Online Supplementary Appendix to: Efficient iBF: Balanced Integration of Fragmented Matching Markets for Welfare Improvement

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Appendix B. Fragmented Matching Markets in Practice

In this section, we provide examples of fragmented matching markets in various contexts and argue that our mechanism has the potential to improve the welfare of those markets. Those markets currently use alternative mechanisms to alleviate the problem of fragmentation, and we will point out problematic features of those alternative methods. In each example, balancedness seems desirable, but the degree to which it is a hard constraint varies across examples. Our solution (efficient iBF) satisfies balancedness, and we view it as a good starting point in improving the respective markets. To accommodate more complicated objective functions in some of those markets and to improve the welfare even further, an analysis that is catered to each market might be necessary. Such an analysis is beyond the scope of the present study but would be fruitful in addressing the problems in those markets.

B.1. **Chinese College Admission.** Chinese college admissions involve a large number of students: In 2023, 12.91 million students took the college entrance exam in China, and 10.42 million students were admitted to regular and vocational colleges.[1] College admissions in China are conducted at the provincial level, where each province ranks students within the province. Most colleges rely on funding from provincial governments and thus, keeping balancedness would be desirable.

In 1952, to better balance regional development, China started allocating college admission quotas to regions, where each region contains several provinces. In 1958, to "facilitate the implementation of principles tailored to local conditions," the allocation of admission quotas was refined to the provincial level.[2] Under the current system, colleges allocate

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[1]Data source: http://www.moe.gov.cn/jyb_xwfb/xw_zt/moe_357/2023/2023_zt09/mtbd/202306/t20230607_1063154.html and https://www.stats.gov.cn/sj/zxfb/202402/t20240228_1947915.html.

[2]See Liu and Li (2014) for more details on the historical change of admission quota allocation policy in China.

their total quotas across provinces, and each province admits students based on these quotas following a version of serial dictatorship.[3,4]

We refer to a mechanism like the one used in this market, i.e., one involving some college seats set aside for non-residents, as the Chinese College Admission (CCA) mechanism. Formally, for each school $s \in S$ and region $r \in R$, we fix the region-specific quota $q_s^r \in \{0, \ldots, q_s\}$ such that $\sum_{r \in R} q_s^r = q_s$. Then, for each region r, the CCA mechanism outputs the outcome of the DA mechanism applied to the students in r and all schools, where for each school s the capacity is set at q_s^r and its priority is such that all students in r and being unmatched are ordered as in s's original priority while all students not in r are deemed unacceptable.[5] In the actual Chinese college admissions, a serial dictatorship with the modified capacities and priorities are often employed. Such a mechanism is a special case of the CCA mechanism in which schools' original priorities are common across schools. The following example highlights potential drawbacks of the CCA mechanism.

Example 7 (Drawbacks of CCA)**.** Let $I = \{i_1, i_2, \tilde{i}_1, \tilde{i}_2\}$ and $S = \{s, \tilde{s}\}$. Let there be two regions, $r = \{i_1, i_2, s\}$, and $\tilde{r} = \{\tilde{i}_1, \tilde{i}_2, \tilde{s}\}$. Each school has the capacity of two. School priorities are given as follows:

$$\succ_s : i_1, i_2, \tilde{i}_1, \tilde{i}_2, \qquad \succ_{\tilde{s}} : \tilde{i}_1, \tilde{i}_2, i_1, i_2.$$

Note that locals are favored.

At each school, suppose that the region-specific quota for each region is one for the CCA mechanism.

First, consider the following student preferences:

$$\succ_{i_1} : \tilde{s}, s, \qquad \succ_{\tilde{i}_1} : s, \tilde{s},$$

$$\succ_{i_2} : \tilde{s}, s, \qquad \succ_{\tilde{i}_2} : s, \tilde{s}.$$

[3]For more details on the policy of quota allocation across different provinces, see https://gaokao.chsi.com.cn/gkxx/zcdh/202403/20240320/2293271442-8.html.

[4]Each province has a centralized admission system in which students submit their preferences after knowing their exam scores. Colleges are partitioned into several tiers. There is a limit to the number of schools students can list in each tier. For the second tier, which includes academic universities, students can list up to 6 to 45 schools. Schools rank students from the same province based on their exam scores. The first-tier schools use either the Boston mechanism or serial dictatorship, while subsequent tiers use serial dictatorship. For more details on the Chinese college admission systems in different provinces in 2023, see https://gaokao.chsi.com.cn/z/gkbmfslq2023/zytb.jsp.

[5]The modified capacities and priorities are merely set as part of the mechanism. Thus, axioms such as fairness are evaluated with respect to the original capacities and priorities.

In this environment, the outcome of the CCA mechanism is:

$$\begin{pmatrix} s & \tilde{s} & \emptyset \\ i_2, \tilde{i}_1 & i_1, \tilde{i}_2 & \emptyset \end{pmatrix}.$$

Our FIG cycles mechanism (with an arbitrary initial iBF and an arbitrary cycle selection rule) produces the following outcome,

$$\begin{pmatrix} s & \tilde{s} & \emptyset \\ \tilde{i}_1, \tilde{i}_2 & i_1, i_2 & \emptyset \end{pmatrix},$$

which one can verify is the unique efficient iBF.

Second, consider the following student preferences:

$$\succ_{i_1}: s, \tilde{s}, \qquad \succ_{\tilde{i}_1}: \tilde{s}, s,$$

$$\succ_{i_2}: s, \tilde{s}, \qquad \succ_{\tilde{i}_2}: \tilde{s}, s.$$

In this environment, the outcome of the CCA mechanism is:

$$\begin{pmatrix} s & \tilde{s} & \emptyset \\ i_1, \tilde{i}_2 & i_2, \tilde{i}_1 & \emptyset \end{pmatrix}.$$

Our FIG cycles mechanism (where the corresponding algorithm uses an arbitrary initial iBF and an arbitrary cycle selection rule) produces the following outcome,

$$\begin{pmatrix} s & \tilde{s} & \emptyset \\ i_1, i_2 & \tilde{i}_1, \tilde{i}_2 & \emptyset \end{pmatrix},$$

which one can verify is the unique efficient iBF.

Third, consider the following student preferences:

$$\succ_{i_1}: s, \qquad \succ_{\tilde{i}_1}: s, \tilde{s},$$

$$\succ_{i_2}: s \qquad \succ_{\tilde{i}_2}: s, \tilde{s}.$$

In this environment, the outcome of the CCA mechanism is:

$$\begin{pmatrix} s & \tilde{s} & \emptyset \\ i_1, \tilde{i}_1 & \tilde{i}_2 & i_2 \end{pmatrix}.$$

Note that this matching is neither balanced nor fair (with respect to school s's original priority).

Our FIG cycles mechanism (where the corresponding algorithm uses an arbitrary initial iBF and an arbitrary cycle selection rule) produces the following outcome,

$$\begin{pmatrix} s & \tilde{s} & \emptyset \\ i_1, i_2 & \tilde{i}_1, \tilde{i}_2 & \emptyset \end{pmatrix},$$

which one can verify is the unique efficient iBF, and hence, in particular, it is balanced and fair.

This example suggests that set aside policies may suffer from efficiency loss because the optimal number of seats to be set aside to different groups depends on preference realization, but the policy needs to decide the number in advance without such information. In the first case of the above example, the number of seats set aside for non-residents is too small, causing residents being rejected from their preferred schools outside of the region of their residence. In the second case, by contrast, the number of seats set aside for non-residents is too large, causing residents being rejected from their preferred schools in the region of their residence. One of the advantages of our FIG cycles mechanisms is that they endogenously determine the number of students who are allocated to schools outside of their residents based on the preferences of the market participants. Another advantage of the FIG cycles mechanisms is that they produce a balanced matching, addressing the drawback highlighted by the third case of the example.

B.2. **Chinese High School Admission.** In most provinces, high school admissions are conducted at the city level. High schools can only admit students within their city. Some specialized high schools, such as those for art or sports, can admit students from the entire province. But they do so by allocating quotas to each city and conducting separate admissions in each city. Chinese middle and primary school admissions are conducted similarly, with almost no inter-municipal school choice allowed.

We use the high school admission system in Hebei province as an example. Other provinces and cities, such as Shaanxi, Yunnan, Inner Mongolia, Hunan, and Beijing, have similar admission systems and restrictions for inter-regional school choices.[6][7]

Hebei province, with its 11 cities, has nearly one million students taking the high school entrance exam each year.[8] Although students across these cities take the same

[6]For Beijing, admissions are conducted at the district level. For more details on Beijing's high school admission system, see https://www.beijing.gov.cn/zhengce/zhengcefagui/202403/t20240319_3594421.html.

[7]The reason why Beijing is listed alongside other provinces is that Beijing is a direct-administered municipality, which has the same administrative hierarchy as a province in China. For more details on direct-administered municipalities in China, see http://english.www.gov.cn/archive/china_abc/2014/08/27/content_281474983873401.htm.

[8]Data source: https://www.toutiao.com/article/7247513553031545379/?&source=m_redirect&wid=1745539044494.

standardized exam, each city independently grades it.[9] Each city has a centralized admission system based on serial dictatorship in which students submit their preferences after knowing their exam scores.[10]

Starting in 2024, Hebei province no longer allows students to apply to high schools in different cities.[11] This change was implemented because increased inter-municipal school choice led to the loss of top students from certain cities, creating imbalances in local education and undermining educational equity.[12]

To understand how our method alleviates the problem of regional imbalance while still retaining the gain from inter-regional transfer, consider the following example.

Example 8. Let $I = \{i_1, i_2, \dots i_{10}, i'_1, i'_2, \dots i'_{10}\}$ and $S = \{s, s'\}$. Let there be two regions, $r = \{i_1, \dots, i_{10}, s\}$, and $r' = \{i'_1, \dots, i'_{10}, s'\}$. Each school has the capacity of ten. Suppose that most students prefer s to s' to being unmatched. More precisely, assume

$$\succ_i\colon s, s' \qquad \text{for all } i \neq i_3, i'_3, i_8, i'_8,$$

$$\succ_i\colon s', s \qquad \text{for all } i = i_3, i'_3, i_8, i'_8.$$

School priorities are given as follows:

$$\succ_s\colon i_1, \dots, i_{10}, i'_1, \dots, i'_{10},$$

$$\succ_{s'}\colon i'_1, \dots, i'_{10}, i_1, \dots, i_{10}.$$

Imagine that the students with indices $1, \dots, 5$ are "good quality" students and others are "bad quality" students. First, we consider Hebei's original policy where they set aside some quota for each region. Specifically, assume that each school reserves three seats for

[9] Since 2022, China has been promoting a standardized exam across all cities within each province, with the goal of implementing this practice to all provinces by 2024. For more details on this practice, see https://news.gmw.cn/2022-04/07/content_35639581.htm. This practice aims to address issues such as uneven exam quality and misconduct due to inadequate local management. See https://www.bjnews.com.cn/detail/1719123366169320.html.

[10] There are several stages of admissions, with each stage containing different types of schools. Students admitted in an earlier stage cannot participate in subsequent stages. Typically, the first stage is for "3 + 4" undergraduate programs and free normal schools, the second stage is for regular high schools, and the third stage is for secondary vocational schools. Within each stage, students can apply to up to 1-5 schools. Schools rank students based on their exam scores, and serial dictatorship is used within each stage. For more details on Hebei's policy, see http://www.hee.gov.cn/col/1410097726928/2024/02/18/1708221286695.html.

[11] The policy allows certain military, arts, and sports specialty schools to admit students across all cities. For detailed policies, see http://www.hebei.gov.cn/columns/580d0301-2e0b-4152-9dd1-7d7f4e0f4980/202406/13/ac3df029-7117-483e-80a7-564941efe894.html.

[12] Source: https://www.gov.cn/xinwen/2014-04/12/content_2657873.htm.

the students from another region. In such a case, the resulting matching μ is such that

$$\mu_s = \{i_1, i_2, i_4, i_5, i_6, i_7, i_9, i'_1, i'_2, i'_4\}, \text{ and } \mu_{s'} = \{i_3, i_8, i_{10}, i'_3, i'_5, i'_6, i'_7, i'_8, i_9, i'_{10}\}.$$

As a result, s has seven good quality students while s' has only three good quality students. Fourteen students are matched with their first choice.

Now, if inter-regional transfer is completely abandoned as in the current practice in Hebei, the resulting matching $\tilde{\mu}$ is such that

$$\tilde{\mu}_s = \{i_1, \ldots, i_{10}\}, \text{ and } \tilde{\mu}_{s'} = \{i'_1, \ldots, i'_{10}\}.$$

As a result, both s and s' have five good quality students each. Ten students are matched with their first choice.

Our mechanism achieves a middle ground between the above two extremes. Specifically, we consider running the FIG cycles algorithm starting with $\tilde{\mu}$, which results in a matching $\hat{\mu}$ such that

$$\hat{\mu}_s = \{i_1, i_2, i_4, \ldots, i_7, i_9, i_{10}, i'_1, i'_2\}, \text{ and } \hat{\mu}_{s'} = \{i'_3, \ldots, i'_{10}, i_3, i_8\}.$$

As a result, s has six good quality students while s' has four good quality students. Fourteen students are matched with their first choice.

We could also modify our algorithm to only allow for cycles involving good quality students to specifically address Hebei's aforementioned concern about the loss of top students from certain cities. In such a case, the resulting matching $\hat{\hat{\mu}}$ is such that

$$\hat{\hat{\mu}}_s = \{i_1, i_2, i_4, \ldots, i_{10}, i'_1\}, \text{ and } \hat{\hat{\mu}}_{s'} = \{i'_2, \ldots, i'_{10}, i_3\}.$$

both s and s' have five good quality students each. Twelve students are matched with their first choice.

As can be seen, our FIG cycles mechanism alleviates the problem of regional imbalance while retaining the efficiency gain due to inter-regional transfers.

B.3. **Chinese Kindergarten Admission.** In major cities in China such as Tianjin and Shanghai, kindergarten admissions are conducted at the district level, and kindergartens can only admit children within their district.[13] A matching mechanism that coordinates the processes of multiple districts would improve efficiency.

[13]For more details on Tianjin, see https://jy.tj.gov.cn/BMFW/rxzs/xqjyslyezs/. For Shanghai, see https://www.shanghai.gov.cn/jcsfbxqjyslyery/index_2.html.

Other major cities, notably Beijing, have some system of interdistrict transfer. Beijing has 16 districts as shown in Figure 8.[14] In 2022, there were 1,989 kindergartens across these districts, enrolling approximately 178,000 new children annually.[15] Each district's kindergarten system is autonomous and locally financed.

Before 2015, kindergarten admissions in Beijing were mostly decentralized. Since then, to "further standardize the admission process," some districts have moved towards centralized admission systems.[16] In 2024, a citywide platform was introduced to manage the collection of information for kindergarten admissions.[17] However, the process as a whole is not well coordinated in that the process remains independent from district to district such that parents can apply to kindergartens in multiple districts and may receive offers from several kindergartens in different districts at the same time.[18] Uncoordinated offers have been shown to cause significant mismatches, which lead to inefficiency, see e.g., Abdulkadiroğlu, Pathak and Roth (2005) and Abdulkadiroğlu, Agarwal and Pathak (2017).

To resolve such inefficiency, a matching mechanism that coordinates the processes of multiple districts is hoped for. Since kindergartens are locally funded, balancedness would be desirable. Our FIG cycles mechanism offers a way to achieve a welfare improvement while keeping the balance in a coordinated manner across the City of Beijing.

B.4. **Japanese Daycare Admission.** In Japan, allocation of slots at accredited daycares are conducted by individual municipal governments and, with few exceptions, a child can only attend a daycare in the municipality of their residence. Below, we list a few notable examples where inefficiency is caused by market fragmentation.

(1) The City of Tokyo is divided into 23 small municipalities, each of which conducts a matching independently. Due to the small sizes of the regions, many families find that daycare centers outside the regions of their residence are among the closest. Moreover, it is often convenient to put their children to a daycare center close to

[14] For a detailed description of Beijing's districts, see: https://www.chinadiscovery.com/beijing-tours/maps/beijing-districts-map.html. For size and population data in 2010, see https://www.stats.gov.cn/sj/tjgb/rkpcgb/dfrkpcgb/202302/t20230206_1902059.html.

[15] Data source: https://jw.beijing.gov.cn/xxgk/shujufab/tongjigaikuang/202303/t20230317_2938666.html. For 2023 data, see https://www.beijing.gov.cn/renwen/bjgk/kjss/202403/t20240322_3597024. html.

[16] Source: https://xueqian.eol.cn/youeryuan/qian/zcxx/201504/t20150415_1247496.shtml.

[17] Platform website: https://ryfw.bjedu.cn/.

[18] Most districts use a version of kindergarten-proposing DA or Boston mechanism, where each district restricts parents to apply to up to 3-6 kindergartens. Priority is given to children who live in the same district as the kindergarten, followed by children from other districts. This priority is not strict, and the ties are broken based on demographic criteria set by the kindergartens. For more details on Beijing's kindergarten admission policy, see https://www.beijing.gov.cn/fuwu/bmfw/bmzt/yyerq/byjy/yey/.

their workplace while many people cross a regional boundary to commute. For these reasons, inter-municipal admission is an appealing option for many families, and it is indeed legal. However, such a practice is currently rare, and when it is implemented, it is done in an uncoordinated manner without an explicit mechanism.[19] Our FIG cycles mechanism has the potential to offer an improvement in a coordinated manner.

(2) The City of Yokohama and the City of Kawasaki are the 2nd and the 7th most populous municipalities, respectively, in Japan, and they share a long city border-15 miles long-from east to west, which makes many residents potentially affected by the issue of fragmentation. In an effort to address the problem of severe daycare slot shortage, the two cities signed the "Collaboration Agreement between Yokohama City and Kawasaki City regarding Childcare Waiting List Measures." Among various policy measures they agreed to, a notable agreement was to open new daycare centers that are jointly operated by the two cities. Following the agreement, each city opened a new daycare center that has a pre-set capacity for the residents from the other city. This can be thought of as the "set-aside" policy as in the CCA mechanism, where our FIG cycles mechanism has a potential to offer an improvement as discussed in Appendix B.1.

In a joint project with CyberAgent Inc., we are working to improve daycare admission.[20] Our effort has resulted in reforms in daycare admission systems of several municipalities in Japan so far. We are currently discussing the issues studied in this paper with several municipalities and discussing the possible use of the FIG cycles mechanism.

Appendix C. Examples

Example 9 (Multiple rounds of FIG cycles)**.** Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$, $R = \{r\}$. Each school has the capacity of one. Student preferences and school priorities are given

[19]In our communication with officials at Bunkyo Ward (one of the municipalities in the City of Tokyo), they wrote "applicants could receive admissions from us as well as from another district. In such a case, we only let the other district know about our result. Since a child cannot be in multiple daycare centers, the officials from the other districts must coordinate with the applicant."

[20]CyberAgent is a major technology firm in Japan that provides a variety of services such as online advertisement and video streaming services. We are working with their AI Lab and GovTech development center. In our joint project, the former provides analysis of data and develops a daycare assignment software including matching algorithms, while the latter engages with municipal governments and puts the proposed reforms to practice.

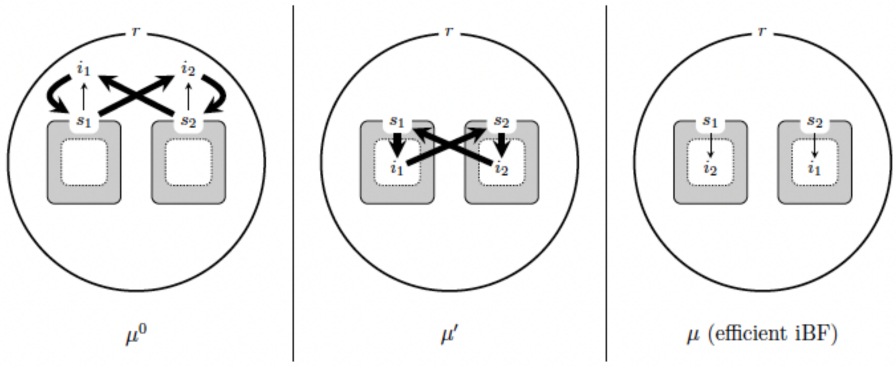

FIGURE 10. Example 9. The thick and thin arrows represent the FIG. μ is a unique efficient iBF, and there is no FIG cycle such that implementing it on μ^0 results in μ. Meanwhile, implementing a FIG cycle on μ^0 (indicated by thick arrows in the left panel of the figure) results in μ', and implementing a FIG cycle on μ' (indicated by thick arrows in the middle panel of the figure) results in μ.

as follows:

$$\succ_{i_1}: s_2, s_1, \qquad \succ_{s_1}: i_1, i_2,$$
$$\succ_{i_2}: s_1, s_2, \qquad \succ_{s_2}: i_2, i_1.$$

Let μ^0 be the empty matching. Starting from μ^0, there are three FIG cycles:

$$\mathcal{F} := (i_1, s_1),\ \mathcal{F}' := (i_2, s_2),\ \text{ and }\ \mathcal{F}'' := (i_1, s_1, i_2, s_2).$$

It is straightforward to see that there is a unique efficient iBF, which is

$$\mu = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2 & i_1 & \emptyset \end{pmatrix}.$$

However, μ is not a matching generated by $(\mu^0, \tilde{\mathcal{F}})$ where $\tilde{\mathcal{F}}$ is either $\mathcal{F}$, $\mathcal{F}'$ or $\mathcal{F}''$. The reason is that, for example, student i_1 does not point to s_2 in the FIG for μ^0 because she is not the top choice of s_2 among those who point to s_2 (s_2's top choice is i_2).

The efficient iBF can be obtained by "two rounds of FIG cycles" starting from μ^0: First, implementing the FIG cycle $\mathcal{F}''$ on μ^0, we obtain:

$$\mu' = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_1 & i_2 & \emptyset \end{pmatrix}.$$

Second, implementing

$$\mathcal{F}''' := (i_1, s_2, i_2, s_1)$$

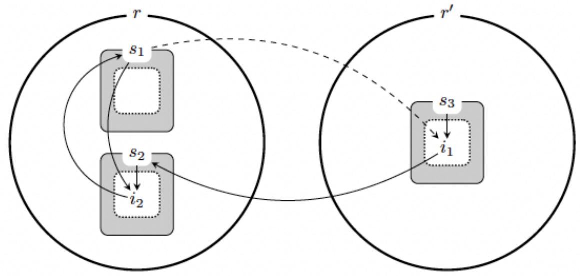

FIGURE 11. Example 10. The solid arrows represent the FIG. Student i_1 lives in r and student i_2 lives in r'. If we allowed s_1 to point to i_1 (the dashed arrow) in the FIG and implemented the resulting cycle (i_2, s_1, i_1, s_2), the balancedness constraint would be violated.

on μ' results in μ. □

Example 10 (Pointing rule for FIG)**.** Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2, s_3\}$, $R = \{r, r'\}$ where $r = \{i_1, s_1, s_2\}$ and $r' = \{i_2, s_3\}$. Each school has the capacity of one. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_2, s_3, s_1, \qquad \succ_{s_1}: i_1, i_2,$$
$$\succ_{i_2}: s_1, s_2, \qquad \succ_{s_2}: i_2, i_1,$$
$$\succ_{s_3}: i_1.$$

Consider the following matching:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ \emptyset & i_2 & i_1 & \emptyset \end{pmatrix}.$$

Note that this matching is a matching generated by $(\mu^0, \mathcal{F})$ where μ^0 denotes the empty matching and we let $\mathcal{F} := (i_1, s_3, i_2, s_2)$. The cycle $\mathcal{F}$ is a FIG cycle and μ is fair.

Now, consider the FIG on μ, which is depicted in Figure 11 by solid arrows. Suppose that we modify case (2b) of the definition of FIG (Definition 3) and hypothetically allow each school with a vacancy to point to *any* student that lives in the same region, irrespective of where that student is currently matched. This means that s_1 points to i_1, although it is *not* allowed in the FIG according to its pointing rule. This additional pointing is depicted by a dotted arrow in Figure 11. With this arrow, there is a cycle $\mathcal{F}' := (i_1, s_2, i_2, s_1)$. However, the matching generated by $(\mu, \mathcal{F}')$ violates the balancedness constraint. The reason is that implementing this cycle on μ would result in one less outflow for region r while the inflow does not change. This happens because the arrow going out of region r starts from a school, not a student. In the definition of the

FIG, all arrows going out of regions are from students. As we explained when providing the intuition for the proof of balancedness in Theorem 1 (see Figure 6), this feature ensures that implementing any FIG cycle on any balanced matching results in a balanced matching. □