in σ if

$$\begin{aligned} v_{A,t}(L,0) &> \max\{v_{A,t}(H,0), v_{A,t}(0,0)\}, \text{ and} \\ v_{B,t}(0,L) &> \max\{v_{B,t}(0,H), v_{B,t}(0,0)\}. \end{aligned}$$

By symmetry, we focus on the case $v_{B,t}(0,L) > \max\{v_{B,t}(0,H), v_{B,t}(0,0)\}$. By continuity of the continuation payoff in time, it suffices to show that $v_{B,t}(0,L) > \max\{v_{B,t}(0,H), v_{B,t}(0,0)\}$ for each $t < t^{**}$.

Since $v_{B,t'}(0,L) > v_{B,t'}(0,H)$ for $t' < t^{**}$ by (36), it suffices to show that, for each $t \leq t^{**}$,

$$v_{B,t}(0,L) > v_{B,t}(0,0) \Leftrightarrow (H+L)(\alpha - 2L(1-\alpha)) - (H-L)\alpha t\lambda > 0. \tag{37}$$

The right hand side of (37) holds because, for each $t \leq t^{**}$,

$$\begin{aligned} &(H+L)(\alpha - 2L(1-\alpha)) - (H-L)\alpha t\lambda \\ &\geq (H+L)(\alpha - 2L(1-\alpha)) - (H-L)\alpha t^*\lambda \quad (\text{since } t \leq t^{**} < t^*) \\ &= (H+L)(\alpha - 2L(1-\alpha)) - (H-L)\alpha \\ &> 0 \end{aligned}$$

because $\alpha \geq \frac{H+L}{H+L+\frac{1}{2}} \geq \frac{H+L}{H+L+1}$. ■

A similar proof shows that neither candidate A nor B stays at $\{0, L, H\}$ at any $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$, respectively.

Lemma 21 *Under σ, there exists $\varepsilon' > 0$ such that, for any $-t \in (-(t^{**} + \varepsilon'), 0]$, no candidate chooses $\{0, L, H\}$ at any $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$, respectively.*

Proof. By continuity of the continuation payoff in time and symmetry, it suffices to show that $\max\{v_{B,t^{**}}(0, L), v_{B,t^{**}}(0, H)\} > v_{B,t^{**}}(0,0)$. In particular, it is sufficient to have $v_{B,t^{**}}(0, L) > v_{B,t^{**}}(0,0)$. This inequality follows from (37). ■

Given this lemma, we have the following result:

Lemma 22 *Under σ, there exists $\varepsilon'' > 0$ such that, for each $-t \in (-(t^{**} + \varepsilon''), -t^{**})$, both candidates A and B take $\{H\}$ at any $h_A \in H_{A,t}(0,0) \cup H_{A,t}(L, 0)$ and $h_B \in H_{B,t}(0,0) \cup H_{B,t}(0, L)$, respectively.*

Proof. Fix $\varepsilon > 0$ such that Lemma 19 holds, and fix $\varepsilon' > 0$ such that Lemma 21 holds. Take $\varepsilon'' \in (0, \min\{\varepsilon, \varepsilon'\})$, and suppose candidate i obtains an opportunity at $-t \in (-(t^{**} + \varepsilon''), -t^{**})$. Let x_i be i's optimal policy at $-t$. By Lemma 21, we have $x_i \in \{\{L, H\}, \{H\}\}$. Consider the following three scenarios that can happen after $-t$: For each $i \in \{A, B\}$,

1. The probability that the two candidates have opportunities in the time interval $(-t, -t^{**})$ is equal to $\left(1 - e^{-\lambda\varepsilon'}\right)^2$.

2. The probability that only candidate j has an opportunity in the time interval $(-t, -t^{**})$ is equal to $e^{-\lambda\varepsilon'}\left(1 - e^{-\lambda\varepsilon'}\right)$. Then, since $x_i \in \{\{L, H\}, \{H\}\}$, Lemmas 17 and 19 (and symmetry) imply that candidate j takes $\{H\}$. Hence, $h_A \in H_{A,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ and $h_B \in H_{B,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ with $(x_{i,t^{**}}, x_{j,t^{**}}) = (x_i, H)$ will be realized at $-t^{**}$. Candidate i strictly prefers $\{H\}$ to $\{L, H\}$ by Lemma 17:

$$v_{i,t^{**}}(H, H) - v_{i,t^{**}}(L, H) > 0.$$

   Note that this inequality holds independently from the choice of ε'.

3. Candidate j does not have an opportunity in the time interval $(-t, -t^{**})$. Conditional on this event, candidate i prefers $\{L, H\}$ to $\{H\}$ at $-t$ if and only if she prefers $\{L, H\}$ to $\{H\}$ at $-t^{**}$. Hence, both $\{H\}$ and $\{L, H\}$ are optimal by the definition of t^{**}.

Since the likelihood ratio of case 2 against case 1 goes to ∞ as $\varepsilon' \to 0$, there exists $\varepsilon' \in (0, \varepsilon)$ such that for each $-t \in (-(t^{**} + \varepsilon'), -t^{**})$, $\{H\}$ is the unique optimal policy set at $-t$. ■

Fix $t > t^{**}$ arbitrarily, and fix the continuation play such that both candidates A and B strictly prefer $\{H\}$ to $\{L, H\}$ and $\{0, L, H\}$ at any $h_A \in H_{A,t'}(0, 0) \cup H_{A,t'}(L, 0)$ and $h_B \in H_{B,t'}(0, 0) \cup H_{B,t'}(0, L)$ for any $-t' \in (-t, -t^{**})$. Then both candidates A and B take $\{H\}$ at any $h_A \in H_{A,t}(0, 0) \cup H_{A,t}(L, 0)$ and $h_B \in H_{B,t}(0, 0) \cup H_{B,t}(0, L)$. To see why, consider the following four scenarios for each $i \in \{A, B\}$:

1. If the two candidates have opportunities in the time interval $(-t, -t^{**})$, then regardless of the strategy at $-t$, $h_A \in H_{A,t^{**}}(H, H)$ and $h_B \in H_{B,t^{**}}(H, H)$ will be realized. In this case, candidate A's spending at $-t$ does not affect i's payoff.

2. If only candidate j has an opportunity in the time interval $(-t, -t^{**})$, then $h_A \in H_{A,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ and $h_B \in H_{B,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ with $(x_{i,t^{**}}, x_{j,t^{**}}) = (x_i, H)$ will be realized at $-t^{**}$, where x_i is i's spending at $-t$. In this case, candidate i strictly prefers $\{H\}$ to $\{L, H\}$ and $\{0, L, H\}$.

3. If only candidate i has an opportunity in the time interval $(-t, -t^{**})$, then $h_A \in H_{A,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ and $h_B \in H_{B,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ with $(x_{i,t^{**}}, x_{j,t^{**}}) = (H, x_j)$ will be realized, where x_j is j's spending at $-t$. In this case, candidate i's spending at $-t$ does not affect i's payoff.

4. If no candidate has an opportunity in the time interval $(-t, -t^{**})$, then $h_A \in H_{A,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ and $h_B \in H_{B,t^{**}}(x_{i,t^{**}}, x_{j,t^{**}})$ with $(x_{i,t^{**}}, x_{j,t^{**}}) = (x_i, x_j)$ will be realized, where (x_i, x_j) is the spending profile at $-t$. In this case, both $\{H\}$ and $\{L, H\}$ are optimal by the definition of t^{**}.

In total, spending H is uniquely optimal, as desired.

Finally, together with Lemma 22, under σ, both candidates A and B strictly prefer $\{H\}$ to $\{L, H\}$ and $\{0, L, H\}$ at any $h_A \in H_{A,t}(0, 0) \cup H_{A,t}(L, 0)$ and $h_B \in H_{B,t}(0, 0) \cup H_{B,t}(0, L)$, respectively, for any $-t \in (-\infty, -t^{**})$.

L.2 Proof of Part 2 of Proposition 6

Fix a PBE σ and an associated belief β. As in part 1 of Proposition 6, it is sufficient to consider the incentives in $H_{A,t}(0,0)$, $H_{A,t}(0,L)$, $H_{B,t}(0,L)$, $H_{A,t}(0,H)$, $H_{B,t}(0,H)$, $H_{A,t}(L,L)$, $H_{A,t}(L,H)$, and $H_{B,t}(L,H)$. Lemmas 17 and 18 still hold. In particular, for $h_A \in H_{A,t}(0,L)$, candidate A always prefers $\{H\}$ to $\{L,H\}$.

Moreover, by the same calculation as (35) and (36), we derive the following: Let σ' be a strategy profile such that candidate A takes $\{H\}$ while candidate B stays at $\{L,H\}$ at histories $h_A \in H_{A,t}(0,L)$ and $h_B \in H_{B,t}(0,L)$ for any $t \geq 0$, respectively. Given the original β, the candidates' payoffs under σ' and β satisfy $\bar{v}_{A,t}(h_A) = v_{A,t}(0,L)$ and $\bar{v}_{B,t}(h_B) = v_{B,t}(0,L)$.

Since there exists $\bar{t} > 0$ such that σ and σ' coincide for $t \in [0,\bar{t})$ as in the proof of part 1, candidate A takes $\{H\}$ and candidate B takes $\{L,H\}$ in σ if

$$v_{A,t}(0,L) < v_{A,t}(H,L) \text{ and } v_{B,t}(0,L) > v_{B,t}(0,H).$$

Again, we have

$$v_{A,t}(0,L) \leq v_{A,t}(H,L) \Leftrightarrow t \leq t^*, \text{ and}$$

$$v_{B,t}(0,L) \geq v_{B,t}(0,H) \Leftrightarrow t \leq t^{**}. \tag{38}$$

In both of these equivalence relationships, the first inequality holds with equality if and only if the second holds with equality.

By (3), we have $t^{**} > t^*$. Hence, under σ, candidate A takes $\{H\}$ as a unique best response at any history $h_A \in H_{A,t}(0,L)$ for all time $-t < -t^*$ while candidate B stays at $\{L,H\}$ as a unique best response at any history $h_B \in H_{B,t}(0,L)$ for all time $-t \leq -t^*$.

For $-t < -t^*$, we show that candidate A stays at $\{0,L,H\}$ for $h_A \in H_{A,t}(0,L)$ and candidate B stays at $\{L,H\}$ for $h_B \in H_{B,t}(0,L)$. First, we show that this is true for $-t$ close to $-t^*$:

Lemma 23 *Under σ, there exists $\varepsilon > 0$ such that, for any $-t \in (-(t^*+\varepsilon), -t^*)$, candidate A chooses $\{0,L,H\}$ for each $h_A \in H_{A,t}(0,L)$ and candidate B chooses $\{L,H\}$ for $h_B \in H_{B,t}(0,L)$.*

Proof. Since $t^{**} > t^*$ and $v_{B,t}(0,L) > v_{B,t}(0,H)$ if $t < t^{**}$, by the continuity of the continuation payoff in time, there exists $\varepsilon_B > 0$ such that, for $h_B \in H_{B,t}(0,L)$ for any $-t \in (-(t^*+\varepsilon_B), -t^*)$, candidate B strictly prefers staying at $\{L,H\}$ at $-t$.

Given this incentive, we consider candidate A's incentive for each $h_A \in H_{A,t}(0, L)$ at $-t \in (-(t^* + \varepsilon_B), -t^*)$. There are the following four cases:

1. Suppose that each of the two candidates receives an opportunity in the time interval $[-t, -t^*)$. In this case, if candidate A always stays at $\{0, L, H\}$ at $-t$, then $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$ because candidate B stays at $\{L, H\}$. Otherwise, $h_A \in H_{A,t^*}(H, H)$ is realized. Since $v_{A,t^*}(0, L) = v_{A,t^*}(H, L) > v_{A,t^*}(H, H)$, she strictly prefers staying at $\{0, L, H\}$.

2. Suppose only candidate B has an opportunity in the time interval $[-t, -t^*)$. In this case, if candidate A stays at $\{0, L, H\}$ at $-t$, $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$ because candidate B stays at $\{L, H\}$. Otherwise, $h_A \in H_{A,t^*}(H, H)$ is realized. Again, she strictly prefers staying at $\{0, L, H\}$.

3. Suppose only candidate A has an opportunity in the time interval $[-t, -t^*)$. In this case, if candidate A always stays at $\{0, L, H\}$, then $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$ because candidate B stays at $\{L, H\}$. Otherwise, $h_A \in H_{A,t^*}(H, L)$ is realized. Since $v_{A,t^*}(0, L) = v_{A,t^*}(H, L)$, she weakly prefers staying at $\{0, L, H\}$ all the time.

4. Suppose no candidate receives an opportunity in the time interval $[-t, -t^*)$. In this case, if candidate A stays at $\{0, L, H\}$ at $-t$, then $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_A \in H_{A,t^*}(H, L)$ is realized. Since $v_{A,t^*}(0, L) = v_{A,t^*}(H, L)$, she weakly prefers staying at $\{0, L, H\}$.

In total, candidate A strictly prefers to choose $\{0, L, H\}$ for each $h_A \in H_{A,t}(0, L)$ at $-t \in (-(t^* + \varepsilon_B), -t^*)$. ■

Given this lemma, we can show the following result:

Lemma 24 *Under σ, for each $-t < -t^*$, candidate A stays at $\{0, L, H\}$ for $h_A \in H_{A,t}(0, L)$ and candidate B stays at $\{L, H\}$ for $h_B \in H_{B,t}(0, L)$. Hence, for $-t < -t^*$, we have*

$$v_{A,t}(h_A) = v_{A,t^*}(0, L) \text{ for } h_A \in H_{A,t}(0, L);$$
$$v_{B,t}(h_B) = v_{B,t^*}(0, L) \text{ for } h_B \in H_{B,t}(0, L).$$

Together with (34), defining

$$\tilde{v}_{i,t}(0, L) := \begin{cases} v_{i,t}(0, L) & \text{for } t \in [0, t^*] \\ v_{i,t^*}(0, L) & \text{for } t \in [t^*, \infty) \end{cases},$$

we have

$$v_{i,t}(h_i) = \tilde{v}_{i,t}(0, L) \text{ for each } i = A, B,\ t \in [0, \infty), \text{ and } h_i \in H_{i,t}(0, L). \tag{39}$$

Proof. Given Lemma 23, it suffices to prove the following claim: For each $-t < -t^*$, given the continuation strategy that candidate A stays at $\{0, L, H\}$ for $h_A \in H_{A,t'}(0, L)$ with $-t' \in (-t, -t^*)$ and candidate B stays at $\{L, H\}$ for $h_B \in H_{B,t'}(0, L)$ with $-t' \in (-t, -t^*)$, candidate A stays at $\{0, L, H\}$ for $h_A \in H_{A,t}(0, L)$ and candidate B stays at $\{L, H\}$ for $h_B \in H_{B,t}(0, L)$. To see why this claim is true, for candidate A, consider the following four scenarios: Since $v_{A,t}(H, L) > v_{A,t}(L, L)$ for each t, she strictly prefers $\{H\}$ to $\{0, L\}$ under the given history. Hence, below we compare $\{0, L, H\}$ and $\{H\}$.

1. Suppose each of the two candidates receives an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate A stays at $\{0, L, H\}$ at $-t$, then $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_A \in H_{A,t^*}(H, H)$ is realized. Since $v_{A,t^*}(0, L) = v_{A,t^*}(H, L) > v_{A,t^*}(H, H)$, she strictly prefers staying at $\{0, L, H\}$.

2. Suppose only candidate B has an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate A stays at $\{0, L, H\}$ at $-t$, $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_A \in H_{A,t^*}(H, H)$ is realized. Again, she strictly prefers staying at $\{0, L, H\}$.

3. Suppose only candidate A has an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate A always stays at $\{0, L, H\}$, then $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_A \in H_{A,t^*}(H, L)$ is realized. Since $v_{A,t^*}(0, L) = v_{A,t^*}(H, L)$, she weakly prefers staying at $\{0, L, H\}$.

4. Suppose no candidate receives an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate A stays at $\{0, L, H\}$ at $-t$, then $h_A \in H_{A,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_A \in H_{A,t^*}(H, L)$ is realized. Since $v_{A,t^*}(0, L) = v_{A,t^*}(H, L)$, she weakly prefers staying at $\{0, L, H\}$.

In total, staying at $\{0, L, H\}$ is uniquely optimal, as desired.

Similarly, for candidate B, consider the following four scenarios:

1. Suppose each of the two candidates has an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate B stays at $\{L, H\}$ at $-t$, then $h_B \in H_{B,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_B \in H_{B,t^*}(H, H)$ is realized. Since $v_{B,t^*}(0, H) > v_{B,t^*}(H, H)$ and $v_{B,t^*}(0, L) > v_{B,t^*}(0, H)$ at $-t^*$ (recall that $t^* < t^{**}$ and $v_{B,\tau}(0, L) > v_{B,\tau}(0, H)$ if $\tau < t^{**}$, and hence $v_{B,t^*}(0, L) > v_{B,t^*}(0, H)$), she strictly prefers staying at $\{L, H\}$.

2. Suppose only candidate A has an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate B stays at $\{L, H\}$ at $-t$, then $h_B \in H_{B,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_B \in H_{B,t^*}(H, H)$ is realized. Again, she strictly prefers staying at $\{L, H\}$.

3. Suppose only candidate B has an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate B always stays at $\{L, H\}$ at $-t$, then $h_B \in H_{B,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_B \in H_{B,t^*}(0, H)$ is realized. She strictly prefers staying at $\{L, H\}$ due to (38) and $t^* < t^{**}$.

4. Suppose no candidate receives an opportunity in the time interval $(-t, -t^*)$. In this case, if candidate B stays at $\{L, H\}$ at $-t$, then $h_B \in H_{B,t^*}(0, L)$ will be realized at $-t^*$. Otherwise, $h_B \in H_{B,t^*}(0, H)$ is realized. She strictly prefers staying at $\{L, H\}$ due to (38).

In total, staying at $\{L, H\}$ is uniquely optimal. ■

Finally, consider $h_A \in H_{A,t}(0, 0)$ and $h_B \in H_{B,t}(0, 0)$:

Lemma 25 *Under σ, for each $-t$, candidates A and B take $\{L, H\}$ for $h_A \in H_{A,t}(0, 0)$ and $h_B \in H_{B,t}(0, 0)$, respectively.*

Proof. Let σ'' be a strategy profile such that, for each candidate $i = A, B$, if the history for i is h_i, i chooses $\{L, H\}$ if $h_i \in H_{i,t}(0, 0)$, and follow σ if $h_i \notin H_{i,t}(0, 0)$. Given the original β, for each $i = A, B$, the candidates' payoffs under σ'' and β can be written as:

$$v_{i,t}(0, 0) := \begin{cases} \frac{1}{2}\alpha + (1-\alpha)(H + (H-L)\lambda t) e^{-\lambda t} & \text{for } t \in [0, t^*] \\ \frac{1}{2}\alpha + (1-\alpha) e^{-1}\left[2H - L + \left(1 - e^{-2(\lambda t - 1)}\right)(H - L)\right] & \text{for } t \in [t^*, \infty) \end{cases}$$

for each $h_i \in H_i(0,0)$. Here, $v_{i,t}(0,0)$ is calculated as follows: For each $t \leq t^*$,

$$v_{i,t}(0,0) = \int_0^t 2\lambda e^{-2\lambda\tau} \frac{v_{A,\tau}(L,0) + V_{A,\tau}(0,L)}{2} d\tau + e^{-2\lambda t}\left(\frac{\alpha}{2} + (1-\alpha) H\right).$$

For $t > t^*$, by Lemma 24, we have

$$v_{A,t}(0,0) = e^{-2\lambda(t-t^*)} v_{A,t^*}(0,0) + \left(1 - e^{-2\lambda(t-t^*)}\right)\left[\frac{1}{2} v_{A,t^*}(L,0) + \frac{1}{2} v_{A,t^*}(0,L)\right].$$

As in part 1, there exists $\bar{t} > 0$ such that, for each $t \in [0, \bar{t})$, at any histories in $H_{A,t}(0,0)$ and $H_{B,t}(0,0)$, σ and σ'' coincide. Together with Lemma 17 and (39), both candidates A and B take $\{L, H\}$ at histories $h_A \in H_{A,t}(0,0)$ and $h_B \in H_{B,t}(0,0)$ under σ if

$$\begin{aligned} \tilde{v}_{A,t}(L,0) &> \max\{v_{A,t}(H,0), v_{A,t}(0,0)\}, \text{ and} \\ \tilde{v}_{B,t}(0,L) &> \max\{v_{B,t}(0,H), v_{B,t}(0,0)\}. \end{aligned}$$

By symmetry, we focus on $\tilde{v}_{B,t}(0,L) > \max\{v_{B,t}(0,H), v_{B,t}(0,0)\}$. By Lemma 24, (36), and $t^* < t^{**}$, we have $\tilde{v}_{B,t}(0,L) > v_{B,t}(0,H)$ for each $t \geq 0$. Hence we are left to show that, for each $t \geq 0$, $\tilde{v}_{B,t}(0,L) > v_{B,t}(0,0)$.

For $-t \geq -t^*$, since $\tilde{v}_{B,t}(0,L) = v_{B,t}(0,0)$, the same proof as (37) implies that candidate B strictly prefers $\{L, H\}$ to $\{0, L, H\}$ for $h_B \in H_{B,t}(0,0)$.

Moreover, $v_{B,t}(0,0)$ is strictly increasing in t and we have

$$v_{B,t}(0,0) < \lim_{t\to\infty} v_{B,t}(0,0) = \frac{1}{2}\alpha + (1-\alpha) e^{-1}(2H - L).$$

Thus, for each $-t < -t^*$, we have

$$\begin{aligned} &\tilde{v}_{B,t}(0,L) - v_{B,t}(0,0) \\ &> v_{B,t^*}(0,L) - \lim_{t\to\infty} v_{B,t}(0,0) \\ &= \frac{L(2\alpha(L+1) - (H+L)(1-\alpha))}{H+L} e^{-1} \geq 0. \end{aligned}$$

Therefore, candidates B and A (by symmetry) prefer $\{L, H\}$ to $\{0, L, H\}$ for each time $-t$ and for each $h_B \in H_{B,t}(0,0)$ and $h_A \in H_{A,t}(0,0)$, respectively. ■

M Proofs for Section 3.5

M.1 Proof of Proposition 7

[The "if" part] We will show that the following is a PBE: each candidate i takes some strategy σ_i^* that satisfies the properties described in the statement of the proposition. After the opponent has entered, then σ_i^* is optimal given (4). Hence, we focus on the history in which the opponent has not entered. We consider the continuation payoff of entering and not entering at each $-t$ (given that each candidate follows σ^* in the continuation play). By symmetry, we only consider candidate R's incentive.

Under (σ_i^*, σ_j^*), the payoff from entering at $-t$ is

$$\begin{aligned}\max_x \left\{-x^R + \exp(-\lambda t)\, x + (1-\exp(-\lambda t))(-x)\right\} &= \max_x \left\{-x^R - (1-2\exp(-\lambda t))\, x\right\} \\ &= \begin{cases} -x^R & \text{if } t \geq \frac{1}{\lambda}\ln 2 \\ -x^R - (1-2\exp(-\lambda t))\, x^R & \text{if } t \leq \frac{1}{\lambda}\ln 2 \end{cases}.\end{aligned}$$

Note that the set of maximizers is $\{0\}$ if $t < \frac{1}{\lambda}\ln 2$, $[0, x^R]$ if $t = \frac{1}{\lambda}\ln 2$, and $\{x^R\}$ if $t > \frac{1}{\lambda}\ln 2$.

In contrast, the continuation payoff of not entering at $-t$ under (σ_i^*, σ_j^*) is $-x^R$ because the following three cases are exhaustive:

1. If candidate L enters next by $-\frac{1}{\lambda}\ln 2$, then R obtains a payoff of $-x^R$ since L enters at 0.

2. If candidate R enters next by $-\frac{1}{\lambda}\ln 2$, then R obtains a payoff of $-x^R$ since R enters at 0.

3. If no candidate enters by $-\frac{1}{\lambda}\ln 2$, then R obtains a payoff of $-x^R$ since, under σ^*, the candidates take symmetric strategies for $t \in \left(-\frac{1}{\lambda}\ln 2, 0\right]$ and they never enter at a policy not in $[-x^R, x^R]$.[24]

Hence, for each $-t < -\frac{1}{\lambda}\ln 2$, both entering at 0 and not entering are optimal; for $-t = -\frac{1}{\lambda}\ln 2$, entering at any policy in $[0, x^R]$ and not entering are optimal; and for $-t > -\frac{1}{\lambda}\ln 2$, entering at x^i is optimal. Hence, (σ_i^*, σ_j^*) is a PBE.

[24] If the current time $-t$ satisfies $t \leq \frac{1}{\lambda}\ln 2$, then this is the only case that happens.

[The "only if" part] First we show that, under the assumption of $p = 1$, (v_A, v_B) is constant-sum. To see this, note that, under any PBE, at any history, each candidate enters at a policy in $[-x^R, x^R]$. This is because, if candidate i enters outside of this interval when the opponent j has not entered, then i will lose for sure since the median voter will prefer j's ideal policy than i's committed policy, and j's best responses are not to enter and to enter at x^j. Candidate i can do strictly better by not entering, which with some strictly positive probability leads i to implement x^i. Therefore, the implemented policy is included in $[-x^R, x^R]$. Restricting our attention to $x \in [-x^R, x^R]$, the game is constant-sum because the sum of two candidates' payoffs when the implemented policy is x is $-\left|x + x^R\right| - \left|x - x^R\right| = -2x^R$.

Now, Theorem 3 implies that, at any time $-t$ under any PBE, the continuation payoff of not entering under (σ_i^*, σ_j^*) that we computed in the "if" part is the continuation payoff of entering, and $-x^R$ is the continuation payoff of not entering. Hence, in any PBE, for each $-t < -\frac{1}{\lambda}\ln 2$, both entering at 0 and not entering are the only optimal actions; for $-t = -\frac{1}{\lambda}\ln 2$, entering at any policy in $[0, x^R]$ and not entering are the only optimal actions; and for $-t > -\frac{1}{\lambda}\ln 2$, entering at x^i is uniquely optimal. Hence, any PBE satisfies the conditions given in the statement of the proposition.

M.2 Proof of Lemma 3

We prove Lemma 3 as well as compute the payoff $v_i(\tilde{p}, \text{enter})$.

Since entering at $x = 0$ ensures that the winning policy is 0, candidate R, when she enters, enters at $x \geq 0$. In addition, given the best response function in (4), we can show that it is suboptimal for R to enter at x with $\tilde{p}x^R < x$:

Claim 1 *For each $x > \tilde{p}x^R$, we have $x \notin \arg\max_{\tilde{x}} v_t(\tilde{p}, \tilde{x})$.*

Proof. Suppose that R receives an opportunity at time $-t$ and enters at x, while L still has not entered. Note that, conditional on this event, if L does not enter until time 0, the median voter votes for candidate R if and only if it is better to vote for candidate R with known policy commitment x than unknown type L:

$$p_0^L\left(-\left|x^L\right|\right) + \left(1 - p_0^L\right)(-0) \leq -x,$$

where p_0^L is the voters' posterior about L being extreme at $-t = 0$. Since in equilibrium each type of candidate L will enter after candidate R enters, if candidate L does not enter, then the voters know that L did not receive an opportunity. Thus, $p_0^L = p_t^L$. Hence, candidate R does not win with a policy $x \geq 0$ with $\tilde{p}x^R < x$.

Therefore, R's payoff from entering at such x is

$$\begin{aligned} & \underbrace{\left(1-e^{-\lambda t}\right)}_{\Pr(L \text{ has an opportunity})} \times \underbrace{\left[(1-\tilde{p})\left(-x^R\right)+\tilde{p}\left(-\left|BR_L(x)-x^R\right|\right)\right]}_{R\text{'s utility when } L \text{ has an opportunity after } R \text{ enters at } x} \\ + & \underbrace{e^{-\lambda t}}_{\Pr(L \text{ does not have an opportunity})} \times \underbrace{\left[\tilde{p}\left(-x^R-x^R\right)+(1-\tilde{p})\left(-x^R\right)\right]}_{\substack{R\text{'s utility when } L \text{ does not enter after } R \text{ enters at } x \\ \text{(note that we consider the case in which } L \text{ wins)}}} . \end{aligned}$$

Note that

$$0 \in \arg\max_x \left[(1-\tilde{p})\left(-x^R\right)+\tilde{p}\left(-\left|BR_L(x)-x^R\right|\right)\right]$$

since, once L has an opportunity after R enters at $x > 0$, he enters at 0 if he is normal while he enters at a best response to x given by (4) if he is extreme. Since

$$(1-\tilde{p})\left(-x^R\right)+\tilde{p}\left(-2x^R\right) < -x^R,$$

R's payoff from entering at 0 is:

$$\begin{aligned} & \underbrace{\left(1-e^{-\lambda t}\right)}_{\Pr(L \text{ has an opportunity})} \times \underbrace{\left[(1-\tilde{p})\left(-x^R\right)+\tilde{p}\left(-x^R\right)\right]}_{R\text{'s utility when } L \text{ has an opportunity after } R \text{ enters at } 0} \\ + & \underbrace{e^{-\lambda t}}_{\Pr(L \text{ does not have an opportunity})} \times \underbrace{\left(-x^R\right)}_{\text{By entering at 0, } R \text{ ensure } -x^R\text{. Note that the winning policy is guaranteed to be 0}} \\ > & \underbrace{\left(1-e^{-\lambda t}\right)}_{\Pr(L \text{ has an opportunity})} \times \underbrace{\left[(1-\tilde{p})\left(-x^R\right)+\tilde{p}\left(-x^R\right)\right]}_{\substack{\text{An upper bound of } R\text{'s utility when } L \text{ has an opportunity after } R \text{ enters at } x \\ \text{since } L \text{ enters at } BR_L(x) \leq 0}} \\ + & \underbrace{e^{-\lambda t}}_{\Pr(L \text{ does not have an opportunity})} \times \underbrace{\left[\tilde{p}\left(-x^R-x^R\right)+(1-\tilde{p})\left(-x^R\right)\right]}_{\substack{R\text{'s utility when } L \text{ does not enter after } R \text{ enters at } x \\ \text{(note that we consider the case in which } L \text{ wins)}}} , \end{aligned}$$

which is the payoff of entering at x. Hence, it is better to enter at 0 rather than x with $\tilde{p}x^R < x$.

■

Given this claim, we focus on R's entering at $x \in \left[0, \tilde{p}x^R\right]$. When candidate R enters at $x \in \left[0, \tilde{p}x^R\right]$ at $-t$, the following two cases can happen:

1. Candidate L receives an opportunity in the time interval $(-t, 0]$.

   (a) If he is extreme, then he will enter at $-x$ and candidate R receives $-x - x^R$. This happens with probability $\tilde{p}\left(1 - e^{-\lambda t}\right)$.

   (b) If he is normal, then he will enter at 0 and candidate R receives $-x^R$. This happens with probability $(1 - \tilde{p})\left(1 - e^{-\lambda t}\right)$.

2. Candidate L does not receive any opportunity in the time interval $(-t, 0]$. This happens with probability $e^{-\lambda t}$. In such a case, the median voter votes for candidate R given $x \in \left[0, \tilde{p}x^R\right]$.

Therefore, the payoff from entering at x at time $-t$ is

$$\tilde{p}\left(1 - e^{-\lambda t}\right)\left(-x - x^R\right) + (1 - \tilde{p})\left(1 - e^{-\lambda t}\right)\left(-x^R\right) - e^{-\lambda t}\left(x^R - x\right)$$

$$= -x^R + \left(e^{-\lambda t} - \tilde{p}\left(1 - e^{-\lambda t}\right)\right)x. \tag{40}$$

If $e^{-\lambda t} > \tilde{p}\left(1 - e^{-\lambda t}\right)$, then R wants to maximize x, so $x = \tilde{p}x^R$ is uniquely optimal; if $e^{-\lambda t} < \tilde{p}\left(1 - e^{-\lambda t}\right)$, then R wants to minimize x, so $x = 0$ is uniquely optimal; if $e^{-\lambda t} = \tilde{p}\left(1 - e^{-\lambda t}\right)$, then R is indifferent among all $x \in \left[0, \tilde{p}x^R\right]$.

M.3 Proof of Lemma 2

Suppose that normal candidate i has an opportunity at $-t$ when the opponent j has not entered. Since there is a positive probability that j is extreme and he has not received any opportunity, for any strategy of j, there is a positive probability that he is extreme.

If i enters at 0, then she obtains a payoff of 0, which is the maximum feasible payoff of this game. If she enters at $x \neq 0$, then her payoff is strictly less than 0 since, if j is extreme and obtains an opportunity in $-\tau \in (-t, 0]$,then he will enter at a policy that is not 0 and win. If she does not enter, then again her payoff is strictly less than 0 since, if no candidate obtains an opportunity in $-\tau \in (-t, 0]$, then j will win with probability $\frac{1}{2}$ and implement her ideal policy which may not be 0. In total, entering at 0 is the unique optimal strategy for i.

M.4 Proof of Lemma 4

Given $\frac{e^{-\lambda t}}{1-e^{-\lambda t}} \leq p(t)$, Lemma 3 implies that R's payoff from entering is $-x^R$. Hence, we are left to show that R's payoff from not entering is greater than $-x^R$.

Since we focus on symmetric PBE and the implemented policy is in $\left[-x^R, x^R\right]$, if L is extreme, then the extreme candidate R obtains $-x^R$ by symmetry. Hence R's payoff from not entering at $-t$ given $p(t) = \tilde{p}$ is given by

$$\begin{aligned} v_t(\tilde{p}, \text{not}) &= \underbrace{\tilde{p}}_{L \text{ is extreme}} \left(-x^R\right) \\ &\quad + \underbrace{(1-\tilde{p})}_{L \text{ is normal}} \left(\begin{array}{c} \underbrace{\left(1-e^{-\lambda t}\right)}_{L \text{ has an opportunity}} \left(-x^R\right) \\ + \underbrace{e^{-\lambda t}}_{L \text{ does not have an opportunity}} \left(\begin{array}{c} q_t \tilde{v}^R \\ + (1-q_t)\left(-\frac{1}{2}x^R\right) \end{array} \right) \end{array} \right), \end{aligned}$$

where q_t is the conditional probability that R enters at some time in $(-t, 0]$ given that L is normal and does not have an opportunity to enter in the time interval $[-t, 0]$, given the equilibrium strategy σ, and $\tilde{v}^R$ is candidate R's expected payoff conditional on the event that R enters at some timing $(-t, 0]$ and L is normal and does not have an opportunity to enter in the time interval $(-t, 0]$.

Since R enters in $\left[0, x^R\right]$ if she enters, we have $\tilde{v}^R \geq -x^R$. Moreover, $1 - q_t \geq e^{-\lambda t}$ (note that $e^{-\lambda t}$ is the probability that R does not obtain any opportunity in $(-t, 0]$) under any σ, we have $v_t(\tilde{p}, \text{not}) > -x^R$.

M.5 Proof of Proposition 8

For each p and T, fix a symmetric PBE. In the following proof, we consider properties of this symmetric PBE. In fact, there exists a unique strategy profile satisfying such properties, and one can show that it is indeed a PBE, which shows the existence of a PBE.

M.5.1 Equilibrium behavior near the deadline

If the deadline is close, then the probability that candidate L will receive an opportunity is close to zero. By (40), if candidate R enters, then she receives

$$\begin{aligned} & -x^R + \max\left\{\left(e^{-\lambda t} - \tilde{p}\left(1 - e^{-\lambda t}\right)\right)\tilde{p}x^R, 0\right\}\Big|_{t=0} \\ & = -\left(1 - \tilde{p}\right)x^R, \end{aligned}$$

where $\tilde{p}$ is R's posterior probability with which L is extreme.

If candidate R does not enter, then since she wins with probability $\frac{1}{2}$ at $-t = 0$, she receives

$$\begin{aligned} & \frac{1}{2}\left\{\tilde{p}\left(-x^R - x^R\right) + \left(1 - \tilde{p}\right)\left(-x^R\right)\right\} + \frac{1}{2} \times 0 \\ & = -\left(1 - \tilde{p}\right)x^R + \frac{1}{2}\left(1 - 3\tilde{p}\right)x^R. \end{aligned}$$

Therefore, candidate R enters at x with $x = \tilde{p}x^R$ if her posterior $\tilde{p}$ about L being extreme is greater than $\frac{1}{3}$. She does not enter if the belief is less than $\frac{1}{3}$ at $-t = 0$. Intuitively, if the probability that candidate L is extreme is very high, then (i) R's payoff when L wins when L has not entered at time 0 is low since L will pick x^L after the election with a higher probability, and (ii) the median voter votes for R even if R takes a large $x < x^R$. Hence, it is more attractive to enter $x > 0$ so that R can win.

For each p and λ, for sufficient large T, we have $\frac{e^{-\lambda t}}{1-e^{-\lambda t}} \leq p \leq p(t)$ for $-t \in \left[-\frac{T}{2}, -T\right]$ and $p\left(\frac{T}{2}\right) > \frac{1}{3}$. Since $p(0) \geq p\left(\frac{T}{2}\right) > \frac{1}{3}$, candidate R enters at x with $x = p(t)\,x^R$ for sufficiently small t.

M.5.2 Backward Induction

Suppose that candidates enter in the time interval $(-t, 0]$. Then we have $p(\tau) = p(t)$ for each $-\tau \in [-t, 0]$. Hence, if $p(t) = \tilde{p}$ and the candidates enter in the time interval $(-t, 0]$, then R's payoff from entering at $-t$ is

$$-x^R + \left(e^{-\lambda t} - \tilde{p}\left(1 - e^{-\lambda t}\right)\right)\tilde{p}x^R,$$

while the payoff from not entering at $-t$ is

$$\begin{aligned}
v_t(\tilde{p}, \text{not}) &= \underbrace{\tilde{p}}_{L \text{ is extreme}} \left(-x^R\right) \\
&+ \underbrace{(1-\tilde{p})}_{L \text{ is normal}} \left(\begin{array}{c} \underbrace{\left(1-e^{-\lambda t}\right)}_{L \text{ has an opportunity}} \left(-x^R\right) \\ + \underbrace{e^{-\lambda t}}_{L \text{ does not have an opportunity}} \left(\underbrace{\left(1-e^{-\lambda t}\right)}_{R \text{ enters at } \tilde{p}x^R} \left(\tilde{p}x^R - x^R\right) + e^{-\lambda t}\left(-\tfrac{1}{2}x^R\right) \right) \end{array} \right) \\
&= -x^R + (1-\tilde{p}) \left(e^{-\lambda t}\tilde{p}x^R - e^{-2\lambda t}\left(1-\tilde{p}+\frac{1}{2}\right) x^R \right).
\end{aligned}$$

Candidate R enters if

$$-x^R + \left(e^{-\lambda t} - \tilde{p}\left(1-e^{-\lambda t}\right)\right)\tilde{p}x^R \geq -x^R + (1-\tilde{p}) \left(e^{-\lambda t}\tilde{p}x^R - e^{-2\lambda t}\left(1-\tilde{p}+\frac{1}{2}\right) x^R \right)$$

$$\Leftrightarrow \quad e^{-\lambda t} = 2\frac{\tilde{p} - \sqrt{\frac{4\tilde{p}^2+3-5\tilde{p}}{2}}}{5\tilde{p}-2\tilde{p}^2-3}\tilde{p}, \tag{41}$$

where the right-hand side is well defined for any $\tilde{p} \in (0,1)$ because $\frac{4\tilde{p}^2+3-5\tilde{p}}{2} \geq \frac{23}{32}$ and $5\tilde{p}-2\tilde{p}^2-3 < 0$ for $\tilde{p} \in (0,1)$.

Note that, if t satisfies (41), we have $\frac{e^{-\lambda t}}{1-e^{-\lambda t}} - \tilde{p} \geq 0$ (and this holds with equality if and only if $\tilde{p} = 0$).

For each p, fix the smallest t such that candidate R weakly prefers not entering at $-t$ given p, and denote it by $t^*(p)$:

$$e^{-\lambda t^*(p)} = 2\frac{p - \sqrt{\frac{4p^2+3-5p}{2}}}{5p-2p^2-3}p. \tag{42}$$

Then, for each p, $-t < -t^*(p)$ and each h_R^t such that $p(t) = p$ and no candidate has entered at $-t$, candidate R's unique best reply is not to enter. To see why, consider the following two scenarios:

1. If L has an opportunity by $-t^*(p)$, then R's action does not affect R's payoff if L is normal. If L is extreme, R is worse off by entering at px than not entering. In particular, if R enters, then her payoff is $-x^R - px^R$. Suppose next that she does not enter until $-t^*(p)$. Then, if L has not entered by $-t^*(p)$, her payoff is $-x^R$ by symmetry. If he enters, in contrast, her

payoff is $-x^R + (1-e^{-\lambda\tilde{t}})p\left(\tilde{t}\right)x^R - e^{-\lambda\tilde{t}}p\left(\tilde{t}\right)x^R$, where L enters at $-\tilde{t} \in (-t, -t^*(p)]$. Note that

$$\begin{aligned}
-x^R + (1-e^{-\lambda\tilde{t}})p\left(\tilde{t}\right)x^R - e^{-\lambda\tilde{t}}p\left(\tilde{t}\right)x^R &\geq -x^R + p\left(\tilde{t}\right)x^R - 2e^{-\lambda\tilde{t}}p\left(\tilde{t}\right)x^R \\
&\geq -x^R + px^R - 2e^{-\lambda t^*(p)}x^R \\
&> -x^R - px^R.
\end{aligned}$$

The last line holds for the following reason: Straightforward algebra shows that $\frac{p-\sqrt{\frac{4p^2+3-5p}{2}}}{5p-2p^2-3} < \frac{1}{4}$, so by (42), we have $2e^{-\lambda t^*(p)} < p$. Hence, R is strictly better off not entering.

2. If L does not have an opportunity until $-t^*(p)$, candidate R is indifferent between entering and not entering (conditional on the event that L does not have an opportunity until $-t^*(p)$, the belief that L is extreme is constant in the time interval $[-t, -t^*(p)]$).

Since there is a positive probability that L is extreme and has an opportunity to enter, it is uniquely optimal for R not to enter at $-t$.

M.5.3 Equilibrium Dynamics

Fix $p \in (0,1)$ arbitrarily. There exists $\bar{T}_1 < \infty$ such that for all $T \geq \bar{T}_1$, we have that at any $-t \in \left[-\frac{T}{2}, -T\right]$, $\frac{e^{-\lambda t}}{1-e^{-\lambda t}} \leq p \leq p\left(t\right)$ holds and so candidate R does not enter. For such T, in any symmetric PBE, $p\left(t\right)$ evolves according to $\frac{d}{dt}p\left(t\right) = -\lambda p\left(t\right)\left(1-p\left(t\right)\right)$ for $t \in [\frac{T}{2}, T]$ with the initial condition $p(T) = p$ since the normal type always enters. Define $\bar{p}_T : [0,T] \to [0,1]$ by the differential equation $\frac{d}{dt}\bar{p}_T\left(t\right) = -\lambda\bar{p}_T\left(t\right)\left(1-\bar{p}_T\left(t\right)\right)$ with the initial condition $\bar{p}_T(T) = p \in (0,1)$.

We will show that there exists $\bar{T} < \infty$ such that for all $T \geq \bar{T}$, there exists a unique $t \in [0,T]$ such that

$$e^{-\lambda t} = 2\frac{\bar{p}_T\left(t\right) - \sqrt{\frac{4\bar{p}_T(t)^2+3-5\bar{p}_T(t)}{2}}}{5\bar{p}_T\left(t\right) - 2\bar{p}_T\left(t\right)^2 - 3}\bar{p}_T\left(t\right). \tag{43}$$

It will be useful to define $f(q) := 2\frac{q-\sqrt{\frac{4q^2+3-5q}{2}}}{5q-2q^2-3}q$ for $q \in [0,1)$, where we note that this is well defined for $q = 0$ as well.

Proof of Existence:

We first show that the right-hand side of (43) is strictly decreasing in t. Since $\bar{p}_T(t)$ is strictly decreasing in t, it suffices to show that $f(q)$ is strictly increasing in $q \in [0,1)$. Note that

$$\frac{d}{dq}f(q) = \frac{\sqrt{2}}{2}\frac{6-5q}{\left(2q^2-5q+3\right)^2\sqrt{4q^2-5q+3}}\left(6q^2-5q-2\sqrt{2}q\sqrt{4q^2-5q+3}+3\right). \tag{44}$$

Since straightforward algebra implies that, for each $q \in [0,1)$,

$$6q^2-5q-2\sqrt{2}q\sqrt{4q^2-5q+3}+3 \in (0,3),$$

$f(q)$ is strictly increasing in $q \in [0,1)$.

Second, we show that there exists $\bar{T}_2 < \infty$ such that, for each $T \geq \bar{T}_2$, $e^{-\lambda T} < f(\bar{p}_T(T))$ holds. This follows since the left-hand side is strictly decreasing in T and converges to 0 as T goes to infinity, while the right-hand side is fixed at $f(p)$ because $\bar{p}_T(T) = p$, and $f(p) > 0$ because f is strictly increasing and $f(0) = 0$.

Third, we show that, at $t = 0$, $e^{-\lambda\times 0} = 1 > f(\bar{p}_T(0))$. This follows since, given that $f(q)$ is strictly increasing in $q \in [0,1)$, an upper bound of the right-hand side is calculated as $\lim_{q\to 1} f(q) = \frac{1}{2}$.

These three observations establish the existence of $t \in [0,T]$ satisfying (43) for each $T \geq \max\left\{\bar{T}_1, \bar{T}_2\right\}$.

Proof of Uniqueness:

Given the existence, for each $T > \max\left\{\bar{T}_1, \bar{T}_2\right\}$, let $t^{**}(T)$ be the largest $t \in [0,T]$ satisfying (43). To show that the solution is unique, it suffices to show that there exists $\bar{T} \in \left[\max\left\{\bar{T}_1, \bar{T}_2\right\}, \infty\right)$ such that for all $T \geq \bar{T}$, for each $t < t^{**}(T)$,

$$\frac{d}{dt}e^{-\lambda t} < \frac{d}{dt}f\left(\bar{p}_T(t)\right).$$

On the one hand, since $t^{**}(T) \leq \frac{T}{2}$ for each $T \geq \max\left\{\bar{T}_1, \bar{T}_2\right\}$, for each $\varepsilon > 0$, there exists $\bar{T}_3 \in \left[\max\left\{\bar{T}_1, \bar{T}_2\right\}, \infty\right)$ such that for all $T \geq \bar{T}_3$, $\left|\bar{p}_T\left(t^{**}(T)\right) - 1\right| < \varepsilon$. Hence, there exists $\bar{T}_4 \in \left[\max\left\{\bar{T}_1, \bar{T}_2\right\}, \infty\right)$ such that for all $T \geq \bar{T}_4$,

$$\left|f\left(\bar{p}_T\left(t^{**}(T)\right)\right) - \lim_{q\to 1} f(q)\right| < \varepsilon.$$

Noting that $\lim_{q\to 1} f(q) = \frac{1}{2}$, this implies that for each $T \geq \bar{T}_4$, we have $e^{-\lambda t} \geq \frac{1}{2} - \varepsilon$ for each

$t \leq t^{**}(T)$ that is a solution to (43). Since $\frac{d}{dt}e^{-\lambda t} = -\lambda e^{-\lambda t}$ and $\frac{d^2}{dt^2}e^{-\lambda t} = \lambda^2 e^{-\lambda t}$, for each $T \geq \bar{T}_4$ and each $t < t^{**}(T)$, we have

$$\frac{d}{dt}e^{-\lambda t} \leq \left.\frac{d}{dt}e^{-\lambda t}\right|_{t=t^{**}(T)} = -\lambda e^{-\lambda t^{**}(T)} \leq -\lambda\left(\frac{1}{2} - \varepsilon\right).$$

On the other hand, the right-hand side of (43) is decreasing in t. Its derivative is

$$\frac{d}{dt}f(\bar{p}_T(T)) = \left(\left.\frac{d}{dq}f(q)\right|_{q=\bar{p}_T(t)}\right) \cdot \left(\frac{d}{dt}\bar{p}_T(t)\right) = -\left(\left.\frac{d}{dq}f(q)\right|_{q=\bar{p}_T(t)}\right) \cdot \lambda \bar{p}_T(t)(1 - \bar{p}_T(t)).$$

Given (44), straightforward algebra yields $\lim_{q \to 1} \frac{d}{dq}f(q) = \frac{1}{16} < \infty$. Recall that, for each $\varepsilon > 0$, there exists $\bar{T}_3 < \infty$ such that, for each $T \geq \bar{T}_3$ and each $t \leq t^{**}(T)$, we have $\bar{p}_T(t) \geq 1 - \varepsilon$. Hence, for each $\varepsilon > 0$, there exists $\bar{T}_5 < \infty$ such that, for each $T \geq \bar{T}_5$, we have $\frac{d}{dt}f(\bar{p}_T(T)) > -\varepsilon$.

In total, there exists $\bar{T} < \infty$ such that, for each $T \geq \bar{T}$ and each $t < t^{**}(T)$, we have

$$\frac{d}{dt}e^{-\lambda t} < \frac{d}{dt}f(\bar{p}_T(T)).$$

Equilibrium Dynamics:

Since the solution to (43) is unique, together with (42), we have the following: For a sufficiently large T such that the solution for (43) is unique, let $t^*(p, \lambda, T)$ be this unique solution. In any symmetric PBE, for each $-t < -t^*(p, \lambda, T)$, extreme candidates do not enter and $p(t) = \bar{p}_T(t)$; and for each $-t \geq -t^*(p, \lambda, T)$, extreme candidate R (or L) enters at $\bar{p}_T(t^*(p, \lambda, T))x^R$ (or $\bar{p}_T(t^*(p, \lambda, T))x^L$) and $p(t) = \bar{p}_T(t^*(p, \lambda, T))$ for each t with $-t \geq -t^*(p, \lambda, T)$.

Finally, note that, for any $p \in (0, 1)$, for sufficiently large $T < \infty$, in any symmetric PBE, extreme candidates do not enter for any $-t \in [-T, \frac{T}{2}]$. Hence, $p(\frac{T}{2}) = \bar{p}_T(\frac{T}{2}) \to 1$ as $T \to \infty$. Since $\lim_{q \to 1} f(q) = \frac{1}{2}$, for any $p \in (0, 1)$ and $\lambda > 0$, we have

$$\lim_{T \to \infty} |\lambda t^*(p, \lambda, T) - \ln 2| = 0.$$

N Proofs for Appendix I.4

N.1 Proof of Proposition 14

Note that, in any PBE, S enters and receives a payoff of 1 if S can move after W enters. In addition, by the same proof as the one for Proposition 2, there exists $\bar{t} > 0$ such that for all time $-t \in (-\bar{t}, 0]$, $Q_t = (N, E)$. Below, we consider the transition of Q_t in the following two cases: $\beta \geq \gamma$ and $\beta < \gamma$.

Since we assume that the positions that S and W enter do not depend on the timing of entry, Assumption 1 of Section 4.1 is satisfied. In addition, Assumption 2 is satisfied. Also, Assumption 3 holds because $v_i^{BR_j} < \sup_{x_i \in \mathcal{X}_i} v_i(x_i, X)$ for each i. Finally, first-mover disadvantage for $i = W$ holds. Moreover, since S does not enter and W enters near the deadline in any PBE, we are in Case 3 for Theorem 1.

N.1.1 Case 1: $\beta \geq \gamma$

Fix a PBE. For all $-t$, W does not enter after S enters if $\beta > \gamma$. If $\beta = \gamma$, then W is indifferent. The following analysis goes through when $\beta = \gamma$ regardless of W's strategy after S enters.

First, let us consider S's incentive. At time $-t$, if W has not entered, S's payoff is $1 - \beta$ if S enters; if S does not enter, then her payoff is $1 - (1 - \alpha) \int_0^t e^{-\lambda_S \tau} \lambda_W e^{-\lambda_W (t-\tau)} d\tau$, given $Q_\tau = (N, E)$ for all $-\tau \in (-t, 0)$. Hence, $\hat{t}_S$ in the notation of Section 4.1 is characterized by the smallest t satisfying the following equation.

$$1 - \beta = 1 - (1 - \alpha) \int_0^t e^{-\lambda_S \tau} \lambda_W e^{-\lambda_W (t-\tau)} d\tau.$$

Defining

$$f_S(t) := \begin{cases} \frac{1}{1-r} \left(e^{-\lambda_S t} - e^{-\lambda_W t} \right) - \frac{\beta}{1-\alpha} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta}{1-\alpha} & \text{if } r = 1 \end{cases}, \tag{45}$$

$\hat{t}_S$ is the smallest positive solution for $f_S(t) = 0$. Recall that we have defined t_S to be such a solution for $f_S(t) = 0$ in Section I.4. Hence, $\hat{t}_S = t_S$. If there is no solution, then we define $t_S = \infty$. Note that the function f_S is continuous, so the smallest positive solution always exists or there is no solution.

Second, let us consider W's incentive. At time $-t$, if S has not entered, W's payoff is

$(1-\alpha)\,e^{-\lambda_S t}$ if W enters; if W does not enter, then his payoff is $(1-\alpha)\int_0^t e^{-\lambda_S \tau}\lambda e^{-\lambda_W(t-\tau)}d\tau$, given $Q_\tau = (N,E)$ for $-\tau \in (-t,0)$. Hence, $\hat{t}_W$ is characterized by the smallest $t>0$ satisfying the following equation.

$$(1-\alpha)\,e^{-\lambda_S t} = (1-\alpha)\int_0^t e^{-\lambda_S \tau}\lambda_W e^{-\lambda_W(t-\tau)}d\tau$$

$\Leftrightarrow$

$$f_W(t)=0, \text{ where } f_W(t) := \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1 \end{cases}.$$

Recall that we define t_W as the smallest solution for $f_W(t)=0$ in Section I.4:

$$\begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t_W} - e^{-\lambda_W t_W}\right) - e^{-\lambda_S t_W} = 0 & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t_W} - e^{-\lambda_W t_W} = 0 & \text{if } r = 1 \end{cases}. \tag{46}$$

Hence, $\hat{t}_W = t_W$. Since $f_W(t)$ is continuous, negative for sufficiently small $t>0$, and positive for sufficiently large t, the smallest positive t such that $f_W(t)=0$ exists.

The transition of Q_t depends on the relationship between t_S and t_W.

Case 1(a): $-t_S < -t_W$. This inequality means that W's cutoff $-t_W$ is closer to the deadline than S's cutoff (if any), $-t_S$. Since Assumption 4 is satisfied, part 1 of Proposition 12 implies that, under the fixed PBE, S does not enter and W enters for $-t \in (-t_W, 0]$, and no candidate enters for $-t \in (-\infty, -t_W)$. Since this argument holds for any PBE, we have the following:

- $Q_t = (N,E)$ for $-t \in (-t_W, 0]$.
- $Q_t = (N,N)$ for $-t \in (-\infty, -t_W)$.

Hence, part 1(a) of Proposition 14 holds.

Case (1)(b): $-t_S > -t_W$. This inequality means that S's cutoff $-t_S$ is closer to the deadline than W's cutoff $-t_W$. By part 2 of Proposition 12, we are in Case 2 for Theorem 1 with $t_0 = t_S + \varepsilon$ for small $\varepsilon > 0$.

Since first-mover disadvantage for W holds, there exists $t_W^* < \infty$ such that $\bar{v}_{W,t_W^*}$ (not) $=$

v_{W,t^*_W}(enter).[25] In particular, since

$$\begin{aligned} v_{W,t^*_W}(\text{enter}) &= (1-\alpha)\, e^{-\lambda_S t}, \\ \bar{v}_{W,t^*_W}(\text{not}) &= \int_0^{t-t_S} e^{-(\lambda_S+\lambda_W)\tau} \left(\lambda_S \beta + \lambda_W (1-\alpha)\, e^{-\lambda_S(t-\tau)}\right) d\tau + e^{-(\lambda_S+\lambda_W)(t-t_S)}\beta, \end{aligned}$$

t^*_W is the smallest $t > 0$ satisfying the following equation.

$$g_W(t) := e^{-(\lambda_S+\lambda_W)(t-t_S)} \left(\frac{1}{1+r}\beta - (1-\alpha)\, e^{-\lambda_S t_S}\right) + \frac{r}{1+r}\beta = 0.$$

Here, we use the fact that S is indifferent between entering and not entering at $-t_S$, which implies that her payoff at $-t_S$ is $1-\beta$ if no candidates have entered by $-t_S$, and thus W's payoff is β if no candidates have entered by $-t_S$.

In contrast, S always prefers entering at $-t < -t_S$ for the following reason. Suppose W enters at all times $-t \in (-t^*, 0]$ and does not enter at all times $-t \in (-\infty, -t^*)$.

For $-t \geq -t^*$, since S (weakly) prefers entering at $-t_S$, if W has not entered by $-t_S$, S's payoff at $-t_S$ is no more than $1-\beta$. Even if S enters by $-t_S$, S gets at most $1-\beta$. That is, W can guarantee β if W does not enter until $-t_S$. The fact that W (strictly) prefers entering implies that W's payoff when W can enter before S is more than β. Therefore, S's payoff when W can enter before S is less than $1-\beta$. In contrast, by entering, S can guarantee a payoff of $1-\beta$. Hence, entering is S's strict best response at $-t \geq -t^*$.

Moreover, since (i) W does not enter before S enters for $-t \in (-\infty, -t^*_W)$ and (ii) W does not enter after S enters, entering is S's strict best response at all times $-t$ (even if W does not enter).

Given the above characterization, under the fixed PBE, S does not enter for $-t \in (-t_S, 0]$ and enters for $-t \in (-\infty, -t_S)$, while W enters for $-t \in (-t^*_W, 0]$ and does not enter for $-t \in (-\infty, -t^*_W)$. Since this argument holds for any PBE, we have the following transition of Q_t:

- $Q_t = (N, E)$ for $-t \in (-t_S, 0]$.
- $Q_t = (E, E)$ for $-t \in (-t^*_W, -t_S)$.
- $Q_t = (E, N)$ for $-t \in (-\infty, -t^*_W)$.

[25] This notation of t^*_W is introduced in t^*_W in Section 4.1.

Hence, part 1(b) of Proposition 14 holds.

N.1.2 Case 2: $\gamma > \beta$

Fix a PBE. For all $-t$, W enters after S enters. The continuation payoff profile when only S enters at $-t$ is given by $(1-\beta_t, \beta_t)$ with

$$\beta_t = \beta + \left(1 - e^{-\lambda_W t}\right)(\gamma - \beta). \tag{47}$$

When we replace β with β_t in (45), the analysis for the case with $\beta \geq \gamma$ implies the following: $\hat{t}_S$ in the notation of Section 4.1 is characterized by the smallest solution for $f_S(t) = 0$, where

$$f_S(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta_t}{1-\alpha} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta_t}{1-\alpha} & \text{if } r = 1 \end{cases}. \tag{48}$$

Recall that we define t_S as the smaller positive solution for $f_S(t) = 0$ in Section I.4. If there is no solution, then we define $t_S = \infty$.

In contrast, $\hat{t}_W$ is characterized by the smallest positive solution for $f_W(t) = 0$, where

$$f_W(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1 \end{cases}.$$

Recall that we define t_W as the smallest solution for $f_W(t) = 0$ in Section I.4:

$$\begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t_W} - e^{-\lambda_W t_W}\right) - e^{-\lambda_S t_W} = 0 & \text{if } r \neq 1, \\ \lambda_W t e^{-\lambda_W t_W} - e^{-\lambda_W t_W} = 0 & \text{if } r = 1. \end{cases}$$

Since $f_W(t)$ is continuous, negative for sufficiently small t, and positive for sufficiently large t, there exists the smallest t such that $f_W(t) = 0$.

The equilibrium dynamics depend on the relationship between t_S and t_W.

Case 2(a): $-t_S < -t_W$. By the same proof as Case 1(a), we can show that, under the fixed PBE, S does not enter and W enters for $-t \in (-t_W, 0]$, and no candidate enters for $-t \in (-\infty, -t_W)$. Since this argument holds for any PBE, we have the following transition of Q_t:

- $Q_t = (N, E)$ for $-t \in (-t_W, 0]$.

- $Q_t = (N, N)$ for $-t \in (-\infty, -t_W)$.

Hence, part 2(a) of Proposition 14 holds.

Case 2(b): $-t_S > -t_W$. This inequality means that S's cutoff $-t_S$ is closer to the deadline than W's cutoff $-t_W$. By part 2 of Proposition 12, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, we are in Case 2 for Theorem 1 with $t_0 = t_S + \varepsilon$. Moreover, since first-mover disadvantage for $i = W$ holds, Proposition 10 pins down the dynamics under the fixed PBE, and since the argument holds for any PBE, we have that the transition of Q_t can be one of the following:

Case 2(b)(i)

- $Q_t = (N, E)$ for $-t \in (-t_S, 0]$.

- $Q_t = (E, E)$ for $-t \in (-t_W^*, -t_S)$.[26]

- $Q_t = (E, N)$ for $-t \in (-\infty, -t_W^*)$.

Case 2(b)(ii)

- $Q_t = (N, E)$ for $-t \in (-t_S, 0]$.

- $Q_t = (E, E)$ for $-t \in (-t_W^*, -t_S)$.

- $Q_t = (E, N)$ for $-t \in (-t_S^{**}, -t_W^*)$.

- $Q_t = (N, N)$ for $-t \in (-\infty, -t_S^{**})$.

Case 2(b)(iii)

- $Q_t = (N, E)$ for $-t \in (-t_S, 0]$.

- $Q_t = (E, E)$ for $-t \in (-t_S^*, -t_S)$.[27]

- $Q_t = (N, E)$ for $-t \in (-t_W^{**}, -t_S^*)$.

[26] This notation of t_W^* is introduced in Section 4.1.
[27] This notation of t_S^* is introduced in Section 4.1.

- $Q_t = (N, N)$ for $-t \in (-\infty, -t_W^{**})$.

We can show that there exists $\bar{r} \leq 1$ such that Case 2(b)(i) is not the case if and only if $r \geq \bar{r}$. To see why, we derive the differential equation that characterizes the transition. Let x_t be W's continuation payoff at time $-t$ when W has entered and S has not entered at $-t$; let y_t be W's continuation payoff at time $-t$ when W has not entered and S has entered at $-t$; and let z_t be W's continuation payoff at time $-t$ when no candidate has entered at $-t$.

Suppose x_t, y_t, and z_t satisfy the following differential equations:

$$\frac{dx_t}{dt} = \lambda_S \left(0 - x_t\right), \tag{49}$$
$$\frac{dy_t}{dt} = \lambda_W \max\left\{\gamma - y_t, 0\right\}, \tag{50}$$
$$\frac{dz_t}{dt} = \lambda_W \max\left\{x_t - z_t, 0\right\} + \lambda_S \min\left\{y_t - z_t, 0\right\}, \tag{51}$$

with the following condition:

$$x_0 = 1 - \alpha, y_0 = \beta, z_0 = 0.$$

Since this system of ordinary differential equations satisfies Lipschitz continuity, there exists a solution. Such a solution is equilibrium payoffs for the following reasons: Equation (49) means that whenever S can enter after W enters, W's payoff is 0. Equation (50) means that when W can enter after S enters, W enters if and only if his payoff for entering, γ, is greater than the payoff for not entering, y_t. In addition, the first term of (51) means that when W can enter, W enters if and only if his payoff for entering, x_t, is greater than the payoff for not entering, z_t. The second term of (51) means that when S can enter, S enters if and only if her payoff for entering, $1 - y_t$, is greater than her payoff for not entering, $1 - z_t$ (that is, y_t is smaller than z_t). Since we have shown the uniqueness of the value function in Proposition 10, the solution for the system of (49), (50), and (51) is the unique equilibrium payoffs.

To show that there exists $\bar{r} \leq 1$ such that $r \geq \bar{r}$ if and only if there exists $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$, we prove the following three claims:

1. [$\bar{r} \leq 1$] For $r \geq 1$, there exists $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$.

2. [cutoff from below] If there does not exist $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all

$-t \in (-\infty, -t_S^{**})$ for (λ_S, λ_W), then such t_S^{**} does not exist for (λ_S', λ_W) with $\lambda_S' < \lambda_S$.

3. [cutoff from above] If there exists $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$ for (λ_S, λ_W), then such t_S^{**} exists for (λ_S', λ_W) with $\lambda_S' > \lambda_S$.

[Proof of "$\bar{r} \leq 1$"] To analyze the conditions under which there exists $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$, let us consider a sufficiently large $t \geq t_W^{**}$. Since $x_t \geq z_t$ if and only if $t \leq t_W^{**}$, as long as $y_t \leq z_t$, we have $\frac{dz_t}{dt} = \lambda_S (y_t - z_t)$. Since $\frac{dz_t}{dt} + \lambda_S z_t = \lambda_S y_t$, we have

$$e^{\lambda_S t} z_t = C + \int_a^t e^{\lambda_S \tau} \lambda_s y_\tau d\tau, \tag{52}$$

where a is the supremum of τ with $x_\tau \geq z_\tau$, and C is determined by the condition $x_a = z_a$. As we have shown above, a is finite, and so is C.

To show that we have $z_t < y_t$ for sufficiently large t for each (λ_S, λ_W) with $\lambda_S \geq \lambda_W$, we consider the following two cases: $r > 1$ and $r = 1$. Suppose first that $r > 1$. The second term of (52) can be explicitly written as follows:

$$\begin{aligned} \int_a^t e^{\lambda_S \tau} \lambda_s y_\tau d\tau &= \int e^{\lambda_S \tau} \lambda_s \left(\beta + \left(1 - e^{-\lambda_W \tau}\right)(\gamma - \beta)\right) d\tau \\ &= \frac{r}{1-r} (\gamma - \beta) e^{(\lambda_S - \lambda_W)t} + \gamma e^{\lambda_S t} \\ &\quad - \frac{r}{1-r} (\gamma - \beta) e^{(\lambda_S - \lambda_W)a} - \gamma e^{\lambda_S a}. \end{aligned}$$

Hence, the payoff z_t is characterized as follows:

$$z_t = C_a e^{-\lambda_S t} + \gamma + \frac{r}{1-r} (\gamma - \beta) e^{-\lambda_W t}, \tag{53}$$

with

$$C_a = C - \frac{r}{1-r} (\gamma - \beta) e^{(\lambda_S - \lambda_W)a} - \gamma e^{\lambda_S a}.$$

In contrast, the payoff y_t is characterized as follows:

$$y_t = \beta + \left(1 - e^{-\lambda_W t}\right)(\gamma - \beta).$$

Therefore, the difference between z_t and y_t (as long as $y_t \leq z_t$) is:

$$z_t - y_t = C_a e^{-\lambda_S t} + \frac{1}{1-r} \left(\gamma - \beta\right) e^{-\lambda_W t}.$$

As a result, whether there exists $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$ or not depends on

$$\lim_{t \to \infty} \left(C_a e^{-\lambda_S t} + \frac{1}{1-r} \left(\gamma - \beta\right) e^{-\lambda_W t} \right). \tag{54}$$

If $r > 1$, there exists $\bar{t} < \infty$ such that for all $t > \bar{t}$, the second term of (54) dominates. Since $r > 1$ for this case, there exists $\hat{t} < \infty$ such that for all $t > \hat{t}$, (54) is negative. That is, there exists $-t_S^{**}$ such that S does not enter for $-t \in (-\infty, -t_S^{**})$.

We now consider the case with $r = 1$. In this case, we can write $\lambda_S = \lambda_W = \lambda$. On the one hand, the second term of (52) can be explicitly written as follows:

$$\int_a^t e^{\lambda_S \tau} \lambda_s y_\tau d\tau = -\lambda \left(t - a\right) \left(\gamma - \beta\right) + \gamma \left(e^{\lambda t} - e^{\lambda a}\right).$$

Hence, the payoff z_t is characterized as

$$z_t = \gamma + e^{-\lambda t} \left(C - \lambda \left(t - a\right) \left(\gamma - \beta\right) - \gamma e^{-\lambda a}\right).$$

On the other hand, again, the payoff y_t is characterized as

$$y_t = \gamma + e^{-\lambda t} \left(\beta - \gamma\right).$$

Therefore, the difference between z_t and y_t (as long as $y_t \leq z_t$) is:

$$z_t - y_t = e^{-\lambda t} \left(C - \lambda \left(t - a\right) \left(\gamma - \beta\right) - \gamma e^{-\lambda a} - \left(\beta - \gamma\right)\right).$$

There exists $\bar{t} < \infty$ such that for all $t > \bar{t}$, the term $-\lambda \left(t - a\right) \left(\gamma - \beta\right)$ dominates the other terms in the parentheses, and so $z_t - y_t < 0$. That is, there exists $-t_S^{**}$ such that S does not enter for $-t \in (-\infty, -t_S^{**})$, as stated in part 2(b)ii of Proposition 14.

[Proof of "cutoff from below"] We show that, if there does not exist $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$ for (λ_S, λ_W), then such t_S^{**} does not exist for (λ'_S, λ_W) with

$\lambda'_S < \lambda_S$.

To show this monotonicity, we first arbitrarily fix λ_W. Note that y_t is independent of λ_S. Let $x_t(\lambda_S)$ and $z_t(\lambda_S)$ be the values of x_t and z_t respectively, given λ_S for the fixed λ_W. There exists $\bar{t} > 0$ such that for all $t \in (0, \bar{t})$, $z_t(\lambda_S) < z_t(\lambda'_S)$. Define $t^* \equiv \inf_t \{z_t(\lambda_S) \geq z_t(\lambda'_S), t > 0\} \in \mathbb{R}_{++} \cup \{+\infty\}$.

If $t^* = +\infty$, then we have $z_t(\lambda_S) \leq z_t(\lambda'_S)$ for all $\lambda'_S < \lambda_S$. Since y_t is independent of λ_S, the proof is complete in this case. Hence, we concentrate on the case with $t^* < \infty$ and will derive a contradiction.

At $-t^*$, it must be the case that $z_{t^*}(\lambda_S) = z_{t^*}(\lambda'_S)$ and $\dot{z}_{t^*}(\lambda_S) \geq \dot{z}_{t^*}(\lambda'_S)$.[28] From $z_{t^*}(\lambda_S) = z_{t^*}(\lambda'_S)$, we have

$$\begin{aligned}
\dot{z}_{t^*}(\lambda_S) &= \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} + \lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}, \\
\dot{z}_{t^*}(\lambda'_S) &= \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda'_S), 0\right\} + \lambda'_S \min\left\{y_{t^*} - z_{t^*}(\lambda'_S), 0\right\} \\
&= \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} + \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.
\end{aligned}$$

Note that, by definition, we have $x_{t^*}(\lambda'_S) > x_{t^*}(\lambda_S)$. Given this inequality, the following two cases are possible:

1. If $x_{t^*}(\lambda_S) \geq z_{t^*}(\lambda_S)$, then we have $\lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} < \lambda_W \max\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\}$. In addition, we have $\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} \leq \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}$. Hence, we have $\dot{z}_{t^*}(\lambda'_S) > \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

2. If $x_{t^*}(\lambda_S) < z_{t^*}(\lambda_S)$, then we consider the following two subcases:

   (a) If $y_{t^*} > z_{t^*}(\lambda_S)$, then we have

$$\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} < \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.$$

[28] The first equality follows from the continuity of z_t with respect to t. The second inequality follows from the first equality and the definition of the derivative: For sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\dot{z}_{t^*}(\lambda_S) &\approx \frac{z_{t^*}(\lambda_S) - z_{t^*-\varepsilon}(\lambda_S)}{\varepsilon}, \\
\dot{z}_{t^*}(\lambda'_S) &\approx \frac{z_{t^*}(\lambda'_S) - z_{t^*-\varepsilon}(\lambda'_S)}{\varepsilon} = \frac{z_{t^*}(\lambda_S) - z_{t^*-\varepsilon}(\lambda'_S)}{\varepsilon}.
\end{aligned}$$

Since $z_{t^*-\varepsilon}(\lambda'_S) > z_{t^*-\varepsilon}(\lambda_S)$, it follows that $\dot{z}_{t^*}(\lambda_S) \geq \dot{z}_{t^*}(\lambda'_S)$.

Since we have

$$\lambda_W \max\left\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\right\} \leq \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\},$$

we have $\dot{z}_{t^*}(\lambda'_S) > \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

(b) If $y_{t^*} \leq z_{t^*}(\lambda_S)$, then we have

$$\lambda_W \max\left\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\right\} = 0$$

and

$$\lambda_S \min\left\{y_{t^*} - z_{t^*}(\lambda_S), 0\right\} = 0.$$

Therefore, $\dot{z}_{t^*}(\lambda_S) = 0$. For $t > t^*$, since $x_t(\lambda_S)$ is decreasing in t and y_t is increasing in t, we have $\dot{z}_t(\lambda_S) = 0$. Together with $y_{t^*} \leq z_{t^*}(\lambda_S)$, we have $y_t < z_t(\lambda_S)$ so S does not enter for $-t < -t^*$. This contradicts the assumption that there does not exist $t_S^{**} \in (t_S, \infty)$ such that S does not enter for all $-t \in (-\infty, -t_S^{**})$ for λ_S.

[Proof of "cutoff from above"] We prove that, if there exists $\bar{T} < \infty$ such that S does not enter for any $t > \bar{T}$ for a pair (λ_S, λ_W), then for any pair (λ'_S, λ_W) with $\lambda'_S > \lambda_S$, there exists $\bar{T}' < \infty$ such that S does not enter for any $t > \bar{T}'$.

This proof is symmetric to the one for "cutoff from below." We first arbitrarily fix λ_W. Again, y_t is independent of λ_S. Let $x_t(\lambda_S)$ and $z_t(\lambda_S)$ be the value of x_t and z_t, respectively, given λ_S for the fixed λ_W. There exists $\bar{t} > 0$ such that for all $t \in (0, \bar{t})$, $z_t(\lambda_S) > z_t(\lambda'_S)$. Define $t^* \equiv \inf_t \left\{z_t(\lambda_S) \leq z_t(\lambda'_S), t > 0\right\} \in \mathbb{R}_{++} \cup \{+\infty\}$.

If $t^* = +\infty$, then we have $z_t(\lambda_S) \geq z_t(\lambda'_S)$ for all $\lambda'_S < \lambda_S$. Since y_t is independent of λ_S, the proof is complete in this case. Hence, we concentrate on the case with $t^* < \infty$.

At $-t^*$, it must be the case that $z_{t^*}(\lambda_S) = z_{t^*}(\lambda'_S)$ and $\dot{z}_{t^*}(\lambda_S) \leq \dot{z}_{t^*}(\lambda'_S)$ by an argument analogous to footnote 28 in the Online Appendix. From $z_{t^*}(\lambda_S) = z_{t^*}(\lambda'_S)$, we have

$$\begin{aligned}
\dot{z}_{t^*}(\lambda_S) &= \lambda_W \max\left\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\right\} + \lambda_S \min\left\{y_{t^*} - z_{t^*}(\lambda_S), 0\right\} \\
\dot{z}_{t^*}(\lambda'_S) &= \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda'_S), 0\right\} + \lambda'_S \min\left\{y_{t^*} - z_{t^*}(\lambda'_S), 0\right\} \\
&= \lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} + \lambda'_S \min\left\{y_{t^*} - z_{t^*}(\lambda_S), 0\right\}.
\end{aligned}$$

Note that, by definition, we have $x_{t^*}(\lambda'_S) < x_{t^*}(\lambda_S)$. Given this inequality, the following two cases are possible:

1. If $x_{t^*}(\lambda_S) > z_{t^*}(\lambda_S)$, then we have $\lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} > \lambda_W \max\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\}$. In addition, we have $\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} \geq \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}$. Hence, we have $\dot{z}_{t^*}(\lambda'_S) < \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

2. If $x_{t^*}(\lambda_S) \leq z_{t^*}(\lambda_S)$, then we have

$$\lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} \leq \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} = 0$$

   and so

$$\lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} = \lambda_W \max\{x_{t^*}(\lambda_S) - z_{t^*}(\lambda_S), 0\} = 0.$$

   We consider the following subcases:

   (a) If $y_{t^*} > z_{t^*}(\lambda_S)$, then we have

$$\lambda_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} > \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\}.$$

   Hence, we have $\dot{z}_{t^*}(\lambda'_S) < \dot{z}_{t^*}(\lambda_S)$. This is a contradiction.

   (b) If $y_{t^*} \leq z_{t^*}(\lambda_S)$, then we have

$$\lambda_W \max\left\{x_{t^*}(\lambda'_S) - z_{t^*}(\lambda_S), 0\right\} = 0$$

   and

$$\lambda'_S \min\left\{y_{t^*} - z_{t^*}(\lambda'_S), 0\right\} = \lambda'_S \min\{y_{t^*} - z_{t^*}(\lambda_S), 0\} = 0.$$

   Therefore, $\dot{z}_{t^*}(\lambda'_S) = 0$. For $t > t^*$, since $x_t(\lambda'_S)$ is decreasing in t and y_t is increasing in t, we have $\dot{z}_t(\lambda'_S) = 0$. Hence, S does not enter for $-t < -t^*$ with λ'_S, as desired.

In the proof above, all the time-cutoffs described are finite and independent of T, as stated in part 3 of Proposition 14.

N.2 Proof of Remark 7

Before proving Proposition 15, we prove Remark 7. It suffices to show that $\phi < 0$ implies $t_S = \infty$.

By definition, we can write

$$f_S(t) = \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta + \left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} & \text{if } r = 1 \end{cases}.$$

If $\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha} \leq 0$, then

$$\begin{aligned} f_S(t) &= \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} \\ &= \frac{1}{1-r}e^{-\lambda_S t} + \left(\frac{\max\{(\gamma-\beta),0\}}{1-\alpha} - \frac{1}{1-r}\right)e^{-\lambda_W t} - \frac{\beta + \{\max(\gamma-\beta),0\}}{1-\alpha} \end{aligned}$$

is always decreasing in t. Since $f_S(0) = -\frac{\beta}{1-\alpha}$, we have $f_S(t) < 0$ for all t. Therefore, we have $t_S = \infty$ as desired.

Hence, for the rest of the proof, we focus on the case in which $\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha} > 0$. Then, the first- and second- order conditions for $f_S(t)$ imply that $f_S(t)$ is single-peaked at

$$t^{\text{peak}} = \begin{cases} \frac{\log\left(\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)}{\lambda_W - \lambda_S} & \text{if } r \neq 1 \\ \frac{1}{\lambda_W}\left(1 - \frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right) & \text{if } r = 1 \end{cases}.$$

For $r \neq 1$, since

$$\begin{aligned} f_S(t) &= \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} \\ &= \left(\frac{1}{1-r}\left(re^{-\lambda_S t} - e^{-\lambda_W t}\right) + \frac{e^{-\lambda_W t}\max\{(\gamma-\beta),0\}}{1-\alpha}\right) \\ &\quad -\frac{1}{1-r}(r-1)e^{-\lambda_S t} - \frac{\beta + \max\{(\gamma-\beta),0\}}{1-\alpha} \\ &= -\frac{1}{\lambda_W}f_S'(t) + e^{-\lambda_S t} - \frac{\max\{\beta,\gamma\}}{1-\alpha}, \end{aligned}$$

substituting $f_S'(t^{\text{peak}}) = 0$ and $t^{\text{peak}} = \frac{\log\left(\frac{1}{r} - \frac{1-r}{r}\frac{\gamma-\beta}{1-\alpha}\right)}{\lambda_W - \lambda_S}$ into $f_S(t)$ yields

$$
\begin{aligned}
f_S(t^{\text{peak}}) &= e^{-\lambda_S \frac{\log\left(\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)}{\lambda_W - \lambda_S}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= e^{-r \frac{\log\left(\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)}{1-r}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= e^{\log\left(\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)^{\frac{r}{r-1}}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= \left(\frac{1}{r} - \frac{1-r}{r}\frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)^{\frac{r}{r-1}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= \phi.
\end{aligned}
$$

Therefore, if $\phi < 0$, then there is no solution for $f_S(t) = 0$ and so $t_S = \infty$, as desired.

For $r = 1$, since

$$
\begin{aligned}
f_S(t) &= \lambda_W t e^{-\lambda_W t} - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} \\
&= -\frac{1}{\lambda_W} f_S'(t) + e^{-\lambda_W t} - \frac{\max\{\beta,\gamma\}}{1-\alpha},
\end{aligned}
$$

substituting $f_S'(t^{\text{peak}}) = 0$ and $t^{\text{peak}} = \frac{1}{\lambda}\left(1 - \frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)$ into $f_S(t)$ yields

$$
\begin{aligned}
f_S(t^{\text{peak}}) &= e^{-\lambda_W \frac{1}{\lambda_W}\left(1 - \frac{\max\{(\gamma-\beta),0\}}{1-\alpha}\right)} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= e^{\frac{\max\{(\gamma-\beta),0\}}{1-\alpha} - 1} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\
&= \phi.
\end{aligned}
$$

Therefore, if $\phi < 0$, then there is no solution for $f_S(t) = 0$ and so $t_S = \infty$ holds, as desired.

N.3 Proof of Proposition 15

Recall that t_S is the smallest positive solution for $f_S(t) = 0$ where

$$
f_S(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - \frac{\beta}{1-\alpha} & \text{if } r = 1 \end{cases}, \tag{55}
$$

while t_W is the smallest solution for $f_W(t) = 0$ where

$$f_W(t) \equiv \begin{cases} \frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t} & \text{if } r \neq 1 \\ \lambda_W t e^{-\lambda_W t} - e^{-\lambda_W t} & \text{if } r = 1 \end{cases}. \tag{56}$$

We prove each part of the proposition in what follows.

N.3.1 Proof of Part 1 of Proposition 15

When we change r, without loss, we keep λ_W fixed and vary λ_S. First, note that, for sufficiently large r, ϕ is negative:

$$\begin{aligned} & \lim_{r\to\infty}\left(\frac{1}{r} - \frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha}, 0\right\}\right)^{\frac{r}{r-1}} - \frac{\max\{\beta,\gamma\}}{1-\alpha} \\ = & -\max\left(\frac{\gamma-\beta}{\alpha-1}, 0\right) - \frac{\max\{\beta,\gamma\}}{1-\alpha} < 0. \end{aligned}$$

Hence, for sufficiently large r, we have $-t_W > -t_S$.

Second, since

$$\begin{aligned} & \lim_{r\to 0}\left(\frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta), 0\}}{1-\alpha}\right) \\ = & \lim_{r\to 0}\frac{1}{1-r}\left(e^{-r\lambda_W t} - e^{-\lambda_W t}\right) - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta), 0\}}{1-\alpha} \\ = & 1 - e^{-\lambda_W t} - \frac{\beta + \left(1 - e^{-\lambda_W t}\right)\max\{(\gamma-\beta), 0\}}{1-\alpha} \end{aligned}$$

and

$$\begin{aligned} & \lim_{r\to 0}\left(\frac{1}{1-r}\left(e^{-\lambda_S t} - e^{-\lambda_W t}\right) - e^{-\lambda_S t}\right) \\ = & \lim_{r\to 0}\left(\frac{1}{1-r}\left(e^{-r\lambda_W t} - e^{-\lambda_W t}\right) - e^{-r\lambda_W t}\right) \\ = & -e^{-t\lambda_W} \end{aligned}$$

hold for each t, $\lim_{r\to 0} t_S < \infty$ and $\lim_{r\to 0} t_W = \infty$. Thus, for sufficiently small r, we have $-t_W < -t_S$.

Therefore, we are left to show that

$$\frac{\partial\left(-t_W\right)}{\partial r}-\frac{\partial\left(-t_S\right)}{\partial r}>0.$$

To this end, in (55) and (56), when λ_S goes up with λ_W fixed, the first terms in $f_S(t)$ and $f_W(t)$ move in the same way while the second terms ($-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ in $f_S\left(t\right)$ and $-e^{-\lambda_S t}$ in $f_W(t)$) become larger only in $f_W(t)$. Hence, we have $\frac{\partial(-t_W)}{\partial r}-\frac{\partial(-t_S)}{\partial r}>0$, as desired.

N.3.2 Proof of Part 2 of Proposition 15

First, note that, for sufficiently large α, ϕ is negative for the following reason: If $r\neq 1$, since

$$\lim_{\alpha\to 1}\left(\frac{1}{r}-\frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}\right)<0,$$

$\phi<0$ in the limit as $\alpha\to 1$. If $r=1$, since

$$\lim_{\alpha\to 1}\left(e^{-\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\right)<0,$$

$\phi<0$ in the limit as $\alpha\to 1$. Hence, for sufficiently large α, we have $-t_W>-t_S$.

Therefore, we are left to show that

$$\frac{\partial\left(-t_W\right)}{\partial \alpha}-\frac{\partial\left(-t_S\right)}{\partial \alpha}>0.$$

In (55) and (56), when α goes up, $f_W(t)$ is unchanged. Hence, we are left to show that $\frac{\partial(-t_S)}{\partial\alpha}<0$, that is, the smallest positive t such that $\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)=\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ (or $\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)=\frac{\beta}{1-\alpha}$ if $r=1$) increases. Notice that $\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)$ is single-peaked at $\frac{\log\lambda_W-\log\lambda_S}{\lambda_W-\lambda_S}$. Since $\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ and $\frac{\beta}{1-\alpha}$ become larger, the smallest positive t such that $\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)=\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ (or $\frac{1}{1-r}\left(e^{-\lambda_S t}-e^{-\lambda_W t}\right)=\frac{\beta}{1-\alpha}$ if $r=1$) increases.

N.3.3 Proof of Part 3 of Proposition 15

First, note that, for sufficiently large β, ϕ is negative: If $r \neq 1$, since

$$\begin{aligned}
\lim_{\beta\to 1}\left(\left(\frac{1}{r}-\frac{1-r}{r}\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}\right)^{\frac{r}{r-1}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\right) &= \left(\frac{1}{r}\right)^{\frac{r}{r-1}}-\frac{1}{1-\alpha} \\
&\leq \max_r\left(\frac{1}{r}\right)^{\frac{r}{r-1}}-\frac{1}{1-\alpha} \\
&= 1-\frac{1}{1-\alpha}<0,
\end{aligned}$$

$\phi<0$ in the limit as $\beta \to 1$. If $r=1$, since

$$\lim_{\beta\to 1}\left(e^{-\max\left\{\frac{\gamma-\beta}{1-\alpha},0\right\}}-\frac{\max\{\beta,\gamma\}}{1-\alpha}\right)=1-\frac{1}{1-\alpha}<0,$$

$\phi<0$ in the limit as $\beta \to 1$. Hence, for sufficiently large β, we have $-t_W > -t_S$.

Therefore, we are left to show that

$$\frac{\partial(-t_W)}{\partial\beta}-\frac{\partial(-t_S)}{\partial\beta}<0.$$

In (55) and (56), when β goes up, the second terms $-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ and $-\frac{\beta}{1-\alpha}$ in $f_S(t)$ become smaller while $f_W(t)$ is unchanged. Hence, by the same proof as in the case where α increases, $-t_W-(-t_S)$ increases. Hence, we have $\frac{\partial(-t_W)}{\partial\beta}-\frac{\partial(-t_S)}{\partial\beta}>0$, as desired.

N.3.4 Proof of Part 4 of Proposition 15

It suffices to show that

$$\begin{aligned}
\frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma} &= 0 \text{ if } \gamma\leq\beta, \\
\frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma} &> 0 \text{ if } \gamma>\beta.
\end{aligned}$$

In (55) and (56), if $\beta\geq\gamma$, then neither $f_S(t)$ nor $f_W(t)$ depends on γ. Hence, we have $\frac{\partial(-t_W)}{\partial\gamma}-\frac{\partial(-t_S)}{\partial\gamma}=0$. Hence, let us focus on the case $\gamma>\beta$. If γ goes up in (55) and (56), then the second term $-\frac{\beta+\left(1-e^{-\lambda_W t}\right)\max\{(\gamma-\beta),0\}}{1-\alpha}$ in $f_S(t)$ becomes smaller while $f_W(t)$ is unchanged. Hence, by the same proof as in part 3 of Proposition 15, $-t_W - (-t_S)$ increases, which means $\frac{\partial(-t_W)}{\partial\gamma} - \frac{\partial(-t_S)}{\partial\gamma} > 0$, as desired.