GENERAL THEORY OF MATCHING UNDER DISTRIBUTIONAL CONSTRAINTS

YUICHIRO KAMADA AND FUHITO KOJIMA

Abstract. Distributional constraints are common features in many real matching markets, such as medical residency matching, school admissions, and teacher assignment. To aid market design in a wide range of applications, we develop a general theory of stability and matching mechanisms under distributional constraints. We show that a stable matching exists, and offer a stable mechanism that is (group) strategy-proof for one side of the market. We prove our results by exploiting a new connection between a matching problem under distributional constraints and a matching problem with contracts.
JEL Classification Numbers: C70, D61, D63.
Keywords: medical residency matching, college and graduate school admission, distributional constraints, efficiency, stability, strategy-proofness, matching with contracts, the rural hospital theorem

Note (August 30th, 2016): Part of this paper is subsumed and superseded by our paper "Stability and Strategy-Proofness for Matching with Constraints: a Necessary and Sufficient Condition."

Date: May 7th, 2015.
Kamada: Haas School of Business, University of California, Berkeley, Berkeley, CA 94720, y.cam.24@gmail.com. Kojima: Department of Economics, Stanford University, Stanford, CA 94305, fkojima@stanford.edu. We are grateful to Mustafa Oguz Afacan, Péter Biró, Eric Budish, Sylvain Chassang, Vincent Crawford, Hisao Endo, Clayton Featherstone, Tamás Fleiner, Drew Fudenberg, Tadashi Hashimoto, John William Hatfield, Toshiaki Iizuka, Rob Irving, Ryo Jinnai, Onur Kesten, Scott Duke Kominers, Hideo Konishi, Mihai Manea, David Manlove, Taisuke Matsubae, Aki Matsui, Yusuke Narita, Muriel Niederle, Parag Pathak, Al Roth, Dan Sasaki, Tayfun Sönmez, Satoru Takahashi, William Thomson, Alexis Akira Toda, Kentaro Tomoeda, Utku Ünver, Jun Wako, Alex Westkamp, Yosuke Yasuda, Bumin Yenmez, and participants at numerous seminars and conferences for helpful comments. Doctors Keisuke Izumi, Yoshiaki Kanno, Masataka Kawana, and Masaaki Nagano answered our questions about medical residency in Japan and introduced us to the relevant medical literature. We are grateful to officials at the Ministry of Health, Labor and Welfare and the Japan Residency Matching Program for discussion. Jin Chen, Irene Hsu, Seung Hoon Lee, Stephen Nei, Bobak Pakzad-Hurson, Neil Prasad, Fanqi Shi, Pete Troyan, Wei Wen, and Rui Yu provided excellent research assistance. Kojima gratefully acknowledges financial support from the Sloan Foundation, as well as financial support from the National Research Foundation through its Global Research Network Grant (NRF- 2013S1A2A2035408).

1. Introduction

The theory of two-sided matching has been extensively studied ever since the seminal contribution by Gale and Shapley (1962), and it has been applied to design clearinghouses in various markets in practice. However, many real matching markets are subject to distributional constraints, and researchers and practitioners are often faced with new challenges from those constraints. Meanwhile, traditional theory cannot be applied because it has assumed away such complications.

An illuminating example of distributional constraints is the "regional cap" policy in Japanese medical residency matching. Under this policy, each region of the country is subject to a regional cap, that is, an upper-bound constraint on the total number of residents assigned in the region. This measure was introduced to regulate the geographical distribution of doctors, which was considered to be concentrated too heavily in urban areas at the expense of rural areas. Policies that are mathematically isomorphic to the regional cap policy can be found in a wide range of contexts, such as graduate school admission in China, college admission in several European countries, residency match in the U.K., and teacher assignment in Scotland.[1]

Motivated by these real-life examples, Kamada and Kojima (2015) study the design of matching markets under distributional constraints. As standard stability may conflict with distributional constraints, they propose a relaxed stability concept. They show that existing mechanisms result in instability and inefficiency and offer a mechanism that finds a stable and efficient matching and is (group) strategy-proof for doctors while respecting the distributional constraints.

A major limitation of that paper, though, is that their stability concept is closely tailored to a particular governmental goal to equalize the numbers of doctors across hospitals beyond target capacities. Although such a goal may be appealing in some contexts as a first-order concern, it may not be appropriate in other applications because hospital

[1]There are a large number of studies in matching problems with various forms of constraints. Examples include Roth (1991) on gender balance in labor markets, Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Ergin and Sönmez (2006), Hafalir, Yenmez, and Yildirim (2013), and Ehlers, Hafalir, Yenmez, and Yildirim (2014) on diversity in schools, Westkamp (2013) on trait-specific college admission, Abraham, Irving, and Manlove (2007) on project-specific quotas in projects-students matching, and Biró, Fleiner, Irving, and Manlove (2010) on college admission with multiple types of tuitions. These models share some similarities with our model, but all of them are independent of our study. The more detailed discussion is found in our companion paper, Kamada and Kojima (2015), so we do not reproduce it here.

capacities in a given region may vary wildly. For example, the maximum and the minimum capacities of hospitals in Tokyo are 69 and 2, respectively (see Figure 1). For public elementary schools in Boston, the maximum and the minimum capacities of schools are 871 and 165, respectively (see Figure 2).[2] In such cases, it may be more appropriate to equalize the *ratio* between the numbers of doctors (beyond the targets) and the capacities across hospitals.

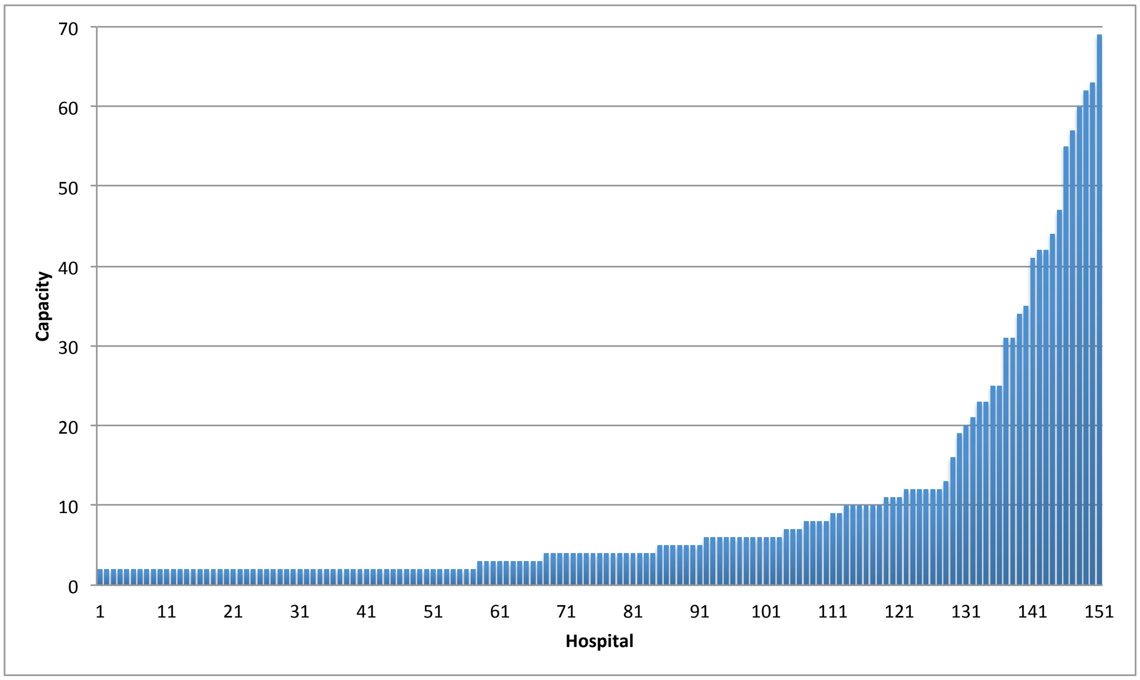

FIGURE 1. The hospital capacities in Tokyo. The data are taken from Japan Residency Matching Program (2013).

There may be other reasonable policy goals as well. For instance, the government may wish to give a priority to some hospitals within a region over others for a variety of reasons.[3] Thus it is clear from these examples that focusing on a particular policy goal limits the practical applicability of the previous analysis.[4]

To accommodate a wide range of policy goals, this paper provides a general theory of matching under distributional constraints. We offer a model in which each region is

---

[2]In Appendix C, we provide further statistics on heterogeneity of capacities in these markets.

[3]In the Japanese medical residency match, a hospital is given preferential treatments if that hospital deploys its doctors to underserved areas (Ministry of Health, Labour and Welfare, 2014).

[4]Some policy goals could be addressed by setting target capacities judiciously. However, it is easy to see that policy goals such as those discussed here cannot be fully expressed simply by picking target capacities.

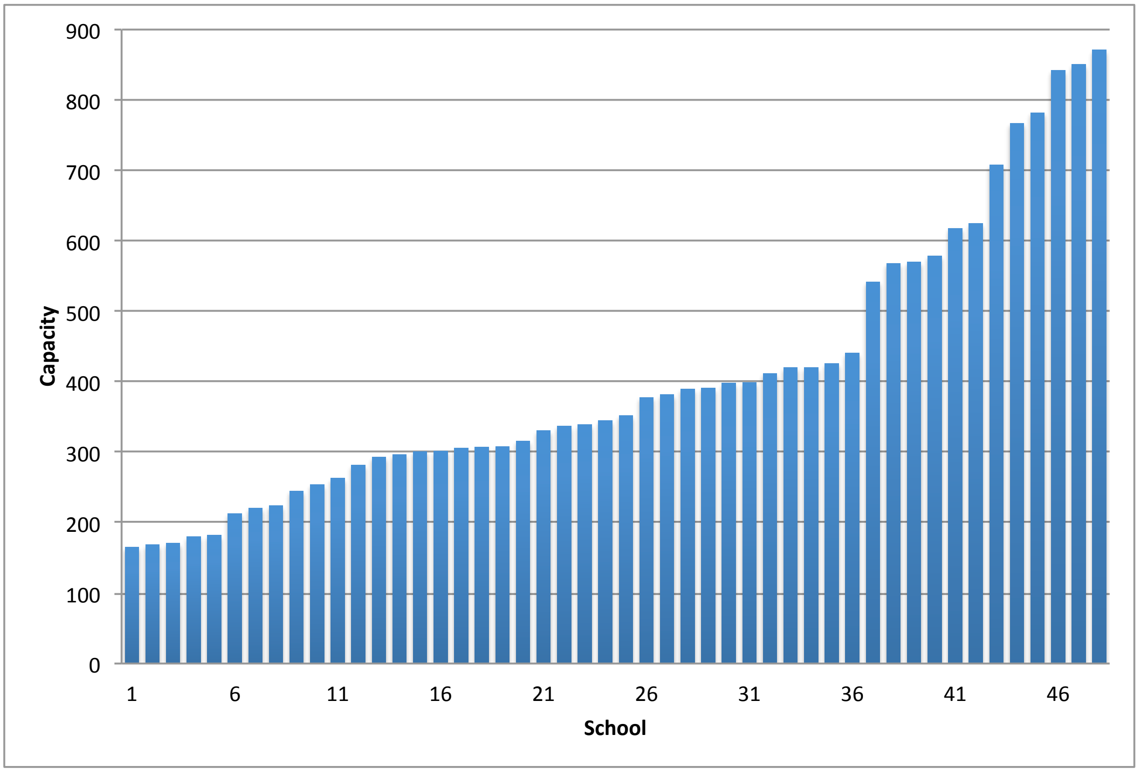

Figure 2. The school capacities in public elementary schools in Boston. The data are taken from Boston Public Schools (2013).

endowed with "regional preferences" over distributions of doctors within the region. The idea behind this modeling approach is to express policy goals as regional preferences, and accommodate different types of policy goals by changing regional preferences.[5] Then we define a stability concept that takes the regional preferences into account. We propose a new class of mechanisms, the flexible deferred acceptance mechanisms, and show that, under some regularity conditions, the mechanism finds a stable (and efficient) matching and it is group strategy-proof for doctors.

Previous research suffers from another limitation. It presumes that there is just one type of constraints, say on geographic distributions, and each hospital belongs to exactly one region such as a prefecture. In practice, however, distributional concerns can entail multiple dimensions. For example, again in Japan, there is not only a concern about regional imbalance of doctors but also that about medical speciality imbalance. This concern is clearly exemplified by a proposal made to the governmental committee meeting, which is titled "Measures to Address Regional Imbalance and Specialty Imbalance of Doctors" (Ministry of Health, Labour and Welfare, 2008). In 2008, Japan took a measure

[5]In Section D we provide various types of regional preferences that represent different policy goals.

that intended to address (only) regional imbalance, but even in 2007, the sense of crisis about specialty imbalance is shared by the private sector as well: In a sensationally entitled article, "Obstetricians Are in Short Supply! Footsteps of Obstetrics Breakdown," NTT Com Research (2007) reports that many obstetrics hospitals have been closed even in urban areas such as Tokyo. In Hungarian college admission, Biró, Fleiner, Irving, and Manlove (2010) point out that there are caps on each field of study as well as the set of state-financed seats.

Given these practical concerns, we further generalize our model to allow for multiple dimensions of distributional concerns. To do this, we consider a situation where there is a hierarchy of distributional constraints.[6] That is, there exist a class of subsets of hospitals that have a hierarchical structure and, for each of those subsets, an upper-bound constraint is imposed.[7] This constraint structure addresses such concerns as those in the previous paragraph, by allowing for caps *both* on the total number of doctors in each region *and* on the number of doctors practicing in each medical specialty in each region.[8] In that setting, we show that a generalization of the flexible deferred acceptance mechanism produces a stable matching (which is defined appropriately), and the mechanism is group strategy-proof for doctors. This generalization enables us to also analyze other types of applications, such as a situation in which there are regional caps not only for each prefecture, but also for each district within each prefecture.

In order to confirm that our general theory subsumes relevant cases, we provide a number of results. First, we show that the stability notion of Kamada and Kojima (2015) is a special case of our stability concept. More specifically, their concept is a case of our stability notion in which the regional preferences satisfy a condition called the "Rawlsian" property. Moreover, when the regional preferences are Rawlsian, our flexible deferred acceptance algorithm reduces to the algorithm of Kamada and Kojima (2015). Thus we establish the main results of Kamada and Kojima (2015) as a special case of our more general results. Furthermore, we demonstrate that a wide range of policy goals can be described by our regional preferences, and the corresponding flexible deferred acceptance algorithm finds a stable matching with respect to those regional preferences.

---

[6]Budish, Che, Kojima, and Milgrom (2013) also study hierarchical constraints, though in the context of object allocation, rather than two-sided matching.

[7]Such a hierarchical structure is called a "laminar family" in the mathematics literature.

[8]Although hierarchical structures exclude some cases, it appears to capture many practical cases. For example, demand for doctors in a particular medical specialty is often in terms of the number *in a given geographical region*, rather than in the entire country, so the constraint structure is hierarchical. Moreover, when the constraint structure is not hierarchical, we show that there need not exist a stable matching.

Although the main motivation for this work is an applied one, we believe that one methodological point is worth emphasizing. Our basic analytical approach is to find an unexpected connection between our model and the "matching with contracts" model (Hatfield and Milgrom, 2005).[9] More specifically, we define a hypothetical matching problem between doctors and *regions* instead of doctors and hospitals; We regard each region as a hypothetical consortium of hospitals that acts as one agent. By imagining that a region (hospital consortium) makes a common employment decision, we can account for interrelated doctor assignments across hospitals within a region, an inevitable feature in markets under distributional constraints. This association necessitates, however, that we distinguish a doctor's matching in different hospitals in the given region. We account for this complication by defining a region's choice function over *contracts* rather than doctors, where a contract specifies a doctor-hospital pair to be matched. Once this connection is established, with some work we show that properties in the matching-with-contract model can be invoked to establish key results in our matching model under distributional constraints, including both the existence of a stable matching and doctor group strategy-proofness of our mechanism.[10] Moreover, this connection allows us to utilize structural properties of stable allocations in matching with contracts. More specifically, we obtain many comparative statics results in markets under distributional constraints as straightforward corollaries of a single new comparative statics result about matching with contracts (Lemma 1).[11] More generally, we envision that analyzing a hypothetical model of matching with contracts may prove to be a useful approach for tackling complex matching problems one may encounter in the future.[12]

[9]Fleiner (2003) considers a framework that generalizes various mathematical results. A special case of his model corresponds to the model of Hatfield and Milgrom (2005), although not all results of the latter (e.g., those concerning incentives) are obtained in the former. See also Crawford and Knoer (1981) who observe that wages can represent general job descriptions in their model, given their assumption that firm preferences satisfy separability.

[10]Specifically, we invoke results by Hatfield and Milgrom (2005), Hatfield and Kojima (2009, 2010), and Hatfield and Kominers (2009, 2012).

[11]This result generalizes existing results such as Gale and Sotomayor (1985a,b), Crawford (1991), and Konishi and Ünver (2006). See also Kelso and Crawford (1982), who derive similar results in a matching model with wages. Echenique and Yenmez (2015) and Chambers and Yenmez (2013) independently obtain similar results in a framework based on choice functions as primitives.

[12]Indeed, after we circulated the first draft of the present paper, this technique was adopted by other studies such as Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014), Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014), and Kojima, Tamura, and Yokoo (2015) in the context of matching with distributional constraints. See Sönmez and Switzer (2013) for a more direct application of matching

The rest of this paper proceeds as follows. In Section 2, we present the model. Section 3 states the main result. In Section 4, we provide a proof of the main result. Section 5 offers a number of discussions. In Section 6, we present a further generalization. Section 7 concludes. Proofs are in the Appendix unless noted otherwise.

2. Model

This section introduces a model of matching under distributional constraints. We describe the model in terms of matching between doctors and hospitals with "regional caps," that is, upper bounds on the number of doctors that can be matched to hospitals in each region. However, the model is applicable to various other situations in and out of the residency matching context. Concrete applications include Chinese graduate school admission, U.K. medical matching, Scottish teacher matching, and college admissions in Ukraine and Hungary.[13]

Our notation and terminology closely follow those of Kamada and Kojima (2015). We reproduce them here for convenience for the reader, while also introducing new ones as needed.

2.1. **Preliminary Definitions.**

Let there be a finite set of doctors D and a finite set of hospitals H. Each doctor d has a strict preference relation $\succ_d$ over the set of hospitals and being unmatched (being unmatched is denoted by $\emptyset$). For any $h, h' \in H \cup \{\emptyset\}$, we write $h \succeq_d h'$ if and only if $h \succ_d h'$ or $h = h'$. Each hospital h has a strict preference relation $\succ_h$ over the set of subsets of doctors. For any $D', D'' \subseteq D$, we write $D' \succeq_h D''$ if and only if $D' \succ_h D''$ or $D' = D''$. We denote by $\succ = (\succ_i)_{i \in D \cup H}$ the preference profile of all doctors and hospitals.

Doctor d is said to be **acceptable** to hospital h if $d \succ_h \emptyset$.[14] Similarly, h is acceptable to d if $h \succ_d \emptyset$. It will turn out that only rankings of acceptable partners matter for our analysis, so we often write only acceptable partners to denote preferences. For example,

$$\succ_d : h, h'$$

means that hospital h is the most preferred, h' is the second most preferred, and h and h' are the only acceptable hospitals under preferences $\succ_d$ of doctor d.

---

with contracts model, where a cadet can be matched with a branch under one of two possible contracts. See also Sönmez (2013) and Kominers and Sönmez (2012).

[13]See Kamada and Kojima (2015) for detailed descriptions.

[14]We denote singleton set $\{x\}$ by x when there is no confusion.

Each hospital $h \in H$ is endowed with a (physical) **capacity** q_h, which is a nonnegative integer. We say that preference relation $\succ_h$ is **responsive with capacity** q_h (Roth, 1985) if

(1) For any $D' \subseteq D$ with $|D'| \le q_h$, $d \in D \setminus D'$ and $d' \in D'$, $(D' \cup d) \setminus d' \succeq_h D'$ if and only if $d \succeq_h d'$,
(2) For any $D' \subseteq D$ with $|D'| \le q_h$ and $d' \in D'$, $D' \succeq_h D' \setminus d'$ if and only if $d' \succeq_h \emptyset$, and
(3) $\emptyset \succ_h D'$ for any $D' \subseteq D$ with $|D'| > q_h$.

In words, preference relation $\succ_h$ is responsive with a capacity if the ranking of a doctor (or keeping a position vacant) is independent of her colleagues, and any set of doctors exceeding its capacity is unacceptable. We assume that preferences of each hospital h are responsive with capacity q_h throughout the paper.

There is a finite set R which we call the set of **regions**. The set of hospitals H is partitioned into hospitals in different regions, that is, $H_r \cap H_{r'} = \emptyset$ if $r \neq r'$ and $H = \cup_{r \in R} H_r$, where H_r denotes the set of hospitals in region $r \in R$. For each $h \in H$, let $r(h)$ denote the region r such that $h \in H_r$. For each region $r \in R$, there is a **regional cap** q_r, which is a nonnegative integer.

A **matching** μ is a mapping that satisfies (i) $\mu_d \in H \cup \{\emptyset\}$ for all $d \in D$, (ii) $\mu_h \subseteq D$ for all $h \in H$, and (iii) for any $d \in D$ and $h \in H$, $\mu_d = h$ if and only if $d \in \mu_h$. That is, a matching simply specifies which doctor is assigned to which hospital (if any). A matching is **feasible** if $|\mu_r| \le q_r$ for all $r \in R$, where $\mu_r = \cup_{h \in H_r} \mu_h$. In other words, feasibility requires that the regional cap for every region is satisfied. This requirement distinguishes the current environment from the standard model without regional caps: We allow for (though do not require) $q_r < \sum_{h \in H_r} q_h$, that is, the regional cap can be smaller than the sum of hospital capacities in the region.

To accommodate the regional caps, we introduce a new stability concept that generalizes the standard notion. For that purpose, we first define two basic concepts. A matching μ is **individually rational** if (i) for each $d \in D$, $\mu_d \succeq_d \emptyset$, and (ii) for each $h \in H$, $d \succeq_h \emptyset$ for all $d \in \mu_h$, and $|\mu_h| \le q_h$. That is, no agent is matched with an unacceptable partner and each hospital's capacity is respected.

Given matching μ, a pair (d, h) of a doctor and a hospital is called a **blocking pair** if $h \succ_d \mu_d$ and either (i) $|\mu_h| < q_h$ and $d \succ_h \emptyset$, or (ii) $d \succ_h d'$ for some $d' \in \mu_h$. In words, a blocking pair is a pair of a doctor and a hospital who want to be matched with each other (possibly rejecting their partners in the prescribed matching) rather than following the proposed matching.

When there are no binding regional caps (in the sense that $q_r \geq \sum_{h \in H_r} q_h$ for every $r \in R$), a matching is said to be stable if it is individually rational and there is no blocking pair. Gale and Shapley (1962) show that there exists a stable matching in that setting. In the presence of binding regional caps, however, there may be no such matching that is feasible (in the sense that all regional caps are respected). Thus in some cases every feasible and individually rational matching may admit a blocking pair.

A **mechanism** φ is a function that maps preference profiles to matchings. The matching under φ at preference profile $\succ$ is denoted $\varphi(\succ)$ and agent i's match is denoted by $\varphi_i(\succ)$ for each $i \in D \cup H$.

A mechanism φ is said to be **strategy-proof** if there does not exist a preference profile $\succ$, an agent $i \in D \cup H$, and preferences $\succ'_i$ of agent i such that

$$\varphi_i(\succ'_i, \succ_{-i}) \succ_i \varphi_i(\succ).$$

That is, no agent has an incentive to misreport her preferences under the mechanism. Strategy-proofness is regarded as a very important property for a mechanism to be successful.[15]

Unfortunately, however, there is no mechanism that produces a stable matching for all possible preference profiles and is strategy-proof even in a market without regional caps, that is, $q_r > |D|$ for all $r \in R$ (Roth, 1982).[16] Given this limitation, we consider the following weakening of the concept requiring incentive compatibility only for doctors. A mechanism φ is said to be **strategy-proof for doctors** if there does not exist a preference profile $\succ$, a doctor $d \in D$, and preferences $\succ'_d$ of doctor d such that

$$\varphi_d(\succ'_d, \succ_{-d}) \succ_d \varphi_d(\succ).$$

A mechanism φ is said to be **group strategy-proof for doctors** if there is no preference profile $\succ$, a subset of doctors $D' \subseteq D$, and a preference profile $(\succ'_{d'})_{d' \in D'}$ of doctors

[15]One good aspect of having strategy-proofness is that the matching authority can actually state it as the property of the algorithm to encourage doctors to reveal their true preferences. For example, the current webpage of the Japan Residency Matching Program (last accessed on May 25, 2010, http://www.jrmp.jp/01-ryui.htm) states, as advice for doctors, that "If you list as your first choice a program which is not actually your first choice, the probability that you end up being matched with some hospital does not increase [...] the probability that you are matched with your actual first choice decreases." In the context of student placement in Boston, strategy-proofness was regarded as a desirable fairness property, in the sense that it provides equal access for children and parents with different degrees of sophistication to strategize (Pathak and Sonmez, 2008).

[16]Remember that a special case of our model in which $q_r > |D|$ for all $r \in R$ is the standard matching model with no binding regional caps.

in D' such that

$$\varphi_d((\succ'_{d'})_{d' \in D'}, (\succ_i)_{i \in D \cup H \setminus D'}) \succ_d \varphi_d(\succ) \text{ for all } d \in D'.$$

That is, no subset of doctors can jointly misreport their preferences to receive a strictly preferred outcome for every member of the coalition under the mechanism.

As this paper analyzes the effect of regional caps in matching markets, it is useful to compare it with the standard matching model without regional caps. Gale and Shapley (1962) consider a matching model without any binding regional cap, which corresponds to a special case of our model in which $q_r > |D|$ for every $r \in R$. In that model, they propose the following **(doctor-proposing) deferred acceptance algorithm**:

- Step 1: Each doctor applies to her first choice hospital. Each hospital rejects the lowest-ranking doctors in excess of its capacity and all unacceptable doctors among those who applied to it, keeping the rest of the doctors temporarily (so doctors not rejected at this step may be rejected in later steps).

In general,

- Step t: Each doctor who was rejected in Step $(t-1)$ applies to her next highest choice (if any). Each hospital considers these doctors *and* doctors who are temporarily held from the previous step together, and rejects the lowest-ranking doctors in excess of its capacity and all unacceptable doctors, keeping the rest of the doctors temporarily (so doctors not rejected at this step may be rejected in later steps).

The algorithm terminates at a step in which no rejection occurs. The algorithm always terminates in a finite number of steps. Gale and Shapley (1962) show that the resulting matching is stable in the standard matching model without any binding regional cap.

Even though there exists no strategy-proof mechanism that produces a stable matching for all possible inputs, the deferred acceptance mechanism is (group) strategy-proof for doctors (Dubins and Freedman, 1981; Roth, 1982).[17] This result has been extended by many subsequent studies, suggesting that the incentive compatibility of the mechanism is quite robust and general.[18]

---

[17]Ergin (2002) defines a stronger version of group strategy-proofness. It requires that no group of doctors can misreport preferences jointly and make some of its members strictly better off without making any of its members strictly worse off. He identifies a necessary and sufficient condition for the deferred acceptance mechanism to satisfy this version of group strategy-proofness.

[18]Researches generalizing (group) strategy-proofness of the mechanism include Abdulkadiroğlu (2005), Hatfield and Milgrom (2005), Martinez, Masso, Neme, and Oviedo (2004), Hatfield and Kojima (2009, 2010), and Hatfield and Kominers (2009, 2012).

Kamada and Kojima (2015) present examples that show that a simple adaptation of the deferred acceptance mechanism results in inefficiency and instability. Motivated by this problem, the current paper presents a theory of stable matching under distributional constraints in the subsequent sections.

2.2. **Model with Regional Preferences.** Let **regional preferences** $\succeq_r$ be a weak ordering over nonnegative-valued integer vectors $W_r := \{w = (w_h)_{h \in H_r} | w_h \in \mathbb{Z}_+\}$. That is, $\succeq_r$ is a binary relation that is complete and transitive (but not necessarily antisymmetric). We write $w \succ_r w'$ if and only if $w \succeq_r w'$ holds but $w' \succeq_r w$ does not. Vectors such as w and w' are interpreted to be supplies of acceptable doctors to the hospitals in region r, but they only specify how many acceptable doctors apply to each hospital and no information is given as to who these doctors are. Given $\succeq_r$, a function $\tilde{\text{Ch}}_r : W_r \to W_r$ is an **associated quasi choice rule** if $\tilde{\text{Ch}}_r(w) \in \arg\max_{\succeq_r}\{w'|w' \leq w\}$ for any non-negative integer vector $w = (w_h)_{h \in H_r}$.[19] We require that the quasi choice rule $\tilde{\text{Ch}}_r$ be **consistent**, that is, $\tilde{\text{Ch}}_r(w) \leq w' \leq w \Rightarrow \tilde{\text{Ch}}_r(w') = \tilde{\text{Ch}}_r(w)$.[20] This condition requires that, if $\tilde{\text{Ch}}_r(w)$ is chosen at w and the supply decreases to $w' \leq w$ but $\tilde{\text{Ch}}_r(w)$ is still available under w', then the same choice $\tilde{\text{Ch}}_r(w)$ should be made under w' as well. Note that there may be more than one quasi choice rule associated with a given weak ordering $\succeq_r$ because the set $\arg\max_{\succeq_r}\{w'|w' \leq w\}$ may not be a singleton for some $\succeq_r$ and w. Note also that there always exists a consistent quasi choice rule.[21] We assume that the regional preferences $\succeq_r$ satisfy the following mild regularity conditions:

(1) $w' \succ_r w$ if $w_h > q_h \geq w'_h$ for some $h \in H_r$ and $w'_{h'} = w_{h'}$ for all $h' \neq h$.

This property says that the region desires no hospital to be forced to be assigned more doctors than its real capacity. This condition implies that, for any w, the component $[\tilde{\text{Ch}}_r(w)]_h$ of $\tilde{\text{Ch}}_r(w)$ for h satisfies $[\tilde{\text{Ch}}_r(w)]_h \leq q_h$ for each $h \in H_r$,

[19] For any two vectors $w = (w_h)_{h \in H_r}$ and $w' = (w'_h)_{h \in H_r}$, we write $w \leq w'$ if and only if $w_h \leq w'_h$ for all $h \in H_r$. We write $w \lneq w'$ if and only if $w \leq w'$ and $w_h < w'_h$ for at least one $h \in H_r$. For any $W'_r \subseteq W_r$, $\arg\max_{\succeq_r} W'_r$ is the set of vectors $w \in W'_r$ such that $w \succeq_r w'$ for all $w' \in W'_r$.

[20] Analogous conditions are used by Blair (1988), Alkan (2002), and Alkan and Gale (2003) in different contexts. In Appendix E, we show that if a regional preference satisfies substitutability and its associated quasi choice rule is acceptant, as defined later, then the quasi choice rule satisfies consistency. Fleiner (2003) and Aygün and Sönmez (2012) prove analogous results although they do not work on substitutability defined over the space of integer vectors.

[21] To see this point consider preferences $\succeq'_r$ such that $w \succ'_r w'$ if $w \succ_r w'$ and $w = w'$ if $w \succeq'_r w'$ and $w' \succeq'_r w$. The quasi choice rule that chooses (the unique element of) $\arg\max_{\succeq'_r}\{w'|w' \leq w\}$ for each w is clearly consistent with $\succeq_r$.

that is, the capacity constraint for each hospital is respected by the (quasi) choice of the region.

(2) $w' \succ_r w$ if $\sum_{h\in H_r} w_h > q_r \geq \sum_{h\in H_r} w'_h$.

This property simply says that region r prefers the total number of doctors in the region to be at most its regional cap. This condition implies that $\sum_{h\in H_r}(\tilde{\text{Ch}}_r(w))_h \leq q_r$ for any w, that is, the regional cap is respected by the (quasi) choice of the region.

(3) If $w' \lneq w \leq q_{H_r} := (q_h)_{h\in H_r}$ and $\sum_{h\in H_r} w_h \leq q_r$, then $w \succ_r w'$.

This condition formalizes the idea that region r prefers to fill as many positions of hospitals in the region as possible so long as doing so does not lead to a violation of the hospitals' real capacities or the regional cap. This requirement implies that any associated quasi choice rule is **acceptant**, that is, for each w, if there exists h such that $[\tilde{\text{Ch}}_r(w)]_h < \min\{q_h, w_h\}$, then $\sum_{h'\in H_r}[\tilde{\text{Ch}}_r(w)]_{h'} = q_r$.[22] This captures the idea that the social planner should not waste caps allocated to the region: If some doctor is rejected by a hospital even though she is acceptable to the hospital and the hospital's capacity is not binding, then the regional cap should be binding.

Definition 1. The regional preferences $\succeq_r$ are **substitutable** if there exists an associated quasi choice rule $\tilde{\text{Ch}}_r$ that satisfies $w \leq w' \Rightarrow \tilde{\text{Ch}}_r(w) \geq \tilde{\text{Ch}}_r(w') \wedge w$.

Notice that the condition in this definition is equivalent to

$$w \leq w' \Rightarrow [\tilde{\text{Ch}}_r(w)]_h \geq \min\{[\tilde{\text{Ch}}_r(w')]_h, w_h\} \text{ for every } h \in H_r. \tag{2.1}$$

This condition says that, when the supply of doctors is increased, the number of accepted doctors at a hospital can increase only when the hospital has accepted all acceptable doctors under the original supply profile. Formally, condition (2.1) is equivalent to

$$w \leq w' \text{ and } [\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h \Rightarrow [\tilde{\text{Ch}}_r(w)]_h = w_h. \tag{2.2}$$

To see that condition (2.1) implies condition (2.2), suppose that $w \leq w'$ and $[\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h$. These assumptions and condition (2.1) imply $[\tilde{\text{Ch}}_r(w)]_h \geq w_h$. Since $[\tilde{\text{Ch}}_r(w)]_h \leq w_h$ holds by the definition of $\tilde{\text{Ch}}_r$, this implies $[\tilde{\text{Ch}}_r(w)]_h = w_h$. To see that condition (2.2) implies condition (2.1), suppose that $w \leq w'$. If $[\tilde{\text{Ch}}_r(w)]_h \geq [\tilde{\text{Ch}}_r(w')]_h$, the conclusion of (2.1) is trivially satisfied. If $[\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h$, then condition (2.2) implies $[\tilde{\text{Ch}}_r(w)]_h = w_h$, thus the conclusion of (2.1) is satisfied.

[22] A similar condition is used by Alkan (2001) and Kojima and Manea (2010) in the context of choice functions over matchings.

This definition of substitutability is analogous to *persistence* by Alkan and Gale (2003), who define the condition on a choice function in a slightly different context. While our definition is similar to substitutability as defined in standard matching models (see Chapter 6 of Roth and Sotomayor (1990) for instance), there are two differences: (i) it is now defined on a region as opposed to a hospital, and (ii) it is defined over vectors that only specify how many doctors apply to hospitals in the region, and it does not distinguish different doctors.

Given a profile of regional preferences $(\succeq_r)_{r \in R}$, stability is defined as follows.

Definition 2. A matching μ is **stable** if it is feasible, individually rational, and if (d, h) is a blocking pair then (i) $|\mu_{r(h)}| = q_{r(h)}$, (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$, and

(iii) either $\mu_d \notin H_{r(h)}$ or $w \succeq_{r(h)} w'$,

where $w_{h'} = |\mu_{h'}|$ for all $h' \in H_{r(h)}$ and $w'_h = w_h + 1$, $w'_{\mu_d} = w_{\mu_d} - 1$ and $w'_{h'} = w_{h'}$ for all other $h' \in H_{r(h)}$.

As stated in the definition, only certain blocking pairs are tolerated under stability. Any blocking pair that may remain is in danger of violating the regional cap since condition (i) implies that the cap for the blocking hospital's region is currently full, and condition (ii) implies that the only blocking involves filling a vacant position.

There are two possible cases under (iii). The first case implies that the blocking doctor is not currently assigned in the hospital's region, so the blocking pair violates the regional cap. The second part of condition (iii) considers blocking pairs within a region (note that $\mu_d \in H_{r(h)}$ holds in the remaining case). It states that if the blocking pair does not improve the doctor distribution in the region with respect to its regional preferences, then it is not regarded as a legitimate block.

The implicit idea behind the definition is that the government or some authority can interfere and prohibit a blocking pair to be executed if regional caps are an issue. Thus, our preferred interpretation is that stability captures a normative notion that it is desirable to implement a matching that respects participants' preferences to the extent possible. Justification of the normative appeal of stability has been established in the recent matching literature, and Kamada and Kojima (2015) offer further discussion on this point, so we refer interested readers to that paper for details.

The way that regional preferences are determined could depend on the policy goal of the region or the social planner. One possibility for regional preferences, studied in detail by Kamada and Kojima (2015), is to prefer distributions of doctors that have "fewer gaps" from the target capacities; see Section 5.1 for detail. Other regional preferences are analyzed in Section 5.2.

Clearly, our stability concept reduces to the standard stability concept of Gale and Shapley (1962) if there are no binding regional caps.

Remark 1. In Kamada and Kojima (2016), we show that any matching satisfying a weaker notion than stability is (constrained) efficient, i.e., there is no feasible matching μ' such that $\mu'_i \succeq_i \mu_i$ for all $i \in D \cup H$ and $\mu'_i \succ_i \mu_i$ for some $i \in D \cup H$.[23] Therefore, a stable matching is efficient for any regional preferences, which provides one normative appeal of our stability concept.

Remark 2. Kamada and Kojima (2016) define a more demanding concept than stability that replaces condition (iii) in Definition 2 with $\mu_d \notin H_{r(h)}$, which they call strong stability. While strong stability also has a natural interpretation, they demonstrate that a strongly stable matching does not necessarily exist, and no mechanism is strategy-proof for doctors and produces a strongly stable matching when there exists one. Given these negative findings, the present paper focuses on stability as defined in Definition 2.

3. Main Result

This section has two goals. The first goal is to demonstrate that a stable matching exists under our general definition of stability under distributional constraints. The second goal is to show that a stable matching can be found by a mechanism that is strategy-proof for doctors. To achieve these goals, we begin by introducing the following (generalized) flexible deferred acceptance algorithm:

The (Generalized) Flexible Deferred Acceptance Algorithm For each region r, fix an associated quasi choice rule $\tilde{\text{Ch}}_r$ which satisfies condition (2.1). Note that the assumption that $\succeq_r$ is substitutable assures the existence of such a quasi choice rule.

(1) Begin with an empty matching, that is, a matching μ such that $\mu_d = \emptyset$ for all $d \in D$.
(2) Choose a doctor d arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
(3) Let d apply to the most preferred hospital $\bar{h}$ at $\succ_d$ among the hospitals that have not rejected d so far. If d is unacceptable to $\bar{h}$, then reject this doctor and go

[23] Since regional caps are a primitive of the environment, we consider a *constrained* efficiency concept.

back to Step 2. Otherwise, let r be the region such that $\bar{h} \in H_r$ and define vector $w = (w_h)_{h \in H_r}$ by
  (a) $w_{\bar{h}}$ is the number of doctors currently held at $\bar{h}$ plus one, and
  (b) w_h is the number of doctors currently held at h if $h \neq \bar{h}$.
(4) Each hospital $h \in H_r$ considers the new applicant d (if $h = \bar{h}$) *and* doctors who are temporarily held from the previous step together. It holds its $(\tilde{\mathrm{Ch}}_r(w))_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to Step 2.

We define the **(generalized) flexible deferred acceptance mechanism** to be a mechanism that produces, for each input, the matching given at the termination of the above algorithm.

This algorithm is a generalization of the deferred acceptance algorithm of Gale and Shapley (1962) to the model with regional caps. The main differences are found in Steps 3 and 4. Unlike the deferred acceptance algorithm, this algorithm limits the number of doctors (tentatively) matched in each region r at q_r. This results in rationing of doctors across hospitals in the region, and the rationing rule is governed by regional preferences $\succeq_r$. Clearly, this mechanism coincides with the standard deferred acceptance algorithm if all the regional caps are large enough and hence non-binding.

With the definition of the flexible deferred acceptance mechanism, we are now ready to present the main result of this paper.

Theorem 1. *Suppose that $\succeq_r$ is substitutable for every $r \in R$. Then the flexible deferred acceptance algorithm stops in finite steps. The mechanism produces a stable matching for any input and is group strategy-proof for doctors.*

This theorem offers a sense in which it is possible to design a desirable mechanism even under distributional constraints and various policy goals. As will be seen in subsequent sections, the class of substitutable regional preferences subsumes the "Rawlsian" regional preferences motivated by a residency matching application (Section 5.1) as well as others (Section 5.2). For each of these cases, the flexible deferred acceptance mechanism finds a stable matching that addresses a given policy goal, while inducing truthful reporting by doctors. Moreover, because stability implies efficiency (Kamada and Kojima, 2016), the algorithm produces an efficient matching.

4. Proof of Theorem 1

This section presents the proof of Theorem 1. Since the proof is somewhat involved, we illustrate a sketch of the proof in Section 4.1, and then present the formal proof in Section 4.2.

4.1. **Sketch of the proof.** Our proof strategy is to connect our matching model with constraints to the "matching with contracts" model (Hatfield and Milgrom, 2005). More specifically, given the original matching model under constraints, we define an "associated model," a hypothetical matching model between doctors and *regions* instead of doctors and hospitals; In the associated model, we regard each region as a hypothetical consortium of hospitals that acts as one agent. By imagining that a region (hospital consortium) makes a coordinated employment decision, we can account for the fact that acceptance of a doctor by a hospital may depend on doctor applications to other hospitals in the same region, an inevitable feature in markets under distributional constraints. This association necessitates, however, that we distinguish a doctor's placements in different hospitals in the given region. We account for this complication by defining a region's choice function over *contracts* rather than doctors, where a contract specifies a doctor-hospital pair to be matched. We construct such a choice function by using two pieces of information: the preferences of all the hospitals in the given region, and regional preferences. The idea is that each hospital's preferences are used for choosing doctors *given the number of allocated slots*, while regional preferences are used to regulate slots allocated to different hospitals in the region. In other words, regional preferences trade off multiple hospitals' desires to accept more doctors, when accepting more is in conflict with the regional cap. With the help of this association, we demonstrate that any stable allocation in the associate model with contracts induces a stable matching in the original model with distributional constraints (Proposition 2).

Once this association is established, with some work we show that the key conditions in the associated model—the substitutes condition and the law of aggregate demand—are satisfied (Proposition 1). This enables us to invoke existing results for matching with contracts, namely that an existing algorithm called the "cumulative offer process" finds a stable allocation, and it is (group) strategy-proof for doctors in the associated model (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2009; Hatfield and Kominers, 2012). Then, we observe that the outcome of the cumulative offer process corresponds to the matching produced by the flexible deferred acceptance algorithm in the original model with constraints (Remark 3). This correspondence establishes that the flexible deferred acceptance mechanism finds a stable matching in the original problem and this algorithm is group strategy-proof for doctors, proving Theorem 1.

The full proof, presented in the next subsection, formalizes this idea. The proof is somewhat involved because one needs to exercise some care when establishing correspondences between the two models and confirming that a property in one model induces the corresponding property in the other. For illustration, our proof approach is represented as a chart in Figure 3.

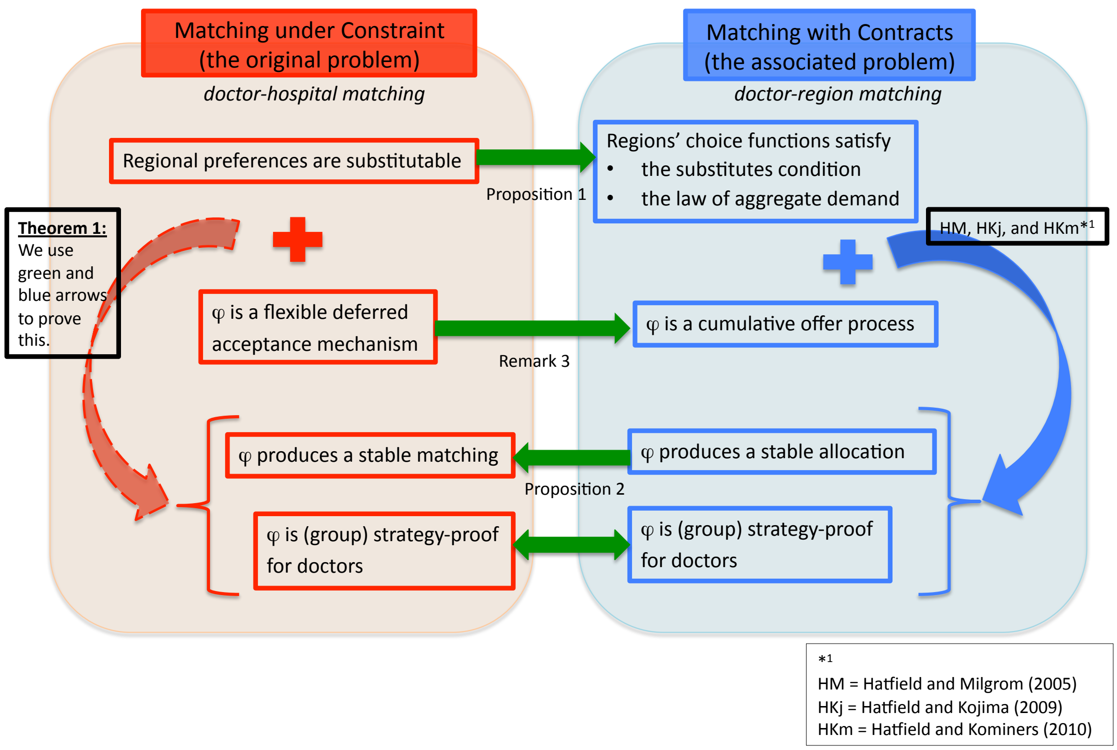

FIGURE 3. Proof sketch for Theorem 1.

4.2. **The complete proof.** Let there be two types of agents, doctors in D and regions in R. Note that we regard a region, instead of a hospital, as an agent in this model. There is a set of contracts $X = D \times H$.

We assume that, for each doctor d, any set of contracts with cardinality two or more is unacceptable, that is, a doctor wants to sign at most one contract. For each doctor d, her preferences $\succ_d$ over $(\{d\} \times H) \cup \{\emptyset\}$ are given as follows.[24] We assume $(d, h) \succ_d (d, h')$

[24]We abuse notation and use the same notation $\succ_d$ for preferences of doctor d both in the original model and in the associated model with contracts.

in this model if and only if $h \succ_d h'$ in the original model, and $(d,h) \succ_d \emptyset$ in this model if and only if $h \succ_d \emptyset$ in the original model.

For each region $r \in R$, we assume that the region has preferences $\succeq_r$ and its associated choice rule $\text{Ch}_r(\cdot)$ over all subsets of $D \times H_r$. For any $X' \subset D \times H_r$, let $w(X') := (w_h(X'))_{h \in H_r}$ be the vector such that $w_h(X') = |\{(d,h) \in X' | d \succ_h \emptyset\}|$. For each X', the chosen set of contracts $\text{Ch}_r(X')$ is defined by

$$\text{Ch}_r(X') = \bigcup_{h \in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D | (d',h) \in X', d' \succeq_h d\}| \leq (\tilde{\text{Ch}}_r(w(X')))_h \right\}. \tag{4.1}$$

That is, each hospital $h \in H_r$ chooses its $(\tilde{\text{Ch}}_r(w(X')))_h$ most preferred contracts available in X'.

We extend the domain of the choice rule to the collection of all subsets of X by setting $\text{Ch}_r(X') = \text{Ch}_r(\{(d,h) \in X' | h \in H_r\})$ for any $X' \subseteq X$.

Definition 3 (Hatfield and Milgrom (2005))**.** Choice rule $\text{Ch}_r(\cdot)$ satisfies the **substitutes condition** if there does not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \text{Ch}_r(X' \cup \{x'\})$ and $x' \in \text{Ch}_r(X' \cup \{x, x'\})$.

In other words, contracts are substitutes if adding a contract to the choice set never induces a region to choose a contract it previously rejected. Hatfield and Milgrom (2005) show that there exists a stable allocation (defined in Definition 5) when contracts are substitutes for every region.

Definition 4 (Hatfield and Milgrom (2005))**.** Choice rule $\text{Ch}_r(\cdot)$ satisfies the **law of aggregate demand** if for all $X' \subseteq X'' \subseteq X$, $|\text{Ch}_r(X')| \leq |\text{Ch}_r(X'')|$.[25]

Proposition 1. *Suppose that $\succeq_r$ is substitutable. Then choice rule $\text{Ch}_r(\cdot)$ defined above satisfies the substitutes condition and the law of aggregate demand.*[26]

Proof. Fix a region $r \in R$. Let $X' \subseteq X$ be a subset of contracts and $x = (d,h) \in X \setminus X'$ where $h \in H_r$. Let $w = w(X')$ and $w' = w(X' \cup x)$. To show that Ch_r satisfies the substitutes condition, we consider a number of cases as follows.

[25]Analogous conditions called cardinal monotonicity and size monotonicity are introduced by Alkan (2002) and Alkan and Gale (2003) for matching models without contracts.

[26]Note that choice rule $\text{Ch}_r(\cdot)$ allows for the possibility that multiple contracts are signed between the same pair of a region and a doctor. Without this possibility, the choice rule may violate the substitutes condition (Sönmez and Switzer, 2013; Sönmez, 2013). Hatfield and Kominers (2013) explore this issue further.

(1) Suppose that $\emptyset \succ_h d$. Then $w' = w$ and, for each $h' \in H_r$, the set of acceptable doctors available at $X' \cup x$ is identical to the one at X'. Therefore, by inspection of the definition of Ch_r, we have $\text{Ch}_r(X' \cup x) = \text{Ch}_r(X')$, satisfying the conclusion of the substitutes condition in this case.

(2) Suppose that $d \succ_h \emptyset$.

(a) Consider a hospital $h' \in H_r \setminus h$. Note that we have $w'_{h'} = w_{h'}$. This and the inequality $[\tilde{\text{Ch}}_r(w')]_{h'} \leq w'_{h'}$ (which always holds by the definition of $\tilde{\text{Ch}}_r$) imply that $[\tilde{\text{Ch}}_r(w')]_{h'} \leq w_{h'}$. Thus we obtain $\min\{[\tilde{\text{Ch}}_r(w')]_{h'}, w_{h'}\} = [\tilde{\text{Ch}}_r(w')]_{h'}$. Since $w' \geq w$ and condition (2.1) holds, this implies that

$$[\tilde{\text{Ch}}_r(w)]_{h'} \geq [\tilde{\text{Ch}}_r(w')]_{h'}. \tag{4.2}$$

Also observe that the set $\{d' \in D | (d', h') \in X'\}$ is identical to $\{d' \in D | (d', h') \in X' \cup x\}$, that is, the sets of doctors that are available to hospital h' are identical under X' and $X' \cup x$. This fact, inequality (4.2), and the definition of Ch_r imply that if $x' = (d', h') \notin \text{Ch}_r(X')$, then $x' \notin \text{Ch}_r(X' \cup x)$, obtaining the conclusion for the substitute condition in this case.

(b) Consider hospital h.

(i) Suppose that $[\tilde{\text{Ch}}_r(w)]_h \geq [\tilde{\text{Ch}}_r(w')]_h$. In this case we follow an argument similar to (but slightly different from) Case (2a): Note that the set $\{d' \in D | (d', h) \in X'\}$ is a subset of $\{d' \in D | (d', h) \in X' \cup x\}$, that is, the set of doctors that are available to hospital h under X' is smaller than under $X' \cup x$. These properties and the definition of Ch_r imply that if $x' = (d', h) \in X' \setminus \text{Ch}_r(X')$, then $x' \in X' \setminus \text{Ch}_r(X' \cup x)$, obtaining the conclusion for the substitute condition in this case.

(ii) Suppose that $[\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h$. This assumption and (2.2) imply $[\tilde{\text{Ch}}_r(w)]_h = w_h$. Thus, by the definition of Ch_r, any contract $(d', h) \in X'$ such that $d' \succ_h \emptyset$ is in $\text{Ch}_r(X')$. Equivalently, if $x' = (d', h) \in X' \setminus \text{Ch}_r(X')$, then $\emptyset \succ_h d'$. Then, again by the definition of Ch_r, it follows that $x' \notin \text{Ch}_r(X' \cup x)$ for any contract $x' = (d', h) \in X' \setminus \text{Ch}_r(X')$. Thus we obtain the conclusion of the substitute condition in this case.

To show that Ch_r satisfies the law of aggregate demand, simply note that $\tilde{\text{Ch}}_r$ is acceptant by assumption. This leads to the desired conclusion. □

A subset X' of $X = D \times H$ is said to be **individually rational** if (1) for any $d \in D$, $|\{(d, h) \in X' | h \in H\}| \leq 1$, and if $(d, h) \in X'$ then $h \succ_d \emptyset$, and (2) for any $r \in R$, $\text{Ch}_r(X') = X' \cap (D \times H_r)$.

Definition 5. A set of contracts $X' \subseteq X$ is a **stable allocation** if

(1) it is individually rational, and
(2) there exists no region $r \in R$, hospital $h \in H_r$, and a doctor $d \in D$ such that $(d,h) \succ_d x$ and $(d,h) \in \text{Ch}_r(X' \cup \{(d,h)\})$, where x is the contract that d receives at X' if any and $\emptyset$ otherwise.

When condition (2) is violated by some (d,h), we say that (d,h) is a **block** of X'. A **doctor-optimal stable allocation** in the matching model with contracts is a stable allocation that every doctor weakly prefers to every other stable allocation (Hatfield and Milgrom, 2005).

Given any individually rational set of contracts X', define a **corresponding matching** $\mu(X')$ in the original model by setting $\mu_d(X') = h$ if and only if $(d,h) \in X'$ and $\mu_d(X') = \emptyset$ if and only if no contract associated with d is in X'. Since each doctor regards any set of contracts with cardinality of at least two as unacceptable, each doctor receives at most one contract at X' and hence $\mu(X')$ is well defined for any individually rational X'.

Proposition 2. *If X' is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

Proof. Suppose that X' is a stable allocation in the associated model with contracts and denote $\mu := \mu(X')$. Individual rationality of μ is obvious from the construction of μ. Suppose that (d,h) is a blocking pair of μ. Denoting $r := r(h)$, by the definition of stability, it suffices to show that the following conditions (4.3) and (4.4) hold if $\mu_d \notin H_r$, and (4.3), (4.4) and (4.5) hold if $\mu_d \in H_r$:

$$|\mu_{H_r}| = q_r, \tag{4.3}$$

$$d' \succ_h d \text{ for all } d' \in \mu_h, \tag{4.4}$$

$$w \succeq_r w', \tag{4.5}$$

where $w = (w_h)_{h \in H_r}$ is defined by $w_{h'} = |\mu_{h'}|$ for all $h' \in H_r$ while $w' = (w'_h)_{h \in H_r}$ is defined by $w'_h = w_h + 1$, $w'_{\mu_d} = w_{\mu_d} - 1$ (if $\mu_d \in H_r$) and $w'_{h'} = w_{h'}$ for all other $h' \in H_r$.

Claim 1. *Conditions (4.3) and (4.4) hold (irrespectively of whether $\mu_d \in H_r$ or not).*

Proof. First note that the assumption that $h \succ_d \mu_d$ implies that $(d,h) \succ_d x$ where x denotes the (possibly empty) contract that d signs under X'. Let $w'' = (w''_h)_{h \in H_r}$ be defined by $w''_h = w_h + 1$ and $w''_{h'} = w_{h'}$ for all other $h' \in H_r$.

(1) Assume by contradiction that condition (4.4) is violated, that is, $d \succ_h d'$ for some $d' \in \mu_h$. First, by consistency of $\tilde{\text{Ch}}_r$, we have $[\tilde{\text{Ch}}_r(w'')]_h \geq [\tilde{\text{Ch}}_r(w)]_h$.[27] That is, weakly more contracts involving h are signed at $X' \cup (d,h)$ than at X'. This property, together with the assumptions that $d \succ_h d'$ and that $(d',h) \in X'$ imply that $(d,h) \in \text{Ch}_r(X' \cup (d,h))$.[28] Thus, together with the above-mentioned property that $(d,h) \succ_d x$, (d,h) is a block of X' in the associated model of matching with contracts, contradicting the assumption that X' is a stable allocation.

(2) Assume by contradiction that condition (4.3) is violated, so that $|\mu_{H_r}| \neq q_r$. Then, since $|\mu_{H_r}| \leq q_r$ by the construction of μ and the assumption that X' is individually rational, it follows that $|\mu_{H_r}| < q_r$. Then $(d,h) \in \text{Ch}_r(X' \cup (d,h))$ because,
   (a) $d \succ_h \emptyset$ by assumption,
   (b) since $\sum_{h \in H_r} w_h = \sum_{h \in H_r} |\mu_h| = |\mu_{H_r}| < q_r$, it follows that $\sum_{h \in H_r} w''_h = \sum_{h \in H_r} w_h + 1 \leq q_r$. Moreover, $|\mu_h| < q_h$ because (d,h) is a blocking pair by assumption and (4.4) holds, so $w''_h = |\mu_h| + 1 \leq q_h$. These properties and the assumption that $\tilde{\text{Ch}}_r$ is acceptant imply that $\tilde{\text{Ch}}_r(w'') = w''$. In particular, this implies that all contracts $(d',h) \in X' \cup (d,h)$ such that $d' \succ_h \emptyset$ is chosen at $\text{Ch}_r(X' \cup (d,h))$.

   Thus, together with the above-mentioned property that $(d,h) \succ_d x$, (d,h) is a block of X' in the associated model of matching with contract, contradicting the assumption that X' is a stable allocation.

□

To finish the proof of the proposition suppose that $\mu_d \in H_r$ and by contradiction that (4.5) fails, that is, $w' \succ_r w$. Then it should be the case that $[\tilde{\text{Ch}}_r(w'')]_h = w''_h = w_h + 1 =$

[27]To show this claim, assume for contradiction that $[\tilde{\text{Ch}}_r(w'')]_h < [\tilde{\text{Ch}}_r(w)]_h$. Then, $[\tilde{\text{Ch}}_r(w'')]_h < [\tilde{\text{Ch}}_r(w)]_h \leq w_h$. Moreover, since $w''_{h'} = w_{h'}$ for every $h' \neq h$ by construction of w'', it follows that $[\tilde{\text{Ch}}_r(w'')]_{h'} \leq w''_{h'} = w_{h'}$. Combining these inequalities, we have that $\tilde{\text{Ch}}_r(w'') \leq w$. Also we have $w \leq w''$ by the definition of w'', so it follows that $\tilde{\text{Ch}}_r(w'') \leq w \leq w''$. Thus, by consistency of $\tilde{\text{Ch}}_r$, we obtain $\tilde{\text{Ch}}_r(w'') = \tilde{\text{Ch}}_r(w)$, a contradiction to the assumption $[\tilde{\text{Ch}}_r(w'')]_h < [\tilde{\text{Ch}}_r(w)]_h$.

[28]The proof of this claim is as follows. $\text{Ch}_r(X')$ induces hospital h to select its $[\tilde{\text{Ch}}_r(w)]_h$ most preferred contracts while $\text{Ch}_r(X' \cup (d,h))$ induces h to select a weakly larger number $[\text{Ch}_r(w'')]_h$ of its most preferred contracts. Since (d',h) is selected as one of the $[\tilde{\text{Ch}}_r(w)]_h$ most preferred contracts for h at X' and $d \succ_h d'$, we conclude that (d,h) should be one of the $[\text{Ch}_r(w'')]_h \geq [\tilde{\text{Ch}}_r(w)]_h$ most preferred contracts at $X' \cup (d,h)$, thus selected at $X' \cup (d,h)$.

$|\mu_h| + 1$.[29] Also we have $|\mu_h| < q_h$ and hence $|\mu_h| + 1 \leq q_h$ and $d \succ_h \emptyset$, so

$$(d, h) \in \mathrm{Ch}_r(X' \cup (d, h)).$$

This relationship, together with the assumption that $h \succ_d \mu_d$, and hence $(d, h) \succ_d x$, is a contradiction to the assumption that X' is stable in the associated model with contracts. □

Remark 3. Each step of the flexible deferred acceptance algorithm corresponds to a step of the cumulative offer process (Hatfield and Milgrom, 2005), that is, at each step, if doctor d proposes to hospital h in the flexible deferred acceptance algorithm, then at the same step of the cumulative offer process, contract (d, h) is proposed. Moreover, the set of doctors accepted for hospitals at a step of the flexible deferred acceptance algorithm corresponds to the set of contracts held at the corresponding step of the cumulative offer process. Therefore, if X' is the allocation that is produced by the cumulative offer process, then $\mu(X')$ is the matching produced by the flexible deferred acceptance algorithm. These observations apply to a more general model presented in Section 6 as well.

Proof of Theorem 1. By Proposition 1, the choice function of each region satisfies the substitutes condition and the law of aggregate demand in the associate model of matching with contracts. By Hatfield and Milgrom (2005), Hatfield and Kojima (2009), and Hatfield and Kominers (2009, 2012), the cumulative offer process with choice functions satisfying these conditions produces a stable allocation and is (group) strategy-proof.[30] The former fact, together with Remark 3 and Proposition 2, implies that the outcome of the flexible deferred acceptance algorithm is a stable matching in the original model. The latter fact and Remark 3 imply that the flexible deferred acceptance mechanism is (group) strategy-proof for doctors. □

[29]To show this claim, assume by contradiction that $[\tilde{\mathrm{Ch}}_r(w'')]_h \leq w_h$. Then, since $w''_{h'} = w_{h'}$ for any $h' \neq h$ by the definition of w'', it follows that $\tilde{\mathrm{Ch}}_r(w'') \leq w \leq w''$. Thus by consistency of $\tilde{\mathrm{Ch}}_r$, we obtain $\tilde{\mathrm{Ch}}_r(w'') = \tilde{\mathrm{Ch}}_r(w)$. But $\tilde{\mathrm{Ch}}_r(w) = w$ because X' is a stable allocation in the associated model of matching with contracts, so $\tilde{\mathrm{Ch}}_r(w'') = w$. This is a contradiction because $w' \leq w''$ and $w' \succ_r w$ while $\tilde{\mathrm{Ch}}_r(w'') \in \arg\max_{\succeq_r}\{w'''|w''' \leq w''\}$.

[30]Aygün and Sönmez (2012) point out that a condition called path-independence (Fleiner, 2003) or irrelevance of rejected contracts (Aygün and Sönmez, 2012) is needed for these conclusions. Aygün and Sönmez (2012) show that the substitutes condition and the law of aggregate demand imply this condition. Since the choice rules in our context satisfy the substitutes condition and the law of aggregate demand, the conclusions go through.

5. Discussion

5.1. **Stability in Kamada and Kojima (2015).** In this section we establish the main result of Kamada and Kojima (2015) by showing that their stability concept can be rewritten by using a substitutable regional preferences.

In Kamada and Kojima (2015), there is an exogenously given (government-imposed) nonnegative integer $\bar{q}_h \leq q_h$ called **target capacity**, for each hospital h such that $\sum_{h \in H_r} \bar{q}_h \leq q_r$ for each region $r \in R$. Given a profile of target capacities, their stability concept is defined as follows.

Definition 6. A matching μ is **stable** if it is feasible, individually rational, and if (d, h) is a blocking pair then (i) $|\mu_{r(h)}| = q_{r(h)}$, (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$, and

(iii) either $\mu_d \notin H_{r(h)}$ or $|\mu'_h| - \bar{q}_h > |\mu'_{\mu_d}| - \bar{q}_{\mu_d}$,

where μ' is the matching such that $\mu'_d = h$ and $\mu'_{d'} = \mu_{d'}$ for all $d' \neq d$.

Kamada and Kojima (2015) define the **flexible deferred acceptance algorithm** in their setting as follows. For each $r \in R$, specify an order of hospitals in region r: Denote $H_r = \{h_1, h_2, \ldots, h_{|H_r|}\}$ and order h_i earlier than h_j if $i < j$. Given this order, consider the following algorithm.

(1) Begin with an empty matching, that is, a matching μ such that $\mu_d = \emptyset$ for all $d \in D$.
(2) Choose a doctor d who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
(3) Let d apply to the most preferred hospital $\bar{h}$ at $\succ_d$ among the hospitals that have not rejected d so far. Let r be the region such that $\bar{h} \in H_r$.
(4) (a) For each $h \in H_r$, let D'_h be the entire set of doctors who have applied to but have not been rejected by h so far and are acceptable to h. For each hospital $h \in H_r$, choose the $\bar{q}_h$ best doctors according to $\succ_h$ from D'_h if they exist, and otherwise choose all doctors in D'_h. Formally, for each $h \in H_r$ choose D''_h such that $D''_h \subset D'_h$, $|D''_h| = \min\{\bar{q}_h, |D'_h|\}$, and $d \succ_h d'$ for any $d \in D''_h$ and $d' \in D'_h \setminus D''_h$.
    (b) Start with a tentative match D''_h for each hospital $h \in H_r$. Hospitals take turns to choose (one doctor at a time) the best remaining doctor in their current applicant pool. Repeat the procedure (starting with h_1, proceeding to $h_2, h_3, \ldots$ and going back to h_1 after the last hospital) until the regional

quota q_r is filled or the capacity of the hospital is filled or no doctor remains to be matched. All other applicants are rejected.[31]

Kamada and Kojima (2015) define the **flexible deferred acceptance mechanism** to be a mechanism that produces, for each input, the matching at the termination of the above algorithm.[32] The following theorem is the main result of Kamada and Kojima (2015).

Proposition 3 (Theorem 2 of Kamada and Kojima (2015))**.** *In the setting of Kamada and Kojima (2015), the flexible deferred acceptance algorithm stops in finite steps. The mechanism produces a stable matching for any input and is group strategy-proof for doctors.*

In the remainder of this section, we establish this result as a corollary of the main result of the present paper, Theorem 1.

To start the analysis, fix a region r. Given the target capacity profile $(\bar{q}_h)_{h\in H_r}$ and the vector $w \in W_r$, define the **ordered excess weight vector** $\eta(w) = (\eta_1(w), ..., \eta_{|H_r|}(w))$ by setting $\eta_i(w)$ to be the i'th lowest value (allowing repetition) of $\{w_h - \bar{q}_h | h \in H_r\}$ (we suppress dependence of η on target capacities). For example, if $w = (w_{h_1}, w_{h_2}, w_{h_3}, w_{h_4}) = (2, 4, 7, 2)$ and $(\bar{q}_{h_1}, \bar{q}_{h_2}, \bar{q}_{h_3}, \bar{q}_{h_4}) = (3, 2, 3, 0)$, then $\eta_1(w) = -1, \eta_2(w) = \eta_3(w) = 2, \eta_4(w) = 4$.

Consider the regional preferences $\succeq_r$ that compare the excess weights lexicographically. More specifically, let $\succeq_r$ be such that $w \succ_r w'$ if and only if there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\eta_j(w) = \eta_j(w')$ for all $j < i$ and $\eta_i(w) > \eta_i(w')$. The associated weak regional preferences $\succeq_r$ are defined by $w \succeq_r w'$ if and only if $w \succ_r w'$ or $\eta(w) = \eta(w')$. We call such regional preferences **Rawlsian**.

Proposition 4. *Stability of Kamada and Kojima (2015), defined in Definition 6, is a special case of the stability in Definition 2 such that the regional preferences of each region are Rawlsian.*

[31]Formally, let $\iota_i = 0$ for all $i \in \{1, 2, \dots, |H_r|\}$. Let $i = 1$.

(i) If either the number of doctors already chosen by the region r as a whole equals q_r, or $\iota_i = 1$, then reject the doctors who were not chosen throughout this step and go back to Step 2.

(ii) Otherwise, let h_i choose the most preferred (acceptable) doctor in D'_{h_i} at $\succ_{h_i}$ among the doctors that have not been chosen by h_i so far, if such a doctor exists and the number of doctors chosen by h_i so far is strictly smaller than q_{h_i}.

(iii) If no new doctor was chosen at Step 4(b)ii, then set $\iota_i = 1$. If a new doctor was chosen at Step 4(b)ii, then set $\iota_j = 0$ for all $j \in \{1, 2, \dots, |H_r|\}$. If $i < |H_r|$ then increment i by one and if $i = |H_r|$ then set i to be 1 and go back to Step 4(b)i.

[32]Theorem 1 and Propositions 4 and 5 show that the algorithm stops in a finite number of steps.

Proof. See Appendix A.1. □

Consider the (generalized) flexible deferred acceptance algorithm in a previous subsection. With the following quasi choice rule, this algorithm is equivalent to the flexible deferred acceptance algorithm in Kamada and Kojima (2015): For each $w' \in W_r$,

$$\tilde{\text{Ch}}_r(w') = \max_{\substack{w = w^k \text{ for some } k \\ \sum_{h \in H_r} w_h \le q_r}} w, \tag{5.1}$$

where $w^0 = (\min\{w'_h, \bar{q}_h\})_{h \in H_r}$ and $w^k \in W_r$ $(k = 1, 2, \dots)$ is defined by

$$w^k_{h_j} = \min\{w'_{h_j}, q_{h_j}, w^{k-1}_{h_j} + \mathbb{I}_{j \equiv k \pmod{|H_r|}}\} \quad \text{for each } j = 1, 2, \dots, |H_r|.$$

Proposition 5. *Rawlsian preferences are substitutable with the associated quasi choice rule (5.1).*

Proof. See Appendix A.1. □

Theorem 1 and Propositions 4 and 5 imply Theorem 2 of Kamada and Kojima (2015).

In Appendix D, we discuss how to allocate target capacities among hospitals in a region, within the Rawlsian framework. There we observe that the allocation problem is similar to the celebrated "bankruptcy problem," and consider several rules studied in that literature.

5.2. **Alternative Criteria.** Although Kamada and Kojima (2015) focus on a particular stability concept and corresponding regional preferences, called Rawlsian preferences, it is quite plausible that some societies may prefer to impose different criteria from the Rawlsian preferences. This section proposes other criteria that seem to be appealing. The following are examples of regional preferences that satisfy substitutability defined in Definition 1. In the following, we assume that $0 \succ_r w$ for any weight vector w such that $\sum_{h \in H_r} w_h > q_r$ or $w_h > q_h$ for some $h \in H_r$. Thus in (1) - (4) below, we assume that any weight vector w satisfies $\sum_{h \in H_r} w_h \le q_r$ and $w_h \le q_h$ for all $h \in H_r$.

(1) "Equal gains": Let the region prefer a distribution that equalizes the weights across hospitals in the region as much as possible. Formally, such a preference, which we call the **equal gains** preferences, can be expressed as the Rawlsian preferences for the special case in which the target capacity for every hospital is set at zero. Since Proposition 5 shows that the Rawlsian preferences are substitutable for any target capacity profile, the equal gains preferences satisfy substitutability.
(2) "Equal Losses": Let the region prefer to equalize the "losses," that is, the differences between the (physical) capacities and the weights across hospitals in the

region. More generally, one could consider the preferences for **equal losses above target capacities**, that is, the regional preferences first prefer to fill as many positions as possible to meet target capacities and then (lexicographically less importantly) prefer to equalize the losses. To formally define such preferences $\succ_r$, recall that $\eta(w)$ denotes the ordered excess weight vector as defined in Section 5.1, and define $\hat{\eta}(w)$ as a $|H_r|$-dimensional vector whose i'th component $\hat{\eta}_i(w)$ is the i'th highest value (allowing repetition) of $\{q_h - w_h | h \in H_r\}$. We let $w \succ_r w'$ if and only if

(a) there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\min\{\eta_j(w), 0\} = \min\{\eta_j(w'), 0\}$ for all $j < i$ and $\min\{\eta_i(w), 0\} > \min\{\eta_i(w'), 0\}$, or

(b) $\min\{\eta_i(w), 0\} = \min\{\eta_i(w'), 0\}$ for every index $i \in \{1, 2, \dots, |H_r|\}$, and there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\hat{\eta}_j(w), = \hat{\eta}_j(w')$ for all $j < i$ and $\hat{\eta}_i(w) < \hat{\eta}_i(w')$.

(3) "Proportional": The **proportional** regional preferences prefer to allocate positions to hospitals in a proportional manner subject to integer constraints. More precisely, define $\tilde{\eta}(w)$ as a $|H_r|$-dimensional vector whose i'th component $\tilde{\eta}_i(w)$ is the i'th lowest value (allowing repetition) of $\{w_h/q_h | h \in H_r\}$. We let $w \succ_r w'$ if there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\tilde{\eta}_j(w), = \tilde{\eta}_j(w')$ for all $j < i$ and $\tilde{\eta}_i(w) > \tilde{\eta}_i(w')$. As above, one could consider preferences for proportional losses as well. Also, these preferences can be generalized so that these concerns enter only above target capacities (this generalization is somewhat tedious but straightforward, and can be done as in Item 2). Finally, when constructing $\tilde{\eta}_i$, we can use a denominator different from q_h.[33]

(4) "Hospital-lexicographic": Let there be a pre-specified order over hospitals, and the region lexicographically prefers filling a slot in a higher-ranked hospital to filling that of a lower-ranked hospital. For instance, the region may desire to fill positions of hospitals that are underserved within the region (say, a prefecture may desire to fill positions of a hospital in a remote island within the prefecture before other hospitals). Formally, **hospital-lexicographic** regional preferences $\succ_r$ are defined as follows. Fix an order over hospitals in r, denoted by $h_1, h_2, \dots,$ and $h_{|H_r|}$. Let $w \succ_r w'$ if and only if there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $w_{h_j} = w'_{h_j}$ for all $j < i$ and $w_{h_i} > w'_{h_i}$. We note that one can also

[33]Moreover, the generalizations mentioned above can be combined. For example, the region may desire to fill capacities above targets proportionally to $q_h - \bar{q}_h$.

consider hospital-lexicographic preferences above targets by using the criterion for hospital-lexicographic preferences for weights above targets.

All the above regional preferences have associated quasi choice rules that satisfy the property that we call "order-respecting." To define this property, let there be a finite sequence of hospitals in region r such that each hospital h appears, potentially repeatedly, q_h times in the sequence, and the total size of the sequence is $\sum_{h\in H_r} q_h$. Consider a quasi choice rule that increases the weights of hospitals one by one following the specified order.[34] Formally, fix a vector $(h_1, h_2, \dots, h_{\sum_{h\in H_r} q_h}) \in (H_r)^{\sum_{h\in H_r} q_h}$ such that $\#\{i \in \{1, 2, \dots, \sum_{h\in H_r} q_h\} | h_i = h\} = q_h$ for each $h \in H_r$, and define $\tilde{\text{Ch}}_r(w)$ through the following algorithm:

(1) Let w^0 be the $|H_r|$-dimensional zero vector, indexed by hospitals in H_r.
(2) For any $t \geq 0$, if $\sum_{h\in H_r} w^t_h = q_r$ or $w^t_h = \min\{q_h, w_h\}$ for all $h \in H_r$, then stop the algorithm and define $\tilde{\text{Ch}}_r(w) = w^t$. If not, define w^{t+1} by:
   (a) If $w^t_{h_{t+1}} < \min\{q_{h_{t+1}}, w_{h_{t+1}}\}$, then let $w^{t+1}_{h_{t+1}} = w^t_{h_{t+1}} + 1$; otherwise, let $w^{t+1}_{h_{t+1}} = w^t_{h_{t+1}}$.
   (b) For every $h \neq h_{t+1}$, let $w^{t+1}_h = w^t_h$.

It is easy to see that any order-respecting quasi choice rule satisfies the condition in the definition of substitutability. Also it is easy to see that, for each of the above regional preferences (1) - (4), there exists an associated quasi choice rule that is order-respecting. By these observations, all of the above regional preferences are substitutable.

Remark 4. In addition, it may be of interest to consider regional preferences involving "subregions": The region prefers to assign no more doctors than a certain number to a subset of the hospitals in the region. Such preferences may arise if the society desires to impose a hierarchy of regional caps, say one cap for a prefecture and one for each district within the prefecture. Or the policy maker may desire to regulate the total number of doctors practicing in each specialty in each prefecture. In general, this type of preferences is outside of the current framework because if a cap of a district in a prefecture is filled while there are remaining seats in the prefecture as a whole, then no more doctor can be accepted to hospitals in the district and this violates the assumption that the associated quasi choice rule of the prefecture is acceptant. For this reason, a further generalization of our model is called for. Such a generalization is done in Section 6.

[34] Order-respecting quasi choice rules are similar to choice functions based on the precedence order of Kominers and Sönmez (2012), although we find no logical relationship between these two concepts.

5.3. **Comparative Statics.** As demonstrated in Section 4, our analytical approach is to construct an associated matching model with contracts and to utilize results from that model to obtain corresponding results in the original market. This connection enables us to exploit structural properties of stable allocations in the matching model with contracts. In particular, we obtain many comparative statics results as corollaries of a single general result in the matching with contract model.

We begin by stating various comparative statics results presented in Kamada and Kojima (2015). They formalize the current practice in Japan, the **Japan Residency Matching Program (JRMP) mechanism**. The JRMP mechanism is a rule that produces the matching resulting from the deferred acceptance algorithm except that, for each hospital h, it uses $\bar{q}_h \leq q_h$ instead of q_h as the hospital's capacity. In words, the JRMP mechanism pretends that the target capacities are actual physical capacities.

The first result establishes comparisons across the flexible deferred acceptance, JRMP, and the (unconstrained) deferred acceptance algorithms:

Proposition 6 (Theorem 3 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015). For any preference profile,*

(1) *Each doctor $d \in D$ weakly prefers a matching produced by the deferred acceptance mechanism to the one produced by the flexible deferred acceptance mechanism to the one produced by the JRMP mechanism.*
(2) *If a doctor is unmatched in the deferred acceptance mechanism, she is unmatched in the flexible deferred acceptance mechanism. If a doctor is unmatched in the flexible deferred acceptance mechanism, she is unmatched in the JRMP mechanism.*

The next result shows that, under the flexible deferred acceptance mechanism, all doctors are made weakly worse off when the regional caps become more stringent. By contrast, the number of doctors matched in a region whose regional cap is unchanged weakly increases when the regional caps of other regions become more stringent.

Proposition 7 (Proposition 3 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015). Fix a picking order in the flexible deferred acceptance mechanism. Let $(q_r)_{r\in R}$ and $(q'_r)_{r\in R}$ be regional caps such that $q'_r \leq q_r$ for each $r \in R$. Then the following statements hold.*

(1) *Each doctor $d \in D$ weakly prefers a matching produced by the flexible deferred acceptance mechanism under regional caps $(q_r)_{r\in R}$ to the one under $(q'_r)_{r\in R}$.*

(2) *For each region r such that $q_r = q'_r$, the number of doctors matched in r at a matching produced by the flexible deferred acceptance mechanism under regional caps $(q'_r)_{r\in R}$ is weakly larger than at the matching under $(q_r)_{r\in R}$.*

Another comparative statics result is about the changes in the imposed constraints under the JRMP mechanism.

Proposition 8 (Proposition 4 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015). Let $(\bar{q}_h)_{h\in H}$ and $(\bar{q}'_h)_{h\in H}$ be target capacities such that $\bar{q}'_h \leq \bar{q}_h$ for each $h \in H$. Then the following statements hold.*[35]

(1) *Each doctor $d \in D$ weakly prefers a matching produced by the JRMP mechanism under target capacities $(\bar{q}_h)_{h\in H}$ to the one under $(\bar{q}'_h)_{h\in H}$.*
(2) *Each hospital $h \in H$ such that $\bar{q}_h = \bar{q}'_h$ weakly prefers a matching produced by the JRMP mechanism under target capacities $(\bar{q}'_h)_{h\in H}$ to the one under $(\bar{q}_h)_{h\in H}$. Moreover, the number of doctors matched to any such h in the former matching is weakly larger than that in the latter.*

The following result, also from Kamada and Kojima (2015), shows that, whenever a hospital or a region is underserved under the flexible deferred acceptance mechanism, the (unconstrained) deferred acceptance mechanism cannot improve the match at such a hospital or a region.

Proposition 9 (Proposition 2 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015).*

(1) *If the number of doctors matched with $h \in H$ in the flexible deferred acceptance mechanism is strictly less than its target capacity, then the set of doctors matched with h under the (unconstrained) deferred acceptance mechanism is a subset of the one under the flexible deferred acceptance mechanism.*
(2) *If the number of doctors matched in $r \in R$ in the flexible deferred acceptance mechanism is strictly less than its regional cap, then each hospital h in r weakly prefers a matching produced by the flexible deferred acceptance mechanism to the one under the (unconstrained) deferred acceptance mechanism. Moreover, the number of doctors matched to any such h in the former matching is weakly larger than that in the latter.*

[35] Since the JRMP mechanism is equivalent to the deferred acceptance mechanism with respect to the target capacities, this result can also be obtained by appealing to the "Capacity Lemma" by Konishi and Ünver (2006), although we obtain these results as corollaries of a more general result, Lemma 1.

We obtain all these results as corollaries of a single general comparative statics result in the matching with contracts model. More specifically, we establish that if the choice function of a region becomes larger in the set inclusion sense, then all doctors are made weakly better off and all other regions are made weakly worse off in the doctor-optimal stable allocation (Lemma 1 in Appendix A.2). Also, Hatfield and Milgrom (2005) show that the outcome of a cumulative offer process is a doctor-optimal allocation. Given these results, we can prove all the above results by demonstrating that all the comparisons above can be interpreted as comparisons of outcomes of cumulative offer processes under different choice functions of regions. The formal statement of the Lemma and proofs of all the results in this section can be found in Appendix A.2.

6. A Generalization for Hierarchies of Regions

This section provides a generalization of the model in Section 2: we consider the situation where there is a hierarchy of regional caps. For instance, one could consider a hierarchy of regional caps, say one cap for a prefecture and one for each district within the prefecture. Or the policy maker may desire to regulate the total number of doctors practicing in each specialty in each prefecture. We show that a generalization of the flexible deferred acceptance mechanism induces a stable matching appropriately defined.

The set of **regions** R is a subset of $2^H \setminus \{\emptyset\}$ such that $\{h\} \in R$ for all $h \in H$ and $H \in R$ (the region H is called the grand region). Further, we assume that the set of regions R is nested (a hierarchy), that is, $r, r' \in R$ implies $r \subseteq r'$ or $r' \subseteq r$ or $r \cap r' = \emptyset$. H_r denotes the set of hospitals in region r (thus we use H_r and r interchangeably for convenience).[36] For each region r, there is a fixed positive integer q_r which we call the **regional cap** for r. For singleton region $\{h\}$ for each $h \in H$, we let $q_{\{h\}} = q_h$.

For any r, $r' \in R$, region r' is said to be an **immediate subregion** of r if $r' \subsetneq r$ and, for any $r'' \in R$, $r'' \subsetneq r$ implies either $r'' \cap r' = \emptyset$ or $r'' \subseteq r'$. It is straightforward to see that any non-singleton region $r \in R$ is partitioned into its immediate subregions. In the remainder, we simply refer to an immediate subregion as a **subregion**. Denote by $S(r)$ the set of subregions of r.

We say that region r is of **depth** k if $|\{r' \in R | r \subseteq r'\}| = k$. Note that the depth of a "smaller" region is larger. The standard model without regional caps can be interpreted as a model with regions of depths less than or equal to 2 (H and singleton sets), and the

[36] The assumption that the set of regions forms a hierarchy is important for our results. See Remark 7 and Appendix F for details.

model in the previous sections has regions of depths less than or equal to 3 (H, "regions," and singleton sets), both with q_H sufficiently large.

Below is an example in which the set of regions forms a hierarchy.

Example 1. There are 6 hospitals, $h_1, h_2, \ldots, h_6$. The regions are

$$R = \{H, r_1, r_2, r_3, r_4, \{h_1\}, \{h_2\}, \{h_3\}, \{h_4\}, \{h_5\}, \{h_6\}\},$$

where $r_1 = \{h_1, h_2\}$, $r_2 = \{h_3, h_4, h_5, h_6\}$, $r_3 = \{h_3, h_4\}$, and $r_4 = \{h_5, h_6\}$. See Figure 4 for a graphical representation. In this example, r_1 and r_2 are the (immediate) subregions

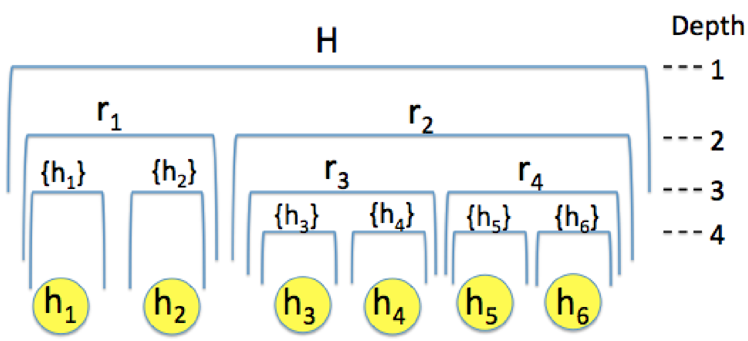

FIGURE 4. A hierarchy of regions in Example 1.

of H, r_3 and r_4 are the (immediate) subregions of r_2, and each singleton region is an (immediate) subregion of r_1 or r_3 or r_4. The depths of regions are as depicted in the figure. For example, the depth of H is 1, that of r_1 is 2, that of $\{h_1\}$ is 3, and that of $\{h_5\}$ is 4. □

Let $\succeq_r$ be a weak ordering over nonnegative-valued integer vectors $W_r := \{w = (w_{r'})_{r' \in S(r)} | w_{r'} \in \mathbb{Z}_+\}$. That is, $\succeq_r$ is a binary relation that is complete and transitive (but not necessarily antisymmetric). We write $w \succ_r w'$ if and only if $w \succeq_r w'$ holds but $w' \succeq_r w$ does not. Vectors such as w and w' will be interpreted to be supplies of acceptable doctors to regions that partition r, but they will only specify how many acceptable doctors apply to each subregion and no information is given as to who these doctors are. Given $\succeq_r$, a function

$$\tilde{\text{Ch}}_r : W_r \times \{0, 1, 2, \ldots, q_r\} \to W_r$$

is an **associated quasi choice rule** if $\tilde{\text{Ch}}_r(w; t) \in \arg\max_{\succeq_r} \{w' | w' \leq w, \sum_{r' \in S(r)} w'_{r'} \leq t\}$ for any non-negative integer vector $w = (w_{r'})_{r' \in S(r)}$ and non-negative integer $t \leq q_r$.[37]

[37]Similarly to footnote 19, for any two vectors $w = (w_{r'})_{r' \in S(r)}$ and $w' = (w'_{r'})_{r' \in S(r)}$, we write $w \leq w'$ if and only if $w_{r'} \leq w'_{r'}$ for all $r' \in S(r)$. We write $w \lneq w'$ if and only if $w \leq w'$ and $w_{r'} < w'_{r'}$ for at

Intuitively, $\tilde{\text{Ch}}_r(w, t)$ is the best vector of numbers of doctors allocated to subregions of r given a vector of numbers w under the constraint that the sum of the number of doctors cannot exceed the quota t.

We assume that the regional preferences $\succeq_r$ satisfy $w \succ_r w'$ if $w' \lneq w$. This condition formalizes the idea that region r prefers to fill as many positions in its subregions as possible. This requirement implies that any associated quasi choice rule is **acceptant** in the sense that, for each w and t, if there exists $r' \in S(r)$ such that $[\tilde{\text{Ch}}_r(w; t,)]_{r'} < w_{r'}$, then $\sum_{r' \in S(r)} [\tilde{\text{Ch}}_{r'}(w; t)]_{r'} = t$. This captures the idea that the social planner should not waste caps allocated to the region.[38]

We now define a restriction on preferences that we will maintain throughout our analysis.

Definition 7. The weak ordering $\succeq_r$ is **substitutable** if there exists an associated quasi choice rule $\tilde{\text{Ch}}_r$ that satisfies

$$w \leq w' \text{ and } t \geq t' \Rightarrow \tilde{\text{Ch}}_r(w; t) \geq \tilde{\text{Ch}}_r(w'; t') \wedge w.$$

Remark 5. Three remarks on the concept of substitutability are in order. First, the condition in the definition of substitutability can be decomposed into two parts, as follows:

$$w \leq w' \Rightarrow \tilde{\text{Ch}}_r(w; t) \geq \tilde{\text{Ch}}_r(w'; t) \wedge w, \text{ and} \tag{6.1}$$

$$t \geq t' \Rightarrow \tilde{\text{Ch}}_r(w; t) \geq \tilde{\text{Ch}}_r(w; t'). \tag{6.2}$$

Condition (6.1) imposes a condition on the quasi choice rule for different vectors w and w' with a fixed parameter t while Condition (6.2) places restrictions for different parameters t and t' with a fixed vector w. The former condition is similar to the standard substitutability condition except that it deals with multiunit supplies (that is, coefficients in w can take integers different from 0 or 1).[39] The latter condition may appear less familiar, and it requires that the choice increase (in the standard vector sense) if the allocated quota is increased. Conditions (6.1) and (6.2) are independent from each other. One

least one $r' \in S(r)$. For any $W'_r \subseteq W_r$, $\arg\max_{\succeq_r} W'_r$ is the set of vectors $w \in W'_r$ such that $w \succeq_r w'$ for all $w' \in W'_r$.

[38]This condition is a variant of the concept of acceptance due to Kojima and Manea (2010).

[39]Condition (6.1) is analogous to *persistence* by Alkan and Gale (2003), who define the condition on a choice function in a slightly different context. While our condition is similar to substitutability as defined in standard matching models (see Chapter 6 of Roth and Sotomayor (1990) for instance), there are two differences: (i) it is now defined on a region as opposed to a hospital, and (ii) it is defined over vectors that only specify how many doctors apply to hospitals in the region, and it does not distinguish different doctors.

might suspect that these conditions are related to responsiveness of preferences, but these conditions do no imply responsiveness. In Appendix E we provide examples to distinguish these conditions.

Second, Condition (6.1) is equivalent to

$$w \le w' \Rightarrow [\tilde{\text{Ch}}_r(w;t)]_{r'} \ge \min\{[\tilde{\text{Ch}}_r(w';t)]_{r'}, w_{r'}\} \text{ for every } r' \in S(r). \tag{6.3}$$

This condition says that, when the supply of doctors is increased, the number of accepted doctors at a hospital can increase only when the hospital has accepted all acceptable doctors under the original supply profile. Formally, condition (6.3) is equivalent to

$$w \le w' \text{ and } [\tilde{\text{Ch}}_r(w;t)]_{r'} < [\tilde{\text{Ch}}_r(w';t)]_{r'} \Rightarrow [\tilde{\text{Ch}}_r(w,t)]_{r'} = w_{r'}. \tag{6.4}$$

To see that condition (6.3) implies condition (6.4), suppose that $w \le w'$ and $[\tilde{\text{Ch}}_r(w;t)]_{r'} < [\tilde{\text{Ch}}_r(w';t)]_{r'}$. These assumptions and condition (6.3) imply $[\tilde{\text{Ch}}_r(w;t)]_{r'} \ge w_{r'}$. Since $[\tilde{\text{Ch}}_r(w;t)]_{r'} \le w_{r'}$ holds by the definition of $\tilde{\text{Ch}}_r$, this implies $[\tilde{\text{Ch}}_r(w;t)]_{r'} = w_{r'}$. To see that condition (6.4) implies condition (6.3), suppose that $w \le w'$. If $[\tilde{\text{Ch}}_r(w;t)]_{r'} \ge [\tilde{\text{Ch}}_r(w';t)]_{r'}$, the conclusion of (6.3) is trivially satisfied. If $[\tilde{\text{Ch}}_r(w;t)]_{r'} < [\tilde{\text{Ch}}_r(w';t)]_{r'}$, then condition (6.4) implies $[\tilde{\text{Ch}}_r(w;t,)]_{r'} = w_{r'}$, thus the conclusion of (6.3) is satisfied.

Finally, substitutability implies the following natural property that we call "consistency": A quasi choice rule $\tilde{\text{Ch}}_r$ is said to be **consistent** if for any t, $\tilde{\text{Ch}}_r(w;t) \le w' \le w \Rightarrow \tilde{\text{Ch}}_r(w';t) = \tilde{\text{Ch}}_r(w;t)$.[40] Consistency requires that, if $\tilde{\text{Ch}}_r(w;t)$ is chosen at w and the supply decreases to $w' \le w$ but $\tilde{\text{Ch}}_r(w;t)$ is still available under w', then the same choice $\tilde{\text{Ch}}_r(w;t)$ should be made under w' as well. Note that there may be more than one consistent quasi choice rule associated with a given weak ordering $\succeq_r$ because the set $\arg\max_{\succeq_r}\{w'|w' \le w, \sum_{r'\in S(r)} w'_{r'} \le t\}$ may not be a singleton for some $\succeq_r$, w, and t. Note also that there always exists a consistent quasi choice rule.[41] We relegate the proof of the fact that substitutability implies consistency to Appendix E. □

Now we define the notion of stability and the **(generalized) flexible deferred acceptance algorithm** in our context where R is a hierarchy. Let $SC(h,h') \in R$ be the **smallest common region** of hospitals h and h', that is, it is a region $r \in R$ with the property that $h, h' \in H_r$, and there is no $r' \in R$ with $r' \subsetneq r$ such that $h, h' \in H_{r'}$. Given $(\succeq_r)_{r\in R}$, stability is defined as follows.

[40]More precisely, it is Condition (6.1) of substitutability that implies consistency.

[41]To see this point consider preferences $\succeq'_r$ such that $w \succ'_r w'$ if $w \succ_r w'$ and $w = w'$ if $w \succeq'_r w'$ and $w' \succeq'_r w$. The quasi choice rule that chooses (the unique element of) $\arg\max_{\succeq'_r}\{w'|w' \le w, \sum_{r'\in S(r)} w'_{r'} \le t\}$ for each w is clearly consistent.

Definition 8. A matching μ is **stable** if it is feasible, individually rational, and if (d,h) is a blocking pair then there exists $r \in R$ with $h \in H_r$ such that (i) $|\mu_r| = q_r$, (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$, and

(iii) either $\mu_d \notin H_r$ or $(w_{r'})_{r' \in S(SC(h,\mu_d))} \succeq_{SC(h,\mu_d)} (w'_{r'})_{r' \in S(SC(h,\mu_d))}$,

where $w_{r'} = \sum_{h' \in r'} |\mu_{h'}|$ for all $r' \in S(r)$ and $w'_{r_h} = w_{r_h} + 1$, $w'_{r_d} = w_{r_d} - 1$ and $w'_{r'} = w_{r'}$ for all other $r' \in S(r)$ where r_h and r_d are subregions of r such that $h \in r'_h$, and $\mu_d \in r_d$.

Remark 6. Condition (iii) of this definition captures the idea behind stability in Definition 2 in that a region's preferences are invoked when a doctor moves within a region whose regional cap is binding (region r in the definition). However, when r is a strict superset of $SC(h,\mu_d)$, we do not invoke region r's regional preferences, but the preferences of $SC(h,\mu_d)$.[42] The use of preferences of $SC(h,\mu_d)$ reflects the following idea: if the regional cap at r is binding then holding fixed the number of doctors matched in r but not in $SC(h,\mu_d)$, there is essentially a binding cap for $SC(h,\mu_d)$. This motivates our use of the regional preferences of $SC(h,\mu_d)$. The reason for not using preferences of r (or any region between r and $SC(h,\mu_d)$) is that the movement of a doctor within the region $SC(h,\mu_d)$ does not affect the distribution of doctors on which preferences of r (or regions of any smaller depth than $SC(h,\mu_d)$) are defined. □

Remark 7. As mentioned in footnote 36, the assumption that the set of regions forms a hierarchy is important for our results. Example 5 in Appendix F shows that a stable matching does not necessarily exist if the set of regions does not form a hierarchy.[43]

We proceed to define a quasi choice rule for the "hospital side," denoted $\tilde{\text{Ch}}$: Let $\tilde{q}_H = q_H$. Given $w = (w_h)_{h \in H}$, we define $v^w_{\{h\}} = \min\{w_h, q_h\}$, and inductively define $v^w_r = \min\{\sum_{r' \in S(r)} v^w_{r'}, q_r\}$. Thus, v^w_r is the maximum number that the input w can allocate to its subregions given the feasibility constraints that w and regional caps of subregions of r impose. Note that v^w_r is weakly increasing in w, that is, $w \geq w'$ implies $v^w_r \geq v^{w'}_r$.

[42] It is important that we allow r to be a strict superset of $SC(h,\mu_d)$. Example 4 in Appendix F points out that, if we further require $r \subseteq SC(h,\mu_d)$ in Definition 8, then there does not need to exist a matching that satisfies this stronger notion of stability.

[43] Under a different setting and stability concept from ours, Biró, Fleiner, Irving, and Manlove (2010) construct a related example that shows nonexistence of a stable matching under non-hierarchical regional caps. The main difference is that their stability concept requires there be no blocking pair by a vacant position of a hospital and a doctor who is outside of the hospital's region (with a certain additional restriction), while our concept does not impose such a requirement.

We inductively define $\tilde{\text{Ch}}(w)$ following a procedure starting from Step 1, where Step k for general k is as follows:

> Step k: If all the regions of depth k are singletons, then let $\tilde{\text{Ch}}(w) = (\tilde{q}^w_{\{h\}})_{h\in H}$ and stop the procedure. For each nonsingleton region r of depth k, set $\tilde{q}^w_{r'} = [\tilde{\text{Ch}}_r((v^w_{r''})_{r''\in S(r)}; \tilde{q}^w_r)]_{r'}$ for each subregion r' of r. Go to Step $k+1$.

Assume that $\succeq_r$ is substitutable for every region r. Now we are ready to define a generalized version of the **flexible deferred acceptance algorithm**:

For each region r, fix an associated quasi choice rule $\tilde{\text{Ch}}_r$ for which conditions (6.1) and (6.2) are satisfied (note that the assumption that $\succeq_r$ is substitutable assures the existence of such a quasi choice rule.)

(1) Begin with an empty matching, that is, a matching μ such that $\mu_d = \emptyset$ for all $d \in D$.

(2) Choose a doctor d arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.

(3) Let d apply to the most preferred hospital $\bar{h}$ at $\succ_d$ among the hospitals that have not rejected d so far. If d is unacceptable to $\bar{h}$, then reject this doctor and go back to Step 2. Otherwise, define vector $w = (w_h)_{h\in H}$ by
   (a) $w_{\bar{h}}$ is the number of doctors currently held at $\bar{h}$ plus one, and
   (b) w_h is the number of doctors currently held at h if $h \neq \bar{h}$.

(4) Each hospital $h \in H$ considers the new applicant d (if $h = \bar{h}$) *and* doctors who are temporarily held from the previous step together. It holds its $[\tilde{\text{Ch}}(w)]_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to Step 2.

We define the **(generalized) flexible deferred acceptance mechanism** to be a mechanism that produces, for each input, the matching given at the termination of the above algorithm.[44]

Theorem 2. *Suppose that $\succeq_r$ is substitutable for every $r \in R$. Then the flexible deferred acceptance algorithm stops in a finite number of steps. The mechanism produces a stable matching for any input and is group strategy-proof for doctors.*

Proof. See Appendix B. □

[44]Note that this algorithm terminates in a finite number of steps. Note also that the outcome of the algorithm is independent of the order in which doctors make their applications during the algorithm.

The formal proof of this result is more involved than that of Theorem 1, and it is relegated to the Appendix. The proof strategy is similar to the one for Theorem 1, in the sense that we relate our model to the model of matching with contracts. In the model with hierarchical regions, however, the allocation of doctors in two regions can be related through a constraint on another region that contains both of the first two regions, so we cannot regard each region as one agent as before. Instead, we regard the set of *all* hospitals as one agent, and define its choice function. With this change in the view, the proof strategy analogous to the previous theorem is employed. More specifically, we establish key conditions in the associated matching with contracts model, and then we demonstrate that those desirable properties imply corresponding properties in the original model with regional caps, establishing the theorem.

7. Conclusion

This paper presented a model of matching under distributional constraints. Building upon an approach of Kamada and Kojima (2015), we defined a stability concept that takes distributional constraints into account. Departing from the previous work, we generalized the model to allow for a variety of policymaker preferences over doctor distributions. We offered a class of mechanisms that produce a stable matching under distributional constraints and various policy preferences, and showed that these mechanisms are (group) strategy-proof for doctors.

It is worth noting that our paper found a new connection between matching with constraints and matching with contracts. This technique was subsequently adopted by other studies such as Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014), Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014), and Kojima, Tamura, and Yokoo (2015). We envision that this approach may prove useful for tackling complex matching problems one may encounter in the future.

In addition to its intrinsic theoretical interest, our major motivation for a general theory was the desire to accommodate various constraints and policy preferences in practice, thus enabling applications to diverse types of real problems. As already mentioned, geographic and other distributional constraints are prevalent in practice; Concrete examples include British and Japanese medical matches, Chinese graduate admission, European college admissions, and Scottish teacher allocation, just to name a few. Although all these markets are subject to distributional constraints, because of differences in details, the same mechanism may be suitable in one market while unfit in another. This is a major reason that a general theory is needed. Moreover, we are quite confident that there are many other markets with specific constraints which have yet to be recognized or addressed in the literature. We hope that this paper provides a useful building block for market design in those undiscovered markets, and stimulates further research in matching under constraints and, more generally, practical market design.

References

Abdulkadiroğlu, A. (2005): "College admissions with affirmative action," *International Journal of Game Theory*, 33(4), 535–549.

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Abraham, D. J., R. Irving, and D. Manlove (2007): "Two algorithms for the student-project allocation problem," *Journal of Discrete Algorithms*, 5, 73–90.

Alkan, A. (2001): "On preferences over subsets and the lattice structure of stable matchings," *Review of Economic Design*, 6(1), 99–111.

——— (2002): "A class of multipartner matching markets with a strong lattice structure," *Economic Theory*, 19(4), 737–746.

Alkan, A., and D. Gale (2003): "Stable Schedule Matching Under Revealed Preferences," *Journal of Economic Theory*, 84, 73–94.

Aygün, O., and T. Sönmez (2012): "Matching with Contracts: The Critical Role of Irrelevance of Rejected Contracts," Discussion paper, Boston College Department of Economics.

Biró, P., T. Fleiner, R. W. Irving, and D. F. Manlove (2010): "The College Admissions problem with lower and common quotas," *Theoretical Computer Science*, 411, 3136–3153.

Blair, C. (1988): "The lattice structure of the set of stable matchings with multiple partners," *Mathematics of operations research*, 13(4), 619–628.

Boston Public Schools (2013): "BPS 5 Year Capital Facilities Master Plan, Phase I Fiscal Years 2014 - 2018 BPS 5 Year Capital Facilities Master Plan, Phase I Fiscal Years 2014-2018," http://bostonpublicschools.org/cms/lib07/MA01906464/Centricity/Domain/111/capital_facility_master 2018_volume_1_final.pdf.

Budish, E., Y.-K. Che, F. Kojima, and P. R. Milgrom (2013): "Designing Random Allocation Mechanisms: Theory and Applications," *American Economic Review*, 103(2), 585–623.

Chambers, C. P., and B. Yenmez (2013): "Choice and Matching," mimeo.

Crawford, V. (1991): "Comparative statics in matching markets," *Journal of Economic Theory*, 54(2), 389–400.

Crawford, V., and E. M. Knoer (1981): "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49, 437–450.

Dubins, L. E., and D. A. Freedman (1981): "Machiavelli and the Gale-Shapley algorithm," *American Mathematical Monthly*, 88, 485–494.

Echenique, F., and B. Yenmez (2015): "How to control controlled school choice," *American Economic Review*, 105(8), 2679–2694.

Ehlers, L., I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim (2014): "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*.

Ergin, H. (2002): "Efficient Resource Allocation on the Basis of Priorities," *Econometrica*, 70, 2489–2498.

Ergin, H., and T. Sönmez (2006): "Games of School Choice under the Boston Mechanism," *Journal of Public Economics*, 90, 215–237.

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28, 103–126.

Gale, D., and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

Gale, D., and M. A. O. Sotomayor (1985a): "Ms Machiavelli and the Stable matching problem," *American Mathematical Monthly*, 92, 261–268.

——— (1985b): "Some Remarks on the Stable Matching Problem," *Discrete Applied Mathematics*, 11, 223–232.

Goto, M., N. Hashimoto, A. Iwasaki, Y. Kawasaki, S. Ueda, Y. Yasuda, and M. Yokoo (2014): "Strategy-proof matching with regional minimum quotas," *AAMAS2014*.

Goto, M., A. Iwasaki, Y. Kawasaki, Y. Yasuda, and M. Yokoo (2014): "Improving Fairness and Efficiency in Matching Markets with Regional Caps: Priority-list based Deferred Acceptance Mechanism," mimeo (the latest version is available at http://mpra.ub.uni-muenchen.de/53409/).

Hafalir, I. E., M. B. Yenmez, and M. A. Yildirim (2013): "Effective affirmative action in school choice," *Theoretical Economics*, 8(2), 325–363.

Hatfield, J. W., and F. Kojima (2009): "Group Incentive Compatibility for Matching with Contracts," *Games and Economic Behavior*, pp. 745–749.

——— (2010): "Substitutes and Stability for Matching with Contracts," *Journal of Economic Theory*, pp. 1704–1723.

Hatfield, J. W., and S. D. Kominers (2009): "Contract Design and Stability in Matching Markets," mimeo.

——— (2012): "Matching in Networks with Bilateral Contracts," *American Economic Journal: Microeconomics*, 4(1), 176–208.

——— (2013): "Hidden Substitutes," Discussion paper, Working paper, Harvard Univ.

Hatfield, J. W., and P. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

Japan Residency Matching Program (2013): "Match Results by Residency Programs," http://www.jrmp.jp/oubosya/2013kekka.pdf.

Kamada, Y., and F. Kojima (2015): "Efficient Matching under Distributional Constraints: Theory and Applications," *American Economic Review*, (1), 67–99.

——— (2016): "Stability Concepts in Matching with Distributional Constraints," mimeo.

Kelso, A., and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.

Kojima, F., and M. Manea (2010): "Axioms for Deferred Acceptance," *Econometrica*, 78(2), 633–653.

Kojima, F., A. Tamura, and M. Yokoo (2015): "Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis," mimeo.

Kominers, S., and T. Sönmez (2012): "Matching with Slot-Specific Priorities: Theory," forthcoming, *Theoretical Economics.*

Konishi, H., and M. U. Ünver (2006): "Games of Capacity Manipulation in the Hospital-Intern Market," *Social Choice and Welfare*, 27, 3–24.

Martinez, R., J. Masso, A. Neme, and J. Oviedo (2004): "On group strategy-proof mechanisms for a many-to-one matching model," *International Journal of Game Theory*, 33, 115–128.

Ministry of Health, Labour and Welfare (2008): "Measures to Address Regional Imbalance and Specialty Imbalance of Doctors (in Japanese)," http://www.mhlw.go.jp/shingi/2008/08/dl/s0823-3e.pdf.

——— (2014): "Frequently Asked Questions about the Medical Residency Program (For Hospitals)," http://www.mhlw.go.jp/stf/seisakunitsuite/bunya/kenkou_iryou/iryou/rinsyo/qa/byouin.html.

NTT Com Research (2007): "Obstetricians Are in Short Supply! Footsteps of Obstetrics Breakdown," http://research.nttcoms.com/database/data/000687/.

Pathak, P., and T. Sonmez (2008): "Leveling the playing field: Sincere and sophisticated players in the Boston mechanism," *The American Economic Review*, 98(4), 1636–1652.

Roth, A. E. (1982): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7, 617–628.

——— (1985): "The College Admission Problem is not Equivalent to the Marriage Problem," *Journal of Economic Theory*, 36, 277–288.

——— (1991): "A Natural Experiment in the Organization of Entry Level Labor Markets: Regional Markets for New Physicians and Surgeons in the U.K.," *American Economic Review*, 81, 415–440.

Roth, A. E., and M. A. O. Sotomayor (1990): *Two-sided Matching: a Study in Game-theoretic Modeling and Analysis*. Econometric Society monographs, Cambridge.

Sönmez, T. (2013): "Bidding for army career specialties: Improving the ROTC branching mechanism," *Journal of Political Economy*, 121(1), 186–219.

Sönmez, T., and T. B. Switzer (2013): "Matching With (Branch-of-Choice) Contracts at the United States Military Academy," *Econometrica*, 81(2), 451–488.

Thomson, W. (2003): "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey," *Mathematical Social Sciences*, 45(3), 249–297.

Westkamp, A. (2013): "An Analysis of the German University Admissions System," *Economic Theory*, 53(3), 561–589.

Appendix A. Proofs for Section 5

A.1. Proofs for Section 5.1.

Proof of Proposition 4. Let μ be a matching and w be defined by $w_{h'} = |\mu_{h'}|$ for each $h' \in H_r$ and w' by $w'_h = w_h + 1$, $w'_{\mu_d} = w_{\mu_d} - 1$, and $w'_{h'} = w_{h'}$ for all $h' \in H_r \setminus \{h, \mu_d\}$. It suffices to show that $w \succeq_r w'$ if and only if $|\mu_h| + 1 - \bar{q}_h > |\mu_{\mu_d}| - 1 - \bar{q}_{\mu_d}$.

Suppose that $|\mu_h|+1-\bar{q}_h > |\mu_{\mu_d}|-1-\bar{q}_{\mu_d}$. This means that $w_h+1-\bar{q}_h > w_{\mu_d}-1-\bar{q}_{\mu_d}$, which is equivalent to either $w_h-\bar{q}_h = w_{\mu_d}-1-\bar{q}_{\mu_d}$ or $w_h-\bar{q}_h \geq w_{\mu_d}-\bar{q}_{\mu_d}$. In the former case, obviously $\eta(w) = \eta(w')$, so $w \succeq_r w'$. In the latter case, $\{h'|w'_{h'}-\bar{q}_{h'} < |\mu_{\mu_d}|-\bar{q}_{\mu_d}\} = \{h'|w_{h'}-\bar{q}_{h'} < |\mu_{\mu_d}|-\bar{q}_{\mu_d}\} \cup \{\mu_d\}$, and $w_{h'} = w'_{h'}$ for all $h' \in \{h'|w_{h'}-\bar{q}_{h'} < |\mu_{\mu_d}|-\bar{q}_{\mu_d}\}$. Thus we obtain $w \succ_r w'$.

If $|\mu_h|+1-\bar{q}_h \leq |\mu_{\mu_d}|-1-\bar{q}_{\mu_d}$, then obviously $w' \succ_r w$. This completes the proof. □

Proof of Proposition 5. It is clear that the quasi choice rule $\tilde{\text{Ch}}_r$ defined in (5.1) satisfies the condition (2.1) for substitutability (as well as consistency and acceptance). Thus in the following, we will show that $\tilde{\text{Ch}}_r$ indeed satisfies $\tilde{\text{Ch}}_r(w) \in \arg\max_{\succeq_r}\{x|x \leq w\}$ for each w. Let $w' = \tilde{\text{Ch}}_r(w)$. Assume by contradiction that $w' \notin \arg\max_{\succeq_r}\{x|x \leq w\}$ and consider an arbitrary $w'' \in \arg\max_{\succeq_r}\{x|x \leq w\}$. Then we have $w'' \succ_r w'$, so there exists i such that $\eta_j(w'') = \eta_j(w')$ for every $j < i$ and $\eta_i(w'') > \eta_i(w')$. Consider the following cases.

1. Suppose $\sum_j \eta_j(w'') > \sum_j \eta_j(w')$. First note that $\sum_j \eta_j(w'') + \sum_h \bar{q}_h = \sum_h w''_h \leq q_r$ because $w'' \in \arg\max_{\succeq_r}\{x|x \leq w\}$. Thus $\sum_h w'_h = \sum_j \eta_j(w') + \sum_h \bar{q}_h < \sum_j \eta_j(w'') + \sum_h \bar{q}_h \leq q_r$. Moreover, the assumption implies that there exists a hospital h such that $w'_h < w''_h \leq \min\{q_h, w_h\}$. These properties contradict the construction of $\tilde{\text{Ch}}_r$.
2. Suppose $\sum_j \eta_j(w'') < \sum_j \eta_j(w')$. First note that $\sum_j \eta_j(w')+\sum_h \bar{q}_h = \sum_h w'_h \leq q_r$ by construction of $\tilde{\text{Ch}}_r$. Thus $\sum_h w''_h = \sum_j \eta_j(w'')+\sum_h \bar{q}_h < \sum_j \eta_j(w')+\sum_h \bar{q}_h \leq q_r$. Moreover, the assumption implies that there exists a hospital h such that $w''_h < w'_h \leq \min\{q_h, w_h\}$. Then, w''' defined by $w'''_h = w''_h + 1$ and $w'''_{h'} = w''_{h'}$ for all $h' \neq h$ satisfies $w''' \leq w$ and $w''' \succ_r w''$, contradicting the assumption that $w'' \in \arg\max_{\succeq_r}\{x|x \leq w\}$.
3. Suppose that $\sum_j \eta_j(w'') = \sum_j \eta_j(w')$. Then there exists some k such that $\eta_k(w'') < \eta_k(w')$. Let $l = \min\{k|\eta_k(w'') < \eta_k(w')\}$ be the smallest of such indices. Then since $l > i$, we have $\eta_i(w') < \eta_i(w'') \leq \eta_l(w'') < \eta_l(w')$. Thus it should be the case that $\eta_i(w') + 2 \leq \eta_l(w')$. By the construction of $\tilde{\text{Ch}}_r$, this inequality holds only if $w'_h = \min\{q_h, w_h\}$, where h is an arbitrarily chosen hospital such that

$w'_h - \bar{q}_h = \eta_i(w')$. Now it should be the case that $w''_h = \min\{q_h, w_h\}$ as well, because otherwise $w'' \notin \arg\max_{\succeq_r}\{x|x \le w\}$.[45] Thus $w'_h = w''_h$. Now consider the modified vectors of both w' and w'' that delete the entries corresponding to h. All the properties described above hold for these new vectors. Proceeding inductively, we obtain $w'_h = w''_h$ for all h, that is, $w' = w''$. This is a contradiction to the assumption that $w' \notin \arg\max_{\succeq_r}\{x|x \le w\}$ and $w'' \in \arg\max_{\succeq_r}\{x|x \le w\}$.

The above cases complete the proof. □

A.2. **Proofs for Section 5.3.** The following result, which applies not only to matching with contract models defined over the set of contracts $D \times H$ but also to those defined over general environments, proves useful.

Lemma 1. *Consider a model of matching with contracts. Fix the set of doctors and regions as well as doctor preferences. Assume that choice rules $Ch := (Ch_r)_{r \in R}$ and $Ch' := (Ch'_r)_{r \in R}$ satisfy $Ch'_r(X') \subseteq Ch_r(X')$ for every subset of contracts X' and region r. Then the following two statements hold:*

(1) *Each doctor weakly prefers the outcome of the cumulative offer process with respect to Ch to the result with respect to Ch'. Hence each doctor weakly prefers the doctor-optimal stable allocation under Ch to the doctor-optimal stable allocation under Ch'.*
(2) *The set of contracts that have been offered up to and including the terminal step of the cumulative offer process under Ch is a subset of the corresponding set under Ch'.*

Proof. Let Y_d and Y'_d be the contracts allocated to d by the cumulative offer processes under Ch and Ch', respectively. Also, let $C(t)$ be the set of contracts that have been offered up to and including step t of the cumulative offer process under Ch, and $C'(t)$ be the corresponding set for the cumulative offer process under Ch'. Let T and T' be the terminal steps for the cumulative offer processes under Ch and Ch', respectively. We first prove Part 2 of the lemma, and then show Part 1.

Part 2: Suppose the contrary, i.e., that $C(T) \not\subseteq C'(T')$. Then there exists a step t' such that $C(t) \subseteq C'(T')$ for all $t < t'$ and $C(t') \not\subseteq C'(T')$ holds. That is, t' is the first step such

[45] The proof that $w'' \notin \arg\max_{\succeq_r}\{x|x \le w\}$ if $w''_h < \min\{q_h, w_h\}$ is as follows. Suppose that $w''_h < \min\{q_h, w_h\}$. Consider w''' defined by $w'''_h = w''_h + 1$, $w'''_{h'} = w''_{h'} - 1$ for some h' such that $w''_{h'} - \bar{q}_{h'} = \eta_i(w'')$, and $w'''_{h''} = w''_{h''}$ for all $h'' \in H_r \setminus \{h, h'\}$. Then we have $w'''_h - \bar{q}_h = w''_h - \bar{q}_h + 1 \le w'_h - \bar{q}_h < w''_{h'} - \bar{q}_{h'}$, where the weak inequality follows because $w''_h < \min\{q_h, w_h\} = w'_h$. The strict inequality implies that $w'_h - \bar{q}_h \le w''_{h'} - 1 - \bar{q}_{h'} = w'''_{h'} - \bar{q}_{h'}$. Hence $w'''_h - \bar{q}_h \le w'''_{h'} - \bar{q}_{h'}$, which implies $w''' \succ_r w''$.

that an application not made in the cumulative offer process under Ch' is made in the cumulative offer process under Ch. Let x be the contract that d offers in this step under Ch. Notice that $Y'_d \succ_d x$. This implies that $Y'_d \neq \emptyset$ and that Y'_d is rejected by r' in some steps of the cumulative offer process under Ch, where r' is the region associated with Y'_d. Let the first of such steps be t''. Since in the cumulative offer process doctors make offers in order of their preferences, $Y'_d \succ_d x$ implies that $t'' < t'$, which in turn implies $C(t'') \subseteq C'(T')$ by the definition of t'.

Now, we show that the set of contracts accepted by r' at step t'' of the cumulative offer process under Ch is a superset of the set of contracts accepted by r' from the application pool $C(t'')$ (which is a subset of $C'(T')$) at step T' of the cumulative offer process under Ch'. To see this, note that if the same application pool $C'(T')$ is given, the set of contracts accepted by r' in the cumulative offer process under Ch is weakly larger than that under Ch' by the assumption that $Ch'_r(X') \subseteq Ch_r(X')$ for all $X' \subseteq X$ and $r \in R$. Since Ch is substitutable, subtracting applications in $C'(T') \setminus C(t'')$ does not shrink the set of contracts accepted by r' within $C(t'')$ at step t'' of the cumulative offer process under Ch, which establishes our claim.

However, this contradicts our earlier conclusion that Y'_d is rejected by r' at step t'' of the cumulative offer process under Ch while she is allocated Y'_d in the cumulative offer process under Ch'. Hence we conclude that $C(T) \subseteq C'(T')$.

Part 1: Now, since in the cumulative offer process each doctor d make offers of contracts in order of her preferences, Y_d is $\emptyset$ or the worst contract for d in the set of contracts associated with d in $C(T)$. Similarly, for each doctor d, Y'_d is $\emptyset$ or the worst contract for d in the set of contracts associated with d in $C'(T')$. If $Y_d \neq \emptyset$, this and $C(T) \subseteq C'(T')$ imply that $Y_d \succeq_d Y'_d$. If $Y_d = \emptyset$, d has applied to all acceptable contracts in the cumulative offer process under Ch. Thus $C(T) \subseteq C'(T')$ implies that she has applied to all acceptable contracts in the algorithm under Ch', too. Let x' be the worst acceptable contract in X for d, and r be a region associated with x'. At this point we already know that Y'_d is either x' or $\emptyset$, and we will show that $Y'_d = \emptyset$ in what follows. Again, $C(T) \subseteq C'(T')$ implies that all applications associated with r in $C(T)$ is in $C'(T')$. In particular, d's application to x' is in $C'(T')$. Since Ch is substitutable, subtracting applications in $C'(T') \setminus C(T)$ does not shrink the set of doctors accepted by r within $C(T)$ at step T of the deferred acceptance, so d not being accepted by r from $C(T)$ at step T of the cumulative offer process under Ch implies that she is not accepted by r from $C'(T')$ in step T' of the process under Ch' either. But since we have shown that d's offer of contract x' to r is in $C'(T')$, this implies that in the cumulative offer process under Ch', x' is rejected by r. Because x' is the worst acceptable contract for d and d's applications are made in order of her preferences, we conclude that $Y'_d = \emptyset$, thus in particular $Y_d \succeq_d Y'_d$.

This shows that each doctor $d \in D$ weakly prefers a contract allocated by the cumulative offer process under Ch to the one under Ch'.

Since the outcome of the cumulative offer process is the doctor-optimal stable allocation, the preceding proof has also shown that the doctor-optimal stable allocation under Ch is weakly more preferred to the doctor-optimal stable allocation under Ch'. □

Lemma 1 is a generalization of a number of existing results. Gale and Sotomayor (1985a,b) establish comparative statics results in one-to-one and many-to-one matching with respect to the extension of an agent's list of her acceptable partners or an addition of an agent to the market, and Crawford (1991) generalizes the results to many-to-many matching. Konishi and Ünver (2006) consider many-to-one matching and obtain a comparative statics result with respect to the changes of hospital capacities.[46] All these changes are special cases of changes in the choice rules, so these results are corollaries of Lemma 1.

Lemma 1 may be of independent interest as the most general comparative statics result known to date. In addition, the lemma implies various results that are directly relevant to the current study of regional caps, such as Propositions 6, 7, 8, and 9 in the main text.

Proof of Proposition 6. **Part 1:** Let $\mathrm{Ch}^F = (\mathrm{Ch}^F_r)_{r\in R}$ be the choice rule associated with the flexible deferred acceptance as defined earlier, that is, for each region $r \in R$ and subset of contracts $X' \subseteq X = D \times H$, the chosen set of contracts $\mathrm{Ch}^F_r(X')$ is defined by

$$\mathrm{Ch}^F_r(X') = \bigcup_{h\in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D | (d',h) \in X', d' \succeq_h d\}| \le (\tilde{\mathrm{Ch}}_r(w(X')))_h \right\},$$

where $\tilde{\mathrm{Ch}}_r$ corresponds to a Rawlsian regional preference of region r and $w(X') = (w_h(X'))_{h\in H_r}$ is the vector such that $w_h(X') = |\{(d,h) \in X' | d \succ_h \emptyset\}|$ (this is a special case of the choice rule (4.1)).

[46] See also Kelso and Crawford (1982) who derive comparative statics results in a matching model with wages, and Hafalir, Yenmez, and Yildirim (2013) and Ehlers, Hafalir, Yenmez, and Yildirim (2014) who study comparative statics in the context of diversity in school choice. Echenique and Yenmez (2015) and Chambers and Yenmez (2013) independently obtain similar results to ours in a framework based on choice functions as primitives.

Moreover, consider choice rules $Ch^D = (Ch^D_r)_{r\in R}$ and $Ch^J = (Ch^J_r)_{r\in R}$ such that, for each X' and r,

$$\mathrm{Ch}^D_r(X') = \bigcup_{h\in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D|(d',h) \in X', d' \succeq_h d\}| \le q_h\right\},$$
$$\mathrm{Ch}^J_r(X') = \bigcup_{h\in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D|(d',h) \in X', d' \succeq_h d\}| \le \bar{q}_h\right\}.$$

Clearly, both Ch^D and Ch^J satisfy the substitute condition and the law of aggregate demand. Moreover, the matchings corresponding to the results of the cumulative offer processes under Ch^D and Ch^J are identical to the results of the deferred acceptance algorithm and the JRMP mechanism, respectively. Because $\min\{\bar{q}_h, w_h\} \le (\tilde{\mathrm{Ch}}_r(w(X')))_h \le q_h$ for all $h \in H_r$ and X', by inspection of the above definitions of the choice rules we obtain $\mathrm{Ch}^J_r(X') \subseteq \mathrm{Ch}^F_r(X') \subseteq \mathrm{Ch}^D_r(X')$ for all X' and r. Thus the desired conclusion follows by Part 1 of Lemma 1.

Part 2: This is a direct corollary of Part 1 and the fact that none of the algorithms considered here matches a doctor to an unacceptable hospital. □

Proof of Proposition 7. Let $\mathrm{Ch} = (\mathrm{Ch}_r)_{r\in R}$ and $\mathrm{Ch}' = (\mathrm{Ch}'_r)_{r\in R}$ be the choice rules associated with the flexible deferred acceptance mechanisms (as defined in the proof of Proposition 6) with respect to $(q_r)_{r\in R}$ and $(q'_r)_{r\in R}$, respectively.

Part 1: Because $q'_r \le q_r$ for each $r \in R$, the definition of these choice rules implies $\mathrm{Ch}'_r(X') \subseteq \mathrm{Ch}_r(X')$ for all X' and r. Hence the desired conclusion follows by Part 1 of Lemma 1.

Part 2: Since $\mathrm{Ch}'_r(X') \subseteq \mathrm{Ch}_r(X')$ for all X' and r as mentioned in the proof of Part 1, Part 2 of Lemma 1 implies that $C(T) \subseteq C'(T')$, where C, T, C', and T' are as defined in Part 2 of the lemma. Note that the sets of contracts allocated to hospitals in r at the conclusions of the cumulative offer processes under Ch and Ch' are given as r's choice from contracts associated with r in $C(T)$ and $C'(T')$, respectively. Because the choice rules satisfy the law of aggregate demand and the set-inclusion relationship $C(T) \subseteq C'(T')$ holds, for any r such that $q_r = q'_r$, the number of doctors matched in r under a matching produced by the flexible deferred acceptance mechanism under regional caps $(q'_r)_{r\in R}$ is weakly larger than in the matching under $(q_r)_{r\in R}$, completing the proof. □

Proof of Proposition 8. Let $\mathrm{Ch} = (\mathrm{Ch}_r)_{r\in R}$ and $\mathrm{Ch}' = (\mathrm{Ch}'_r)_{r\in R}$ be the choice rules associated with the JRMP mechanisms (as defined in the proof of Proposition 6) with respect to $(\bar{q}_h)_{h\in H}$ and $(\bar{q}'_h)_{h\in H}$, respectively.

Part 1: Because $\bar{q}'_h \leq \bar{q}_h$ for each $h \in H$, the definition of these choice rules implies $\mathrm{Ch}'_r(X') \subseteq \mathrm{Ch}_r(X')$ for all X' and r. Hence the desired conclusion follows by Part 1 of Lemma 1.

Part 2: Since $\mathrm{Ch}'_r(X') \subseteq \mathrm{Ch}_r(X')$ for all X' and r as mentioned in the proof of Part 1, Part 2 of Lemma 1 implies that $C(T) \subseteq C'(T')$, where C, T, C', and T' are as defined in Part 2 of Lemma 1. Note that the matchings for h at the conclusions of the cumulative offer processes under Ch and Ch' are given as h's most preferred acceptable doctors up to $\bar{q}_h = \bar{q}'_h$ from contracts associated with h in $C(T)$ and $C'(T')$, respectively. Thus the set-inclusion relationship $C(T) \subseteq C'(T')$ implies both of the statements of Part 2. □

Proof of Proposition 9. **Part 1:** First, by Part 2 of Lemma 1 and the proof of Proposition 6, the set of contracts that have been offered up to and including the terminal step under the deferred acceptance mechanism is a subset of the one under the flexible deferred acceptance mechanism. Second, by the construction of the flexible deferred acceptance algorithm, and the assumption that hospital h's target capacity is not filled, under the flexible deferred acceptance mechanism h is matched to every doctor who is acceptable to h and who applied to h in some step of the algorithm. These two facts imply the conclusion.

Part 2: First, by Part 2 of Lemma 1 and the proof of Proposition 6, the set of contracts that have been offered up to and including the terminal step under the deferred acceptance mechanism is a subset of the one under the flexible deferred acceptance mechanism. Second, by the construction of the flexible deferred acceptance algorithm, and the assumption that region r's regional cap is not filled, under the flexible deferred acceptance mechanism any hospital h in region r is matched to every doctor who is acceptable and who is among the most preferred q_h doctors who applied to h in some step of the algorithm. These two facts imply the conclusion. □

Appendix B. Proof of Theorem 2

It is useful to relate our model to a (many-to-many) matching model with contracts (Hatfield and Milgrom, 2005). Let there be two types of agents, doctors in D and the "hospital side" (thus there are $|D| + 1$ agents in total). Note that we regard the hospital side, instead of each hospital, as an agent in this model. There is a set of contracts $X = D \times H$.

We assume that, for each doctor d, any set of contracts with cardinality two or more is unacceptable, that is, a doctor can sign at most one contract. For each doctor d, her preferences $\succ_d$ over $(\{d\} \times H) \cup \{\emptyset\}$ are given as follows.[47] We assume $(d,h) \succ_d (d,h')$ in this model if and only if $h \succ_d h'$ in the original model, and $(d,h) \succ_d \emptyset$ in this model if and only if $h \succ_d \emptyset$ in the original model.

For the hospital side, we assume that it has preferences $\succeq$ and its associated choice rule $\text{Ch}(\cdot)$ over all subsets of $D \times H$. For any $X' \subset D \times H$, let $w(X') := (w_h(X'))_{h \in H}$ be the vector such that $w_h(X') = |\{(d,h) \in X' | d \succ_h \emptyset\}|$. For each X', the chosen set of contracts $\text{Ch}(X')$ is defined by

$$\text{Ch}(X') = \bigcup_{h \in H} \left\{ (d,h) \in X' \;\middle|\; |\{d' \in D | (d',h) \in X', d' \succeq_h d\}| \leq [\tilde{\text{Ch}}(w(X'))]_h \right\}.$$

That is, each hospital $h \in H$ chooses its $[\tilde{\text{Ch}}(w(X'))]_h$ most preferred contracts from acceptable contracts in X'.

Definition 9 (Hatfield and Milgrom (2005))**.** Choice rule $\text{Ch}(\cdot)$ satisfies the **substitutes condition** if there do not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \text{Ch}(X' \cup \{x'\})$ and $x' \in \text{Ch}(X' \cup \{x, x'\})$.

In other words, contracts are substitutes if adding a contract to the choice set never induces a region to choose a contract it previously rejected. Hatfield and Milgrom (2005) show that there exists a stable allocation (defined in Definition 11) when contracts are substitutes for the hospital side.

Definition 10 (Hatfield and Milgrom (2005))**.** Choice rule $\text{Ch}(\cdot)$ satisfies the **law of aggregate demand** if for all $X' \subseteq X'' \subseteq X$, $|\text{Ch}(X')| \leq |\text{Ch}(X'')|$.

Proposition 10. *Suppose that $\succeq_r$ is substitutable for all $r \in R$.*

(1) *Choice rule* $\text{Ch}(\cdot)$ *defined above satisfies the substitutes condition*
(2) *Choice rule* $\text{Ch}(\cdot)$ *defined above satisfies the law of aggregate demand.*

Proof. **Part 1.** Fix $X' \subset X$. Suppose to the contrary, i.e., that there exist X', (d,h) and (d',h') such that $(d',h') \notin \text{Ch}(X' \cup \{(d',h')\})$ and $(d',h') \in \text{Ch}(X' \cup \{(d,h),(d',h')\})$. We will lead to a contradiction.

Let $w' = w(X' \cup \{(d',h')\})$ and $w'' = w(X' \cup \{(d,h),(d',h')\})$. The proof consists of three steps.

Step 1: In this step we observe that $\tilde{q}^{w'}_{\{h'\}} < \tilde{q}^{w''}_{\{h'\}}$. To see this, note that otherwise we would have $\tilde{q}^{w'}_{\{h'\}} \geq \tilde{q}^{w''}_{\{h'\}}$, hence by the definition of Ch we must have [Ch($X' \cup$

[47] We abuse notation and use the same notation $\succ_d$ for preferences of doctor d both in the original model and in the associated model with contracts.

$\{(d',h')\})]_{h'} \supseteq [\text{Ch}(X' \cup \{(d,h),(d',h')\})]_{h'} \setminus \{(d,h)\}$. This contradicts $(d',h') \notin \text{Ch}(X' \cup \{(d',h')\})$ and $(d',h') \in \text{Ch}(X' \cup \{(d,h),(d',h')\})$.

Step 2: Consider any r such that $h' \in r$. Let $\tilde{q}_r^{w'}$ and $\tilde{q}_r^{w''}$ be as defined in the procedure to compute $\tilde{\text{Ch}}(w')$ and $\tilde{\text{Ch}}(w'')$, respectively. Let $r' \in S(r)$ be the subregion such that $h' \in r'$. Suppose $\tilde{q}_{r'}^{w'} < \tilde{q}_{r'}^{w''}$. We will show that $\tilde{q}_r^{w'} < \tilde{q}_r^{w''}$. To see this, suppose the contrary, i.e., that $\tilde{q}_r^{w'} \geq \tilde{q}_r^{w''}$. Let $v' := (v_{r''}^{w'})_{r'' \in S(r)}$ and $v'' := (v_{r''}^{w''})_{r'' \in S(r)}$. Since $w' \leq w''$ and $v_{r''}^{w}$ is weakly increasing in w for any region r'', it follows that $v' \leq v''$. This and substitutability of $\succeq_r$ imply

$$[\tilde{\text{Ch}}_r(v'; \tilde{q}_r^{w'})]_{r'} \geq \min\{[\tilde{\text{Ch}}_r(v''; \tilde{q}_r^{w''})]_{r'}, v'_{r'}\}.$$

Since we assume $\tilde{q}_{r'}^{w'} < \tilde{q}_{r'}^{w''}$, or equivalently

$$[\tilde{\text{Ch}}_r(v'; \tilde{q}_r^{w'})]_{r'} < [\tilde{\text{Ch}}_r(v''; \tilde{q}_r^{w''})]_{r'},$$

this means $[\tilde{\text{Ch}}_r(v'; \tilde{q}_r^{w'})]_{r'} \geq v'_{r'}$. But then by $[\tilde{\text{Ch}}_r(v'; \tilde{q}_r^{w'})]_{r'} \leq v'_{r'}$ (from the definition of $\tilde{\text{Ch}}$) we have $[\tilde{\text{Ch}}_r(v'; \tilde{q}_r^{w'})]_{r'} = v'_{r'}$. But this contradicts the assumption that $(d',h') \notin \text{Ch}(X' \cup \{(d',h')\})$, while d' is acceptable to h' (because $(d',h') \in \text{Ch}(X' \cup \{(d,h),(d',h')\})$). Thus we must have that $\tilde{q}_r^{w'} < \tilde{q}_r^{w''}$.

Step 3: Step 1 and an iterative use of Step 2 imply that $\tilde{q}_H^{w'} < \tilde{q}_H^{w''}$. But we specified $\tilde{q}_H^{w}$ for any w to be equal to q_H, so this is a contradiction.

Part 2. To show that Ch satisfies the law of aggregate demand, let $X' \subseteq X$ and (d,h) be a contract such that $d \succ_h \emptyset$. We shall show that $|\text{Ch}(X')| \leq |\text{Ch}(X' \cup \{(d,h)\})|$. To show this, denote $w = w(X')$ and $w' = w(X' \cup \{(d,h)\})$. By definition of $w(\cdot)$, we have that $w'_h = w_h + 1$ and $w'_{h'} = w_{h'}$ for all $h' \neq h$. Consider the following cases.

(1) Suppose $\sum_{r' \in S(r)} v_{r'}^{w} \geq q_r$ for some $r \in R$ such that $h \in r$. Then we have:

Claim 2. *$v_{r'}^{w'} = v_{r'}^{w}$ unless $r' \subsetneq r$.*

Proof. Let r' be a region that does not satisfy $r' \subsetneq r$. First, note that if $r' \cap r = \emptyset$, then the conclusion holds by the definitions of $v_{r'}^{w}$ and $v_{r'}^{w'}$ because $w'_{h'} = w_{h'}$ for any $h' \notin r$. Second, consider r' such that $r \subseteq r'$ (since R is hierarchical, these cases exhaust all possibilities). Since $v_r^{w} = \min\{\sum_{r' \in S(r)} v_{r'}^{w}, q_r\}$, the assumption $\sum_{r' \in S(r)} v_{r'}^{w} \geq q_r$ implies $v_r(w) = q_r$. By the same argument, we also obtain $v_r(w') = q_r$. Thus, for any r' such that $r \subseteq r'$, we inductively obtain $v_{r'}^{w'} = v_{r'}^{w}$. □

The relation $v_{r'}^{w'} = v_{r'}^{w}$ for all $r' \subsetneq r$ implies that, together with the construction of $\tilde{\mathrm{Ch}}$,

$$[\tilde{\mathrm{Ch}}(w')]_{h'} = [\tilde{\mathrm{Ch}}(w)]_{h'} \text{ for any } h' \notin r. \tag{B.1}$$

To consider hospitals in r, first observe that r satisfies $\sum_{r'\in S(r)} v_{r'}^{w} \geq q_r$ by assumption, so $v_r^w = \min\{\sum_{r'\in S(r)} v_{r'}^{w}, q_r\} = q_r$, and similarly $v_r^{w'} = q_r$, so $v_r^w = v_r^{w'}$. Therefore, by construction of $\tilde{\mathrm{Ch}}$, we also have $v_{r'}^{w} = v_{r'}^{w'}$ for any region r' such that $r \subseteq r'$. This implies $\tilde{q}_r^w = \tilde{q}_r^{w'}$, where $\tilde{q}_r^w$ and $\tilde{q}_r^{w'}$ are the assigned regional caps on r under weight vectors w and w', respectively, in the algorithm to construct $\tilde{\mathrm{Ch}}$.

Now note the following: For any $r' \in R$, since $v_{r'}^{w}$ is defined as $\min\{\sum_{r''\in S(r')} v_{r''}^{w}, q_{r'}\}$ and all regional preferences are acceptant, the entire assigned regional cap $\tilde{q}_{r'}^{w}$ is allocated to some subregion of r', that is, $\bar{q}_{r'}^{w} = \sum_{r''\in S(r')} \bar{q}_{r''}^{w}$. Similarly we also have $\bar{q}_{r'}^{w'} = \sum_{r''\in S(r')} \bar{q}_{r''}^{w'}$. This is the case for not only for $r' = r$ but also for all subregions of r, their further subregions, and so forth. Going forward until this reasoning reaches the singleton sets, we obtain relation

$$\sum_{h'\in r}[\tilde{\mathrm{Ch}}(w')]_{h'} = \sum_{h'\in r}[\tilde{\mathrm{Ch}}(w)]_{h'}. \tag{B.2}$$

By (B.1) and (B.2), we conclude that

$$|\mathrm{Ch}(X')| = \sum_{h'\in H}[\tilde{\mathrm{Ch}}(w)]_{h'} = \sum_{h'\in H}[\tilde{\mathrm{Ch}}(w')]_{h'} = |\mathrm{Ch}(X' \cup \{(d,h)\})|,$$

completing the proof for this case.

(2) Suppose $\sum_{r'\in S(r)} v_{r'}^{w} < q_r$ for all $r \in R$ such that $h \in r$. Then the regional cap for r is not binding for any r such that $h \in r$, so we have

$$[\tilde{\mathrm{Ch}}(w')]_{h} = [\tilde{\mathrm{Ch}}(w)]_{h} + 1. \tag{B.3}$$

In addition, the following claim holds.

Claim 3. $[\tilde{\mathrm{Ch}}(w')]_{h'} = [\tilde{\mathrm{Ch}}(w)]_{h'}$, *for all* $h' \neq h$.

Proof. First, note that $v_r^{w'} = v_r^w + 1$ for all r such that $h \in r$ because the regional cap for r is not binding for any such r. Then, consider the largest region H. By assumption, q_H has not been reached under w, that is, $\sum_{r'\in S(H)} v_{r'}^{w} < q_H$. Thus, since $\tilde{\mathrm{Ch}}_H$ is acceptant, the entire vector $(v_{r'}(w))_{r'\in S(H)}$ is accepted by $\tilde{\mathrm{Ch}}_H$, that is, $\tilde{q}_{r'}^{w} = v_{r'}^{w}$. Hence, for any $r' \in S(H)$ such that $h \notin r'$, both its assigned regional cap and all v's in their regions are identical under w and w', that is, $\tilde{q}_{r'}^{w} = \tilde{q}_{r'}^{w'}$ and $w'_{h'} = w_{h'}$ for all $h' \in r'$. So, for any hospital $h' \in r'$, the claim holds.

Now, consider $r \in S(H)$ such that $h \in r$. By the above argument, the assigned regional cap has increased by one in w' compared to w. But since r's regional cap q_r has not been binding under w, all the v's in the subregions of r are accepted in both w and w'. This means that (1) for each subregion r' of r such that $h \notin r'$, it gets the same assigned regional cap and v's, so the conclusion of the claim holds for these regions, and (2) for the subregion r' of r such that $h \in r'$, its assigned regional cap is increased by one in w' compared to w, and its regional cap $q_{r'}$ has not been binding. And (2) guarantees that we can follow the same argument inductively, so the conclusion holds for all $h \neq h'$. □

By equation (B.3) and Claim 3, we obtain

$$|\text{Ch}(X' \cup \{(d,h)\})| = \sum_{h' \in H} [\tilde{\text{Ch}}(w')]_{h'} = \sum_{h' \in H} [\tilde{\text{Ch}}(w)]_{h'} + 1 = |\text{Ch}(X')| + 1,$$

so we obtain $|\text{Ch}(X' \cup \{(d,h)\})| > |\text{Ch}(X')|$, completing the proof.

□

A subset X' of $X = D \times H$ is said to be **individually rational** if (1) for any $d \in D$, $|\{(d,h) \in X' | h \in H\}| \leq 1$, and if $(d,h) \in X'$ then $h \succ_d \emptyset$, and (2) $\text{Ch}(X') = X'$.

Definition 11. A set of contracts $X' \subseteq X$ is a **stable allocation** if

(1) it is individually rational, and
(2) there exists no hospital $h \in H$ and a doctor $d \in D$ such that $(d,h) \succ_d x$ and $(d,h) \in \text{Ch}(X' \cup \{(d,h)\})$, where x is the contract that d receives at X' if any and $\emptyset$ otherwise.

When condition (2) is violated by some (d,h), we say that (d,h) is a **block** of X'.

Given any individually rational set of contracts X', define a **corresponding matching** $\mu(X')$ in the original model by setting $\mu_d(X') = h$ if and only if $(d,h) \in X'$ and $\mu_d(X') = \emptyset$ if and only if no contract associated with d is in X'. For any individually rational X', $\mu(X')$ is well-defined because each doctor receives at most one contract at such X'.

Proposition 11. *Suppose that $\succeq_r$ is substitutable for all $r \in R$. If X' is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

Proof. Suppose that X' is a stable allocation in the associated model with contracts and denote $\mu := \mu(X')$. Individual rationality of μ is obvious from the construction of μ. Suppose that (d,h) is a blocking pair of μ. By the definition of stability, it suffices to show that there exists a region r that includes h such that the following conditions (B.4), (B.5), and $\mu_d \notin H_r$ hold, or (B.4), (B.5), (B.6), and $h, \mu_d \in r$ hold:

$$|\mu_{H_r}| = q_r, \tag{B.4}$$

$$d' \succ_h d \text{ for all } d' \in \mu_h, \tag{B.5}$$

$$(w_{r''})_{r'' \in S(SC(h,\mu_d))} \succeq_{SC(h,\mu_d)} (w'_{r''})_{r'' \in S(SC(h,\mu_d))}, \tag{B.6}$$

where for any region r' we write $w_{r''} = \sum_{h' \in r''} |\mu_{h'}|$ for all $r'' \in S(r')$ and $w'_{r_h} = w_{r_h} + 1$, $w'_{r_d} = w_{r_d} - 1$ and $w'_{r''} = w_{r''}$ for all other $r'' \in S(r')$ where $r_h, r_d \in S(r)$, $h \in r_h$, and $\mu_d \in r_d$. Let $w = (w_h)_{h \in H}$.

For each region r that includes h, let $w''_{r'} = w_{r'} + 1$ for r' such that $h \in r'$ and $w''_{r''} = w_{r''}$ for all other $r'' \in S(r)$. Let $w'' = (w''_h)_{h \in H}$.

Claim 4. *Condition (B.5) holds, and there exists r that includes h such that Condition (B.4) holds.*

Proof. First note that the assumption that $h \succ_d \mu_d$ implies that $(d, h) \succ_d x$ where x denotes the (possibly empty) contract that d signs under X'.

(1) Assume by contradiction that condition (B.5) is violated, that is, $d \succ_h d'$ for some $d' \in \mu_h$. First, note that $[\tilde{\text{Ch}}(w'')]_h \geq [\tilde{\text{Ch}}(w)]_h$. That is, weakly more contracts involving h are signed at $X' \cup (d, h)$ than at X'. This is because for any r and $r' \in S(r)$ such that $h \in r'$,

$$[\tilde{\text{Ch}}_r((v^{w''}_{r''})_{r'' \in S(r)}; \tilde{q}_r)]_{r'} \geq [\tilde{\text{Ch}}_r((v^{w}_{r''})_{r'' \in S(r)}; \tilde{q}'_r)]_{r'} \text{ if } \tilde{q}_r \geq \tilde{q}'_r. \tag{B.7}$$

To see this, first note that $[\tilde{\text{Ch}}_r((v^{w}_{r''})_{r'' \in S(r)}; \tilde{q}_r)]_{r'} \geq [\tilde{\text{Ch}}_r((v^{w}_{r''})_{r'' \in S(r)}; \tilde{q}'_r)]_{r'}$ by substitutability of $\succeq_r$. Also, by consistency of $\tilde{\text{Ch}}_r$ and $v^{w''}_{r''} \geq v^{w}_{r''}$ for every region r'', the inequality

$$[\tilde{\text{Ch}}_r((v^{w''}_{r''})_{r'' \in S(r)}; \tilde{q}_r)]_{r'} \geq [\tilde{\text{Ch}}_r((v^{w}_{r''})_{r'' \in S(r)}; \tilde{q}_r)]_{r'}$$

follows,[48] showing condition (B.7). An iterative use of condition (B.7) gives us the desired result that $[\tilde{\mathrm{Ch}}(w'')]_h \geq [\tilde{\mathrm{Ch}}(w)]_h$. This property, together with the assumptions that $d \succ_h d'$ and that $(d', h) \in X'$ imply that $(d, h) \in \mathrm{Ch}(X' \cup (d, h))$.[49] Thus, together with the above-mentioned property that $(d, h) \succ_d x$, (d, h) is a block of X' in the associated model of matching with contracts, contradicting the assumption that X' is a stable allocation.

(2) Assume by contradiction that condition (B.4) is violated, so that $|\mu_{H_r}| \neq q_r$ for every r that includes h. Then, for such r, since $|\mu_{H_r}| \leq q_r$ by the construction of μ and the assumption that X' is individually rational, it follows that $|\mu_{H_r}| < q_r$. Then $(d, h) \in \mathrm{Ch}(X' \cup (d, h))$ because,
   (a) $d \succ_h \emptyset$ by assumption,
   (b) since $\sum_{r' \in S(r)} w_{r'} = \sum_{h \in H_r} |\mu_h| = |\mu_{H_r}| < q_r$, it follows that $\sum_{r' \in S(r)} w''_{r'} = \sum_{r' \in S(r)} w_{r'} + 1 \leq q_r$. This property and the fact that $\tilde{\mathrm{Ch}}_r$ is acceptant and the definition of the function $v_{r'}$ for regions r' imply that $\tilde{\mathrm{Ch}}(w'') = w''$. In particular, this implies that every contract $(d', h) \in X' \cup (d, h)$ such that $d' \succ_h \emptyset$ is chosen at $\mathrm{Ch}(X' \cup (d, h))$.

   Thus, together with the above-mentioned property that $(d, h) \succ_d x$, (d, h) is a block of X' in the associated model of matching with contract, contradicting the assumption that X' is a stable allocation.

□

To finish the proof of the proposition suppose for contradiction that there is no r that includes h such that (B.4), (B.5), and $\mu_d \notin H_r$ hold, and that condition (B.6) fails. That is, we suppose $(w'_{r''})_{r'' \in S(SC(h, \mu_d))} \succ_{SC(h, \mu_d)} (w_{r''})_{r'' \in S(SC(h, \mu_d))}$. Then it must be the case that $[\tilde{\mathrm{Ch}}_r((v^{w''}_{r''})_{r'' \in S(SC(h, \mu_d))}; \tilde{q}^{w''}_{SC(h, \mu_d)})]_{r'} = w''_{r'} = w_{r'} + 1 = |\mu_h| + 1$, where $h \in r'$ and

[48]To show this claim, let $v = (v^{w}_{r''})_{r'' \in S(r)}$ and $v'' = (v^{w''}_{r''})_{r'' \in S(r)}$ for notational simplicity and assume for contradiction that $[\tilde{\mathrm{Ch}}_r(v''; \tilde{q}_r)]_{r'} < [\tilde{\mathrm{Ch}}_r(v; \tilde{q}_r)]_{r'}$. Then, $[\tilde{\mathrm{Ch}}_r(v''; \tilde{q}_r)]_{r'} < [\tilde{\mathrm{Ch}}_r(v; \tilde{q}_r)]_{r'} \leq v_{r'}$. Moreover, since $v''_{r''} = v_{r''}$ for every $r'' \neq r'$ by the construction of v'', it follows that $[\tilde{\mathrm{Ch}}_r(v'')]_{r''} \leq v''_{r''} = v_{r''}$. Combining these inequalities, we have that $\tilde{\mathrm{Ch}}_r(v'') \leq v$. Also we have $v \leq v''$ by the definition of v'', so it follows that $\tilde{\mathrm{Ch}}_r(v'') \leq v \leq v''$. Thus, by consistency of $\tilde{\mathrm{Ch}}_r$, we obtain $\tilde{\mathrm{Ch}}_r(v'') = \tilde{\mathrm{Ch}}_r(v)$, a contradiction to the assumption $[\tilde{\mathrm{Ch}}_r(v'')]_{r'} < [\tilde{\mathrm{Ch}}_r(v)]_{r'}$.

[49]The proof of this claim is as follows. $\mathrm{Ch}(X')$ induces hospital h to select its $[\tilde{\mathrm{Ch}}(w)]_h$ most preferred contracts while $\mathrm{Ch}(X' \cup (d, h))$ induces h to select a weakly larger number $[\mathrm{Ch}(w'')]_h$ of its most preferred contracts. Since (d', h) is selected as one of the $[\tilde{\mathrm{Ch}}(w)]_h$ most preferred contracts for h at X' and $d \succ_h d'$, we conclude that (d, h) must be one of the $[\mathrm{Ch}(w'')]_h$ $(\geq [\tilde{\mathrm{Ch}}(w)]_h)$ most preferred contracts at $X' \cup (d, h)$, thus selected at $X' \cup (d, h)$.

$\tilde{q}^{w''}_{SC(h,\mu_d)}$ is as defined in the procedure to compute $\tilde{\text{Ch}}(w'')$.[50] Note that for all r' such that $h \in r'$ and $r' \subsetneq SC(h,\mu_d)$, it follows that $\mu_d \notin H_{r'}$. Also note that (B.5) is satisfied by Claim 4. Therefore we have $|\mu_{r'}| < q_{r'}$ for all $r' \subsetneq SC(h,\mu_d)$ that includes h by assumption and hence $|\mu_{r'}| + 1 \leq q_{r'}$ for all such r'. Moreover we have $d \succ_h \emptyset$, thus

$$(d,h) \in \text{Ch}(X' \cup (d,h)).$$

This relationship, together with the assumption that $h \succ_d \mu_d$, and hence $(d,h) \succ_d x$, is a contradiction to the assumption that X' is stable in the associated model with contracts. □

Proof of Theorem 2. By Proposition 10, the choice rule $\text{Ch}(\cdot)$ satisfies the substitutes condition and the law of aggregate demand in the associate model of matching with contracts. By Hatfield and Milgrom (2005), Hatfield and Kojima (2009), and Hatfield and Kominers (2012), the cumulative offer process with choice rules satisfying these conditions produces a stable allocation and is (group) strategy-proof. The former fact, together with Remark 3 and Proposition 11, implies that the outcome of the flexible deferred acceptance algorithm is a stable matching in the original model. The latter fact and Remark 3 imply that the flexible deferred acceptance mechanism is (group) strategy-proof for doctors. □

Appendix C. Further Statistics on Heterogeneity of Capacities

Across prefectures in Japan, the mean and the median of the ratios of the maximum and the minimum hospital capacities are 20.98 and 19, respectively (see Figure 5). The mean

[50] To show this claim, assume for contradiction that $[\tilde{\text{Ch}}_{SC(h,\mu_d)}((v^{w''}_{r''})_{r'' \in S(SC(h,\mu_d))}; \tilde{q}^{w''}_{SC(h,\mu_d)})]_{r'} \leq w_{r'}$ where $h \in r'$. Let $v := (v^{w}_{r''})_{r'' \in S(SC(h,\mu_d))}$ and $v'' := (v^{w''}_{r''})_{r'' \in S(SC(h,\mu_d))}$. Since $w''_{r''} = w_{r''}$ for any $r'' \neq r'$ by the definition of w'', it follows that

$$\tilde{\text{Ch}}_{SC(h,\mu_d)}(v''; \tilde{q}^{w''}_{SC(h,\mu_d)}) \leq (w_{r''})_{r'' \in SC(h,\mu_d)} \leq (w''_{r''})_{r'' \in SC(h,\mu_d)}.$$

But $\tilde{\text{Ch}}_{SC(h,\mu_d)}(v; \tilde{q}^{w}_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$ because X' is a stable allocation in the associated model of matching with contracts, which in particular implies $v = (w_{r''})_{r'' \in SC(h,\mu_d)}$. Since $v \leq v''$, this means that

$$\tilde{\text{Ch}}_{SC(h,\mu_d)}(v''; \tilde{q}^{w''}_{SC(h,\mu_d)}) \leq v \leq v''.$$

Thus by consistency of $\tilde{\text{Ch}}_{SC(h,\mu_d)}$, we obtain

$$\tilde{\text{Ch}}_{SC(h,\mu_d)}(v''; \tilde{q}^{w''}_{SC(h,\mu_d)}) = \tilde{\text{Ch}}_{SC(h,\mu_d)}(v; \tilde{q}^{w''}_{SC(h,\mu_d)}).$$

But again by $\tilde{\text{Ch}}_{SC(h,\mu_d)}(v; \tilde{q}^{w}_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$, by substitutability we obtain $\tilde{\text{Ch}}_{SC(h,\mu_d)}(v; \tilde{q}^{w''}_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$, thus $\tilde{\text{Ch}}_{SC(h,\mu_d)}(v''; \tilde{q}^{w''}_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$. This is a contradiction because $(w'_{r''})_{r'' \in SC(h,\mu_d)} \leq (w''_{r''})_{r'' \in SC(h,\mu_d)} = v''$ and $(w'_{r''})_{r'' \in SC(h,\mu_d)} \succ_{SC(h,\mu_d)} (w_{r''})_{r'' \in SC(h,\mu_d)}$ while $\tilde{\text{Ch}}_{SC(h,\mu_d)}(v''; \tilde{q}^{w''}_{SC(h,\mu_d)}) \in \arg\max_{\succeq_{SC(h,\mu_d)}} \{(w'''_{r''})_{r'' \in SC(h,\mu_d)} | (w'''_{r''})_{r'' \in SC(h,\mu_d)} \leq v'', \sum_{r'' \in S(SC(h,\mu_d))} w'''_{r''} \leq \tilde{q}^{w''}_{SC(h,\mu_d)}\}$.

and the median of the Gini coefficients across prefectures are both 0.48, showing that the heterogeneity of hospital capacities is quite significant.[51] Capacities differ substantially in the school choice context as well; see Table 1 that reports data from Boston Public Schools (2013). The ratios of the maximum and the minimum of school capacities range from 1.80 to 16.19 with the median of 5.28, and all the Gini coefficients are no less than 0.10.

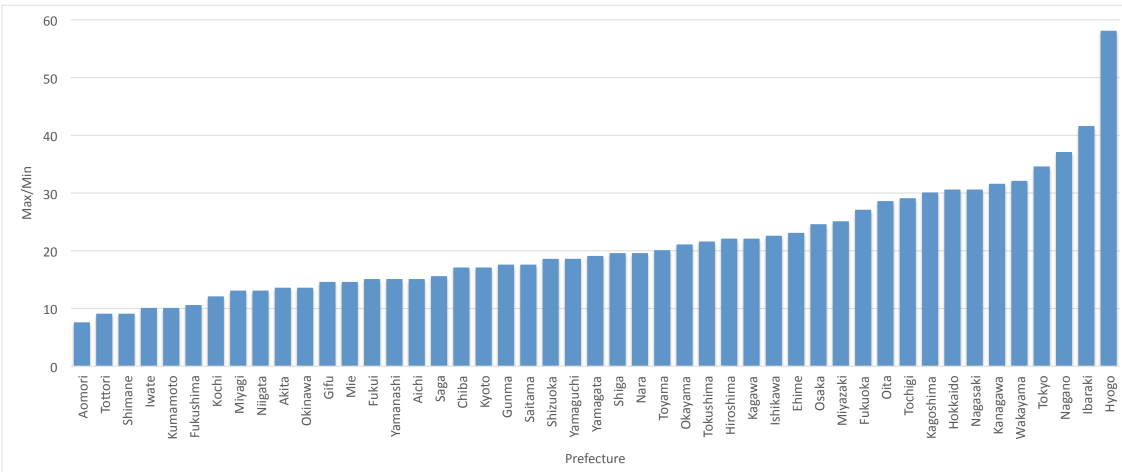

FIGURE 5. The ratios of the maximum and the minimum hospital capacities across prefectures.

Category	Maximum	Minimum	Max/Min	Gini Coefficient
Early Learning Center	234	109	2.15	0.10
Elementary School	871	165	5.28	0.25
Exam School	2323	1291	1.80	0.13
High School	1457	90	16.19	0.32
Kindergarten–Eight	960	132	7.27	0.27
Middle School	760	288	2.64	0.15
Special/Alternative Education	297	25	11.88	0.31

TABLE 1. Heterogeneity in size by school category.

APPENDIX D. ALLOCATING TARGET CAPACITIES

A problem related to, but distinct from, our discussion in Section 5.2 is how to allocate target capacities among hospitals in a region, within the simple, Rawlsian framework of

[51]The data are taken from Japan Residency Matching Program (2013)

Kamada and Kojima (2015). We will not try to provide a final answer to the normative question of how to do so for several reasons. First, there may be different ways to specify the quasi choice rule even given the same target capacity profile, as we have seen in this section, and in fact there may be reasonable quasi choice rules that do not even presuppose the existence of target capacities. Second, even if we fix a quasi choice rule, the relation between target capacities and the desirability of the resulting outcome is ambiguous. An example in Kamada and Kojima (2015) shows that the effect on hospital welfare is ambiguous.[52]

Despite these reservations, hospitals may still find having higher targets intuitively appealing in practice, so the problem seems to be practically important. Motivated by this observation, we present several methods to allocate target capacities that seem to be reasonable.

To do so, our starting point is to point out that the problem of allocating target capacities is similar to the celebrated "bankruptcy problem" (see Thomson (2003)). This is a useful association in the sense that, in the bankruptcy problem, there are known analyses (e.g., axiomatic characterizations) of various allocation rules, which can be utilized to judge which rule is appropriate for a given application.

To make this association, recall that in the standard bankruptcy problem, there is a divisible asset and agents whose claims sum up to (weakly) more than the amount of the available asset. Our problem is a discrete analogue of the bankruptcy problem. The regional cap q_r is an asset, hospitals in region r are agents, and physical capacity q_h is the claim of hospital h. Just as in the bankruptcy problem, the sum of the physical capacities may exceed the available regional cap, so the target capacity profile $(\bar{q}_h)_{h\in H_r}$ needs to be decided subject to the constraint $\sum_{h\in H_r} \bar{q}_h \leq q_r$.

This association suggests adaptations of well-known solutions in the bankruptcy problem to our problem, with the modification due to the fact that both the asset and the claims are discrete in our problem. The following are a few examples (in the following, we assume $\sum_{h\in H_r} q_h \geq q_r$; otherwise set $\bar{q}_h = q_h$ for all h).

(1) "Constrained Equal Awards Rule": This solution allocates the targets as equally as possible except that, for any hospital, it does not allocate a target larger than the capacity. This rule is called the **constrained equal awards rule** in the

[52]Example 9 in Kamada and Kojima (2015) shows that the effect on hospital welfare is ambiguous. In fact, Example 15 in Kamada and Kojima (2015) shows that the same conclusion holds even if hospitals or doctors have homogeneous preferences, which are strong restrictions that often lead to strong conclusions in matching.

literature on the bankruptcy problem. In our context, this solution should be modified because all the targets need to be integers. Formally, a constrained equal awards rule in our context can be defined as follows:

(a) Find λ that satisfies $\sum_{h \in H_r} \min\{\lambda, q_h\} = q_r$.
(b) For each $h \in H_r$, if $\lambda > q_h$, then set $\bar{q}_h = q_h$. Otherwise, set $\bar{q}_h$ to be either $\lfloor \lambda \rfloor$ (the largest integer no larger than λ) or $\lfloor \lambda \rfloor + 1$, subject to the constraint that $\sum_{h \in H_r} \bar{q}_h = q_r$.

The rule to decide which hospital receives $\lfloor \lambda \rfloor$ or $\lfloor \lambda \rfloor + 1$ is arbitrary: For any decision rule, the resulting target profiles satisfy conditions assumed in Kamada and Kojima (2015). The decision can also use randomization, which may help achieve ex ante fairness.

(2) "Constrained Equal Losses Rule": This solution allocates the targets in such a way that it equates losses (that is, differences between the capacities and the targets) as much as possible, except that none of the allocated targets can be strictly smaller than zero. This rule is called the **constrained equal losses rule** in the literature on the bankruptcy problem. As in the case of the constrained equal awards rule, this solution should be modified because all the targets need to be integers. Formally, a constrained equal losses rule in our context can be defined as follows:

(a) Find λ that satisfies $\sum_{h \in H_r} \max\{q_h - \lambda, 0\} = q_r$.
(b) For each $h \in H_r$, if $q_h - \lambda < 0$, then set $\bar{q}_h = 0$. Otherwise, set $\bar{q}_h$ to be either $q_h - \lfloor \lambda \rfloor$ or $q_h - \lfloor \lambda \rfloor - 1$, subject to the constraint that $\sum_{h \in H_r} \bar{q}_h = q_r$.

As in the constrained equal awards rule, the rule to decide which hospital receives $q_h - \lfloor \lambda \rfloor$ or $q_h - \lfloor \lambda \rfloor - 1$ is arbitrary: For any decision rule, the resulting target profiles satisfy conditions assumed in Kamada and Kojima (2015). The decision can also use randomization, which may help achieve ex ante fairness.

(3) "Proportional Rule": This solution allocates the targets in a manner that is as proportional as possible to the hospitals' capacities. This rule is called the **proportional rule** in the literature on the bankruptcy problem. As in the case of the previous rules, this solution should be modified because all the targets need to be integers. Formally, a proportional rule in our context can be defined as follows:

(a) Find λ that satisfies $\sum_{h \in H_r} \lambda q_h = q_r$.
(b) For each $h \in H_r$, set $\bar{q}_h$ be either $\lfloor \lambda q_h \rfloor$ or $\lfloor \lambda q_h \rfloor + 1$, subject to the constraint that $\sum_{h \in H_r} \bar{q}_h = q_r$.

As in the previous cases, the rule to decide which a hospital receives $\lfloor \lambda q_h \rfloor$ or $\lfloor \lambda q_h \rfloor + 1$ is arbitrary: For any decision rule, the resulting target profiles satisfy conditions assumed in Kamada and Kojima (2015). The decision can also use randomization, which may help achieve ex ante fairness.

The proportional rule seems to be fairly appealing in practice. This rule is used in Japanese residency match and Chinese graduate school admission, for example.

Appendix E. Discussion on Substitutability

In this section we aim to deepen our understanding of substitutability conditions. First we study the relationship between substitutability and consistency, and then we show that conditions (6.1) and (6.2) are independent.

Claim 5. *Condition (6.1) implies consistency.*[53]

Proof. Fix $\succeq$ and its associated quasi choice rule $\tilde{\text{Ch}}_r$, and suppose that for some t, $\tilde{\text{Ch}}_r(w';t) \le w \le w'$. Suppose also that condition (6.1) holds. We will prove $\tilde{\text{Ch}}_r(w;t) = \tilde{\text{Ch}}_r(w';t)$. Condition (6.1) implies $w \le w' \Rightarrow \tilde{\text{Ch}}_r(w;t) \ge \tilde{\text{Ch}}_r(w';t) \wedge w$. Since $\tilde{\text{Ch}}_r(w';t) \le w$ implies $\tilde{\text{Ch}}_r(w';t) \wedge w = \tilde{\text{Ch}}_r(w';t)$, this means that $\tilde{\text{Ch}}_r(w';t) \le \tilde{\text{Ch}}_r(w;t) \le w'$. If $\tilde{\text{Ch}}_r(w;t) \ne \tilde{\text{Ch}}_r(w';t)$ then by the assumption that $\tilde{\text{Ch}}_r$ is acceptant, we must have $\tilde{\text{Ch}}_r(w;t) \succ_r \tilde{\text{Ch}}_r(w';t)$. But then $\tilde{\text{Ch}}_r(w';t)$ cannot be an element of $\arg\max_{\succeq_r}\{w''|w'' \le w', \sum_{r' \in S(r)} w''_{r'} \le t\}$ because $\tilde{\text{Ch}}_r(w;t) \in \{w''|w'' \le w', \sum_{r' \in S(r)} w''_{r'} \le t\}$. Hence we have $\tilde{\text{Ch}}_r(w';t) = \tilde{\text{Ch}}_r(w;t)$. □

Example 2 (Regional preferences that violate (6.1) while satisfying (6.2))**.** There is a grand region r in which two hospitals reside. The capacity of each hospital is 2. Region r's preferences are as follows.

$$\succ_r: (2,2), (2,1), (1,2), (2,0), (0,2), (1,1), (1,0), (0,1), (0,0).$$

One can check by inspection that condition (6.2) and consistency are satisfied. To show that (6.1) is not satisfied, observe first that there is a unique associated choice rule (since preferences are strict), and denote it by $\tilde{\text{Ch}}_r$. The above preferences imply that $\tilde{\text{Ch}}_r((1,2);2) = (0,2)$ and $\tilde{\text{Ch}}_r((2,2);2) = (2,0)$. But this is a contradiction to (6.1) because $(1,2) \le (2,2)$ but $\tilde{\text{Ch}}_r((1,2);2) \ge \tilde{\text{Ch}}_r((2,2);2) \wedge (1,2)$ does not hold (the left hand side is $(0,2)$ while the right hand side is $(1,0)$). □

[53] Fleiner (2003) and Aygün and Sönmez (2012) prove analogous results although they do not work on substitutability defined over the space of integer vectors.

Example 3 (Regional preferences that violate (6.2) while satisfying (6.1))**.** There is a grand region r in which three hospitals reside. The capacity of each hospital is 1. Region r's preferences are as follows.

$$\succ_r: (1,1,1),(1,1,0),(1,0,1),(0,1,1),(0,0,1),(0,1,0),(1,0,0),(0,0,0).$$

One can check by inspection that condition (6.1) (and hence consistency by Claim 5) are satisfied. To show that (6.2) is not satisfied, observe first that there is a unique associated choice rule (since preferences are strict), and denote it by $\tilde{\text{Ch}}_r$. The above preferences imply that $\tilde{\text{Ch}}_r((1,1,1);1) = (0,0,1)$ and $\tilde{\text{Ch}}_r((1,1,1);2) = (1,1,0)$. But this is a contradiction to (6.2) because $1 \leq 2$ but $\tilde{\text{Ch}}_r((1,1,1);1) \leq \tilde{\text{Ch}}_r((1,1,1);2)$ does not hold (the left hand side is $(0,0,1)$ while the right hand side is $(1,1,0)$). □

Appendix F. Discussion on Stability for Hierarchical Regions

The following example shows that there exists no matching that satisfies a certain strengthening of the stability concept (see footnote 42).

Example 4 (Stable matchings do not necessarily exist under a stronger definition)**.** Suppose that in the definition of stability (Definition 8), we further require that $r \subseteq SC(h, \mu_d)$. We demonstrate that there does not necessarily exist a stable matching under this notion.

There is a grand region r in which two subregions r' and r'' exist. Two hospitals h_1 and h_2 reside in r', and one hospital h_3 resides in r''. The capacity of each hospital is 1. The regional caps are 1 for r, 2 for r', and 1 for r''. Regional preferences are as follows.

$$\succ_r : (1,0),(0,1),(0,0),$$
$$\succ_{r'} : (0,2),(1,1),(2,0),(0,1),(1,0),(0,0).$$

There are two doctors d_1 and d_2. Preferences are as follows:

$$\succ_{d_1} : h_1, h_2, \qquad \succ_{d_2}: h_2, h_1,$$
$$\succ_{h_1} : d_2, d_1, \qquad \succ_{h_2}: d_1, d_2,$$

and preferences of h_3 are arbitrary.

To show that there exists no stable matching under the stronger definition, first note that the matching in which all doctors are unmatched is clearly unstable because, for example, pair (d_1, h_1) is a valid blocking pair. Also note that no matching under which both of the two doctors are matched is stable because the regional cap for the grand region r is one. Thus we are left with the cases in which only one doctor is matched to a hospital.

(1) Consider a matching μ such that $\mu_{d_1} = h_1$. Pair (d_2, h_1) is a blocking pair and, because $d_2 \succ_{h_1} d_1$, this is a legitimate blocking pair, showing that μ is unstable.
(2) Consider a matching μ such that $\mu_{d_1} = h_2$. First, note that pair (d_1, h_1) is a blocking pair. Moreover, since $SC(h_1, \mu_d) = r'$, we only need to check whether the cap $q_{r'}$ of region r' and the cap $q_{\{h_1\}}$ of the region $\{h_1\}$ are binding. Because $|\mu_{r'}| = 1 < 2 = q_{r'}$ and $|\mu_{h_1}| = 0 < 1 = q_{h_1}$, the regional caps are not binding. Hence the conditions in the stability concept are not satisfied, showing that showing that μ is unstable.
(3) Consider a matching μ such that $\mu_{d_1} = h_3$. Since h_3 is unacceptable to d_1, μ is unstable.

Any matching in which d_2 is matched to a hospital can be shown to be unstable in a symmetric manner. Hence, there does not exist any stable matching under the stronger definition. □

The next example shows that a stable matching does not necessarily exist if the set of regions violates the assumption of a hierarchical structure.

Example 5 (Non-Hierarchical Regions)**.** Suppose that there are three hospitals, h_1, h_2, and h_3. Suppose that regions are not hierarchical, and

$$R = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_2, h_3\}, \{h_3, h_1\}, \{h_1, h_2, h_3\}\}.$$

Each region's regional cap is 1. There are two doctors, d_1 and d_2, and preferences are as follows:

$$\succ_{d_1}: h_1, h_2, h_3, \qquad \succ_{d_2}: h_3, h_1, h_2,$$

$$\succ_{h_1}: d_2, d_1, \qquad \succ_{h_2}: d_2, d_1; \qquad \succ_{h_3}: d_1, d_2.$$

Regional preferences for binary regions are that $\{h_1, h_2\}$ prefers a doctor to be in h_1 rather than h_2, $\{h_2, h_3\}$ prefers a doctor to be in h_2 rather than h_3, and $\{h_3, h_1\}$ prefers a doctor to be in h_3 rather than h_1.

Given the above specification, we show that there is no stable matching. First it is straightforward to see that there is no stable matching in which zero or two doctors are matched. So consider the case in which one doctor is matched. By the definition of stability, no hospital is matched to its second-choice doctor in any stable matching. This leaves us with only three possibilities: $\mu_{d_2} = h_1$, $\mu_{d_2} = h_2$, and $\mu_{d_1} = h_3$.

In the first case, (d_2, h_3) is a blocking pair, and from regional preferences of $\{h_3, h_1\}$, the existence of such a blocking pair violates stability. In the second case, (d_2, h_1) is a blocking pair, and from regional preferences of $\{h_1, h_2\}$, the existence of such a blocking pair violates stability. Finally, in the third case, (d_1, h_2) is a blocking pair, and from regional preferences of $\{h_2, h_3\}$, the existence of such a blocking pair violates stability. Hence there is no stable matching. □