ACCOMMODATING VARIOUS POLICY GOALS IN MATCHING WITH CONSTRAINTS

YUICHIRO KAMADA AND FUHITO KOJIMA

Abstract. Distributional constraints are common features in many real matching markets, such as medical residency matching, school admissions, and teacher assignment. We present a model of matching with constraints that accommodates a wide variety of policy goals, and apply that model to the setting of Kamada and Kojima (2015). We also formalize a number of other policy goals to show that they are subsumed by our model. We prove several comparative statics results such as showing that a mechanism we propose is a Pareto improvement for doctors upon the constrained medical matching mechanism currently used in Japan.

JEL Classification Numbers: C70, D47, D61, D63.

Keywords: matching, distributional constraints, Rawls, stability, strategy-proofness, comparative statics

---

Date: May 23, 2019.

Kamada: Haas School of Business, University of California, Berkeley, Berkeley, CA 94720, y.cam.24@gmail.com. Kojima: Department of Economics, Stanford University, Stanford, CA 94305, fkojima@stanford.edu. An earlier version of this paper was circulated under the title "General Theory of Matching under Distributional Constraints." We are grateful to an anonymous referee, Mustafa Oguz Afacan, Péter Biró, Eric Budish, Sylvain Chassang, Vincent Crawford, Hisao Endo, Clayton Featherstone, Tamás Fleiner, Drew Fudenberg, Tadashi Hashimoto, John William Hatfield, Toshiaki Iizuka, Rob Irving, Ryo Jinnai, Michihiro Kandori, Onur Kesten, Scott Duke Kominers, Hideo Konishi, Mihai Manea, David Manlove, Taisuke Matsubae, Aki Matsui, Yusuke Narita, Muriel Niederle, Parag Pathak, Al Roth, Dan Sasaki, Tayfun Sönmez, Satoru Takahashi, William Thomson, Alexis Akira Toda, Kentaro Tomoeda, Utku Ünver, Jun Wako, Alex Westkamp, Yosuke Yasuda, Bumin Yenmez, and participants at numerous seminars and conferences for helpful comments. Doctors Keisuke Izumi, Yoshiaki Kanno, Masataka Kawana, and Masaaki Nagano answered our questions about medical residency in Japan and introduced us to the relevant medical literature. We are grateful to officials at the Ministry of Health, Labor and Welfare and the Japan Residency Matching Program for discussion. Jin Chen, Yue Fan, Irene Hsu, Seung Hoon Lee, Stephen Nei, Bobak Pakzad-Hurson, Neil Prasad, Fanqi Shi, Pete Troyan, Wei Wen, and Rui Yu provided excellent research assistance. Kojima gratefully acknowledges financial support from the Sloan Foundation, as well as financial support from the National Research Foundation through its Global Research Network Grant (NRF- 2013S1A2A2035408).

1. Introduction

Many real matching markets are subject to distributional constraints. For example, under the "regional cap" policy in Japanese medical residency matching, each region of the country is subject to a regional cap. That is, each region is assigned an upper-bound constraint on the total number of residents placed in the region. This policy was introduced to regulate the geographical distribution of doctors, which was considered to be concentrated too heavily in urban areas at the expense of rural areas. Measures that are mathematically isomorphic to the regional cap policy can be found in a wide range of contexts, such as graduate school admission in China, college admission in several European countries, residency match in the U.K., and teacher assignment in Scotland.[1]

Motivated by these real-life examples, Kamada and Kojima (2015) study the design of matching markets under distributional constraints. As standard stability may conflict with distributional constraints, they propose a relaxed stability concept. They show that existing mechanisms result in instability and inefficiency and offer a mechanism that finds a stable and efficient matching and is (group) strategy-proof for doctors while respecting the distributional constraints.

A major limitation of that paper, though, is that their stability concept is closely tailored to a particular governmental goal to equalize the numbers of doctors across hospitals beyond target capacities. Although such a goal may be appealing in some contexts as a first-order concern, it may not be appropriate in other applications because hospital capacities in a given region may vary wildly. For example, the maximum and the minimum capacities of hospitals in Tokyo are 69 and 2, respectively (see Figure 1). For public elementary schools in Boston, the maximum and the minimum capacities of schools are 871 and 165, respectively (see Figure 2).[2] In such cases, it may be more appropriate to equalize the *ratio* between the numbers of doctors (beyond the targets) and the capacities across hospitals.

[1]There are a large number of studies in matching problems with various forms of constraints. Examples include Roth (1991) on gender balance in labor markets, Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Ergin and Sönmez (2006), Hafalir, Yenmez, and Yildirim (2013), and Ehlers, Hafalir, Yenmez, and Yildirim (2014) on diversity in schools, Westkamp (2013) on trait-specific college admission, Abraham, Irving, and Manlove (2007) on project-specific quotas in projects-students matching, and Biró, Fleiner, Irving, and Manlove (2010) on college admission with multiple types of tuitions. These models share some similarities with our model, but all of them are independent of our study. More detailed discussions are found in our companion paper, Kamada and Kojima (2015), so we do not reproduce it here.

[2]In Appendix C, we provide further statistics on heterogeneity of capacities in these markets.

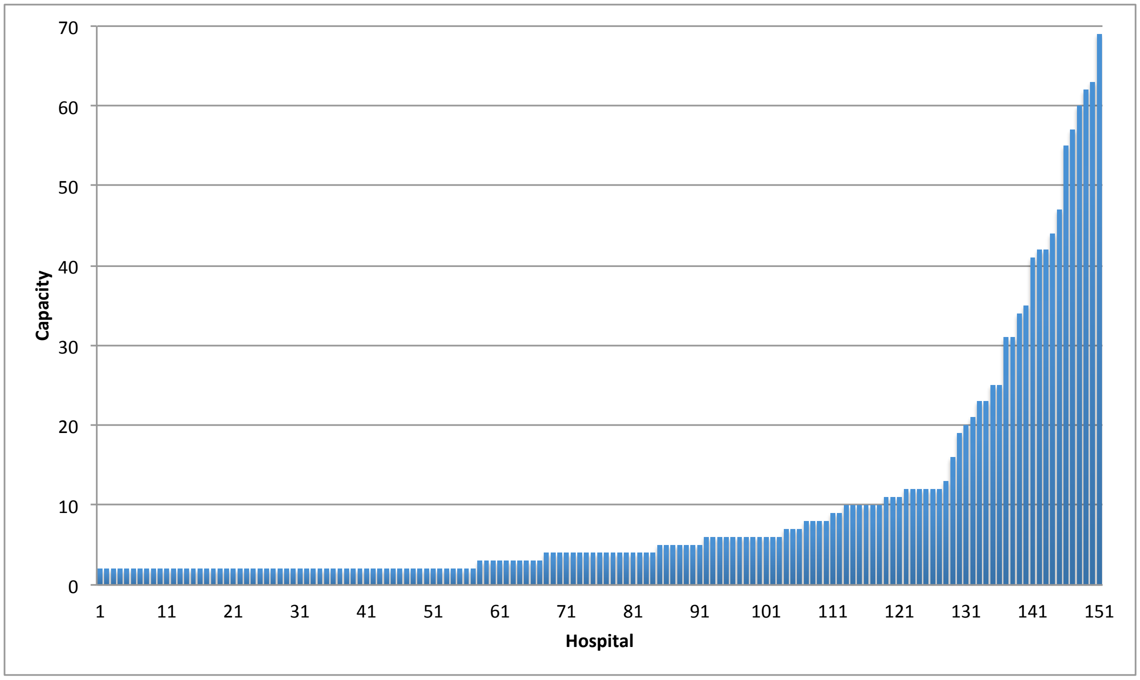

Figure 1. The hospital capacities in Tokyo. The data are taken from Japan Residency Matching Program (2013).

There may be other reasonable policy goals as well. For instance, the government may wish to give a priority to some hospitals within a region over others for a variety of reasons.[3] Thus it is clear from these examples that focusing on a particular policy goal limits the practical applicability of the model of Kamada and Kojima (2015).[4]

To accommodate a wide range of policy goals, we offer a model in which each region is endowed with "regional preferences" over distributions of doctors within the region. The idea behind this modeling approach is to express policy goals as regional preferences, and accommodate different types of policy goals by changing regional preferences.[5] Then we define a stability concept that takes the regional preferences into account. A result by Kamada and Kojima (2018) implies that, under some regularity conditions, their *flexible deferred acceptance mechanism* finds a stable (and efficient) matching and it is group strategy-proof for doctors.

[3]In the Japanese medical residency match, a hospital is given preferential treatments if that hospital deploys its doctors to underserved areas (Ministry of Health, Labour and Welfare, 2014).

[4]Some policy goals could be addressed by setting target capacities judiciously. However, it is easy to see that policy goals such as those discussed here cannot be fully expressed simply by picking target capacities.

[5]In Section D we provide various types of regional preferences that represent different policy goals.

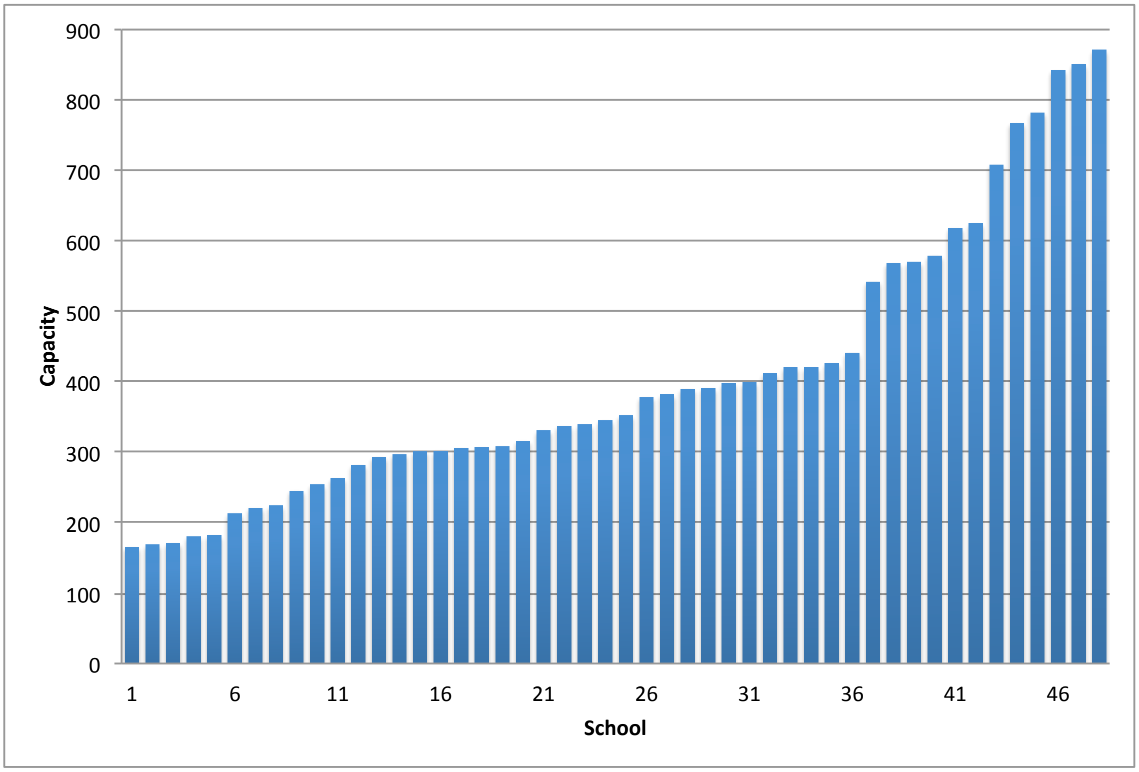

FIGURE 2. The school capacities in public elementary schools in Boston. The data are taken from Boston Public Schools (2013).

This paper offers three sets of results. First, we show that the stability notion of Kamada and Kojima (2015) is a special case of our stability concept. More specifically, their concept is a case of our stability notion in which the regional preferences satisfy a condition called the "Rawlsian" property. Moreover, when the regional preferences are Rawlsian, our flexible deferred acceptance mechanism reduces to the mechanism of Kamada and Kojima (2015). Thus we establish the main results of Kamada and Kojima (2015) as a special case.

Second, we formalize a wide range of policy goals to demonstrate that they can be described by our regional preferences. Consequently, we establish that the corresponding flexible deferred acceptance mechanisms find a stable matching with respect to those regional preferences.

Third, we prove several policy-relevant comparative statics results. Among other things, we compare a mechanism currently used in Japan with the flexible deferred acceptance mechanism and establish that all doctors are weakly better off under the latter mechanism. Still, we also establish that all doctors are weakly worse off under the flexible deferred acceptance mechanism than under the unconstrained deferred acceptance mechanism. We note that we obtain all of our comparative statics results as straightforward corollaries of a single new comparative statics result.[6]

The rest of this paper proceeds as follows. In Section 2, we present the model. Section 3 states preliminary result. In Section 4, we offer our analysis. Section 5 concludes. Proofs are in the Appendix unless noted otherwise.

2. Model

This section introduces a model of matching under distributional constraints. We describe the model in terms of matching between doctors and hospitals with "regional caps," that is, upper bounds on the number of doctors that can be matched to hospitals in each region. However, the model is applicable to various other situations in and out of the residency matching context. Concrete applications include Chinese graduate school admission, U.K. medical matching, Scottish teacher matching, and college admissions in Ukraine and Hungary.[7]

Our notation and terminology closely follow those of Kamada and Kojima (2015, 2018).

2.1. **Preliminary Definitions.**

Let there be a finite set of doctors D and a finite set of hospitals H. Each doctor d has a strict preference relation $\succ_d$ over the set of hospitals and being unmatched (being unmatched is denoted by $\emptyset$). For any $h, h' \in H \cup \{\emptyset\}$, we write $h \succeq_d h'$ if and only if $h \succ_d h'$ or $h = h'$. Each hospital h has a strict preference relation $\succ_h$ over the set of subsets of doctors. For any $D', D'' \subseteq D$, we write $D' \succeq_h D''$ if and only if $D' \succ_h D''$ or $D' = D''$. We denote by $\succ = (\succ_i)_{i \in D \cup H}$ the preference profile of all doctors and hospitals.

Each hospital $h \in H$ is endowed with a (physical) **capacity** q_h, which is a nonnegative integer. We say that preference relation $\succ_h$ is **responsive with capacity** q_h (Roth, 1985) if

(1) For any $D' \subseteq D$ with $|D'| \leq q_h$, $d \in D \setminus D'$ and $d' \in D'$, $(D' \cup d) \setminus d' \succeq_h D'$ if and only if $d \succeq_h d'$,
(2) For any $D' \subseteq D$ with $|D'| \leq q_h$ and $d' \in D'$, $D' \succeq_h D' \setminus d'$ if and only if $d' \succeq_h \emptyset$, and
(3) $\emptyset \succ_h D'$ for any $D' \subseteq D$ with $|D'| > q_h$.

[6]Our general comparative statics result extends existing results such as Gale and Sotomayor (1985a,b), Crawford (1991), and Konishi and Ünver (2006). See also Kelso and Crawford (1982), who derive similar results in a matching model with wages. Echenique and Yenmez (2015) and Chambers and Yenmez (2017) independently obtain similar results in a framework based on choice functions as primitives.

[7]See Kamada and Kojima (2015) for detailed descriptions.

In words, preference relation $\succ_h$ is responsive with a capacity if the ranking of a doctor (or keeping a position vacant) is independent of her colleagues, and any set of doctors exceeding its capacity is less preferred to the outside option. We assume that preferences of each hospital h are responsive with capacity q_h throughout the paper.

Doctor d is said to be **acceptable** to hospital h if $d \succ_h \emptyset$.[8] Similarly, h is acceptable to d if $h \succ_d \emptyset$. It will turn out that only rankings of acceptable partners matter for our analysis, so we often write only acceptable partners to denote preferences. For example,

$$\succ_d : h, h'$$

means that hospital h is the most preferred, h' is the second most preferred, and h and h' are the only acceptable hospitals under preferences $\succ_d$ of doctor d.

There is a finite set R which we call the set of **regions**. The set of hospitals H is partitioned into hospitals in different regions, that is, $H_r \cap H_{r'} = \emptyset$ if $r \neq r'$ and $H = \cup_{r \in R} H_r$, where H_r denotes the set of hospitals in region $r \in R$. For each $h \in H$, let $r(h)$ denote the region r such that $h \in H_r$. For each region $r \in R$, there is a **regional cap** q_r, which is a nonnegative integer.

A **matching** μ is a mapping from $D \cup H$, where we write "μ_i" for "$\mu(i)$" for each $i \in D \cup H$. This mapping satisfies (i) $\mu_d \in H \cup \{\emptyset\}$ for all $d \in D$, (ii) $\mu_h \subseteq D$ for all $h \in H$, and (iii) for any $d \in D$ and $h \in H$, $\mu_d = h$ if and only if $d \in \mu_h$. That is, a matching simply specifies which doctor is assigned to which hospital (if any). A matching is **feasible** if $|\mu_r| \leq q_r$ for all $r \in R$, where $\mu_r = \cup_{h \in H_r} \mu_h$. In other words, feasibility requires that the regional cap for every region is satisfied. This requirement distinguishes the current environment from the standard model without regional caps: We allow for (though do not require) $q_r < \sum_{h \in H_r} q_h$, that is, the regional cap can be smaller than the sum of hospital capacities in the region.

To accommodate the regional caps, we introduce a concept that generalizes the standard stability notion. For that purpose, we first define two basic concepts. A matching μ is **individually rational** if (i) for each $d \in D$, $\mu_d \succeq_d \emptyset$, and (ii) for each $h \in H$, $d \succeq_h \emptyset$ for all $d \in \mu_h$, and $|\mu_h| \leq q_h$. That is, no agent is matched with an unacceptable partner and each hospital's capacity is respected.

Given matching μ, a pair (d, h) of a doctor and a hospital is called a **blocking pair** if $h \succ_d \mu_d$ and either (i) $|\mu_h| < q_h$ and $d \succ_h \emptyset$, or (ii) $d \succ_h d'$ for some $d' \in \mu_h$. In words, a blocking pair is a pair of a doctor and a hospital who want to be matched with each

---

[8]We denote singleton set $\{x\}$ by x when there is no confusion.

other (possibly rejecting their partners in the prescribed matching) rather than following the proposed matching.

When there are no binding regional caps (in the sense that $q_r > \sum_{h \in H_r} q_h$ for every $r \in R$), a matching is said to be stable if it is individually rational and there is no blocking pair. Gale and Shapley (1962) show that there exists a stable matching in that setting. In the presence of binding regional caps, however, there may be no such matching that is feasible (in the sense that all regional caps are respected). Thus in some cases every feasible and individually rational matching may admit a blocking pair.

A **mechanism** φ is a function that maps preference profiles to matchings. The matching under φ at preference profile $\succ$ is denoted $\varphi(\succ)$ and agent i's match is denoted by $\varphi_i(\succ)$ for each $i \in D \cup H$.

A mechanism φ is said to be strategy-proof for doctors if there does not exist a preference profile $\succ$, a doctor $d \in D$, and preferences $\succ'_d$ of doctor d such that $\varphi_d(\succ'_d, \succ_{-d}) \succ_d \varphi_d(\succ)$. A mechanism φ is said to be **group strategy-proof for doctors** if there is no preference profile $\succ$, a subset of doctors $D' \subseteq D$, and a preference profile $(\succ'_{d'})_{d' \in D'}$ of doctors in D' such that

$$\varphi_d((\succ'_{d'})_{d' \in D'}, (\succ_i)_{i \in D \cup H \setminus D'}) \succ_d \varphi_d(\succ) \text{ for all } d \in D'.$$

That is, no subset of doctors can jointly misreport their preferences to receive a strictly preferred outcome for every member of the coalition under the mechanism.

As this paper analyzes the effect of regional caps in matching markets, it is useful to compare it with the standard matching model without regional caps. Gale and Shapley (1962) consider a matching model without any binding regional cap, which corresponds to a special case of our model in which $q_r > \sum_{h \in H_r} q_h$ for every $r \in R$. In that model, they propose the following **(doctor-proposing) deferred acceptance algorithm**:

- Step 1: Each doctor applies to her first choice hospital. Each hospital rejects the lowest-ranking doctors in excess of its capacity and all unacceptable doctors among those who applied to it, keeping the rest of the doctors temporarily (so doctors not rejected at this step may be rejected in later steps).

In general,

- Step t: Each doctor who was rejected in Step $(t-1)$ applies to her next highest choice (if any). Each hospital considers these doctors *and* doctors who are temporarily held from the previous step together, and rejects the lowest-ranking doctors in excess of its capacity and all unacceptable doctors, keeping the rest of the doctors temporarily (so doctors not rejected at this step may be rejected in later steps).

The algorithm terminates at a step in which no rejection occurs. The algorithm always terminates in a finite number of steps. Gale and Shapley (1962) show that the resulting matching is stable in the standard matching model without any binding regional cap.

Even though there exists no strategy-proof mechanism that produces a stable matching for all possible inputs, the deferred acceptance mechanism is (group) strategy-proof for doctors (Dubins and Freedman, 1981; Roth, 1982). This result has been extended by many subsequent studies, suggesting that the incentive compatibility of the mechanism is quite robust and general.[9]

Kamada and Kojima (2015) present examples that show that a simple adaptation of the deferred acceptance mechanism results in inefficiency and instability. Motivated by this problem, the current paper presents a theory of stable matching under distributional constraints in the subsequent sections.

2.2. **Model with Regional Preferences.** Let **regional preferences** $\succeq_r$ be a weak ordering over nonnegative-valued integer vectors $W_r := \{w = (w_h)_{h \in H_r} | w_h \in \mathbb{Z}_+\}$. That is, $\succeq_r$ is a binary relation that is complete and transitive (but not necessarily antisymmetric). We write $w \succ_r w'$ if and only if $w \succeq_r w'$ holds but $w' \succeq_r w$ does not. Vectors such as w and w' are interpreted to be supplies of acceptable doctors to the hospitals in region r, but they only specify how many acceptable doctors apply to each hospital and no information is given as to who these doctors are. Given $\succeq_r$, a function $\tilde{\text{Ch}}_r : W_r \to W_r$ is an **associated quasi choice rule** if $\tilde{\text{Ch}}_r(w) \in \arg\max_{\succeq_r} \{w' | w' \leq w\}$ for any non-negative integer vector $w = (w_h)_{h \in H_r}$.[10] We require that the quasi choice rule $\tilde{\text{Ch}}_r$ be **consistent**, that is, $\tilde{\text{Ch}}_r(w) \leq w' \leq w \Rightarrow \tilde{\text{Ch}}_r(w') = \tilde{\text{Ch}}_r(w)$.[11] This condition requires that, if $\tilde{\text{Ch}}_r(w)$ is chosen at w and the supply decreases to $w' \leq w$ but $\tilde{\text{Ch}}_r(w)$ is still available under w',

[9]Researches generalizing (group) strategy-proofness of the mechanism include Abdulkadiroğlu (2005), Hatfield and Milgrom (2005), Martinez, Masso, Neme, and Oviedo (2004), Hatfield and Kojima (2009, 2010), and Hatfield and Kominers (2012).

[10]For any two vectors $w = (w_h)_{h \in H_r}$ and $w' = (w'_h)_{h \in H_r}$, we write $w \leq w'$ if and only if $w_h \leq w'_h$ for all $h \in H_r$. We write $w \lneq w'$ if and only if $w \leq w'$ and $w_h < w'_h$ for at least one $h \in H_r$. For any $W'_r \subseteq W_r$, $\arg\max_{\succeq_r} W'_r$ is the set of vectors $w \in W'_r$ such that $w \succeq_r w'$ for all $w' \in W'_r$.

[11]Analogous conditions are used by Blair (1988), Alkan (2002), and Alkan and Gale (2003) in different contexts. Kamada and Kojima (2018) show that if a regional preference satisfies substitutability and its associated quasi choice rule is acceptant, as defined later, then the quasi choice rule satisfies consistency. Fleiner (2003) and Aygün and Sönmez (2012) prove analogous results although they do not work on substitutability defined over the space of integer vectors.

then the same choice $\tilde{\mathrm{Ch}}_r(w)$ should be made under w' as well. Note that there may be more than one quasi choice rule associated with a given weak ordering $\succeq_r$ because the set $\arg\max_{\succeq_r}\{w'|w' \le w\}$ may not be a singleton for some $\succeq_r$ and w. Note also that there always exists a consistent quasi choice rule.[12] We assume that the regional preferences $\succeq_r$ satisfy the following mild regularity conditions:

(1) $w' \succ_r w$ if $w_h > q_h \ge w'_h$ for some $h \in H_r$ and $w'_{h'} = w_{h'}$ for all $h' \ne h$.

This property says that the region desires no hospital to be forced to be assigned more doctors than its real capacity. This condition implies that, for any w, the component $[\tilde{\mathrm{Ch}}_r(w)]_h$ of $\tilde{\mathrm{Ch}}_r(w)$ for h satisfies $[\tilde{\mathrm{Ch}}_r(w)]_h \le q_h$ for each $h \in H_r$, that is, the capacity constraint for each hospital is respected by the (quasi) choice of the region.

(2) $w' \succ_r w$ if $\sum_{h\in H_r} w_h > q_r \ge \sum_{h\in H_r} w'_h$.

This property simply says that region r prefers the total number of doctors in the region to be at most its regional cap. This condition implies that $\sum_{h\in H_r}(\tilde{\mathrm{Ch}}_r(w))_h \le q_r$ for any w, that is, the regional cap is respected by the (quasi) choice of the region.

(3) If $w' \lneq w \le q_{H_r} := (q_h)_{h\in H_r}$ and $\sum_{h\in H_r} w_h \le q_r$, then $w \succ_r w'$.

This condition formalizes the idea that region r prefers to fill as many positions of hospitals in the region as possible so long as doing so does not lead to a violation of the hospitals' real capacities or the regional cap. This requirement implies that any associated quasi choice rule is **acceptant**, that is, for each w, if there exists h such that $[\tilde{\mathrm{Ch}}_r(w)]_h < \min\{q_h, w_h\}$, then $\sum_{h'\in H_r}[\tilde{\mathrm{Ch}}_r(w)]_{h'} = q_r$.[13] This captures the idea that the social planner should not waste caps allocated to the region: If some doctor is rejected by a hospital even though she is acceptable to the hospital and the hospital's capacity is not binding, then the regional cap should be binding.

Definition 1. The regional preferences $\succeq_r$ are **substitutable** if there exists an associated quasi choice rule $\tilde{\mathrm{Ch}}_r$ that satisfies $w \le w' \Rightarrow \tilde{\mathrm{Ch}}_r(w) \ge \tilde{\mathrm{Ch}}_r(w') \wedge w$.[14]

Notice that the condition in this definition is equivalent to

$$w \le w' \Rightarrow [\tilde{\mathrm{Ch}}_r(w)]_h \ge \min\{[\tilde{\mathrm{Ch}}_r(w')]_h, w_h\} \text{ for every } h \in H_r. \tag{2.1}$$

[12] See Kamada and Kojima (2018) for the detail.

[13] A similar condition is used by Alkan (2001) and Kojima and Manea (2010) in the context of choice functions over matchings.

[14] For any two vectors $w, w' \in W_r$, $w \wedge w'$ is defined as a vector $(\min\{w_h, w'_h\})_{h\in H_r} \in W_r$.

This condition says that, when the supply of doctors is increased, the number of accepted doctors at a hospital can increase only when the hospital has accepted all acceptable doctors under the original supply profile.[15] Formally, condition (2.1) is equivalent to

$$w \leq w' \text{ and } [\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h \Rightarrow [\tilde{\text{Ch}}_r(w)]_h = w_h. \tag{2.2}$$

To see that condition (2.1) implies condition (2.2), suppose that $w \leq w'$ and $[\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h$. These assumptions and condition (2.1) imply $[\tilde{\text{Ch}}_r(w)]_h \geq w_h$. Since $[\tilde{\text{Ch}}_r(w)]_h \leq w_h$ holds by the definition of $\tilde{\text{Ch}}_r$, this implies $[\tilde{\text{Ch}}_r(w)]_h = w_h$. To see that condition (2.2) implies condition (2.1), suppose that $w \leq w'$. If $[\tilde{\text{Ch}}_r(w)]_h \geq [\tilde{\text{Ch}}_r(w')]_h$, the conclusion of (2.1) is trivially satisfied. If $[\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h$, then condition (2.2) implies $[\tilde{\text{Ch}}_r(w)]_h = w_h$, thus the conclusion of (2.1) is satisfied.

Given a profile of regional preferences $(\succeq_r)_{r \in R}$, stability is defined as follows.

Definition 2. A matching μ is **stable** if it is feasible, individually rational, and if (d, h) is a blocking pair then (i) $|\mu_{r(h)}| = q_{r(h)}$, (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$, and

(iii) either $\mu_d \notin H_{r(h)}$ or $w \succeq_{r(h)} w'$,

where $w_{h'} = |\mu_{h'}|$ for all $h' \in H_{r(h)}$ and $w'_h = w_h + 1$, $w'_{\mu_d} = w_{\mu_d} - 1$ and $w'_{h'} = w_{h'}$ for all other $h' \in H_{r(h)}$.

As stated in the definition, only certain blocking pairs are tolerated under stability. Any blocking pair that may remain is in danger of violating the regional cap since condition (i) implies that the cap for the blocking hospital's region is currently full, and condition (ii) implies that the only blocking involves filling a vacant position.

There are two possible cases under (iii). The first case implies that the blocking doctor is not currently assigned in the hospital's region, so the blocking pair violates the regional cap. The second part of condition (iii) considers blocking pairs within a region (note that $\mu_d \in H_{r(h)}$ holds in the remaining case). It states that if the blocking pair does not improve the doctor distribution in the region with respect to its regional preferences, then it is not regarded as a legitimate block.[16]

The way that regional preferences are determined could depend on the policy goal of the region or the social planner. One possibility for regional preferences, studied in detail by Kamada and Kojima (2015), is to prefer distributions of doctors that have "fewer gaps" from the target capacities; see Section 4.1 for detail. Other regional preferences are analyzed in Section 4.2.

[15]This definition of substitutability is analogous to *persistence* by Alkan and Gale (2003).

[16]See Kamada and Kojima (2015, 2018) for a fuller discussion of the interpretation of our stability concept.

Clearly, our stability concept reduces to the standard stability concept of Gale and Shapley (1962) if there are no binding regional caps. Kamada and Kojima (2017) show that a stable matching is constrained efficient, i.e., there is no feasible matching μ' such that $\mu'_i \succeq_i \mu_i$ for all $i \in D \cup H$ and $\mu'_i \succ_i \mu_i$ for some $i \in D \cup H$.

3. Preliminaries

This section has two goals. The first goal is to demonstrate that a stable matching exists under our general definition of stability under distributional constraints. The second goal is to show that a stable matching can be found by a mechanism that is strategy-proof for doctors. To achieve these goals, we begin by introducing the following (generalized) flexible deferred acceptance algorithm:

The (Generalized) Flexible Deferred Acceptance Algorithm For each region r, fix an associated quasi choice rule $\tilde{\text{Ch}}_r$ which satisfies condition (2.1). Note that the assumption that $\succeq_r$ is substitutable assures the existence of such a quasi choice rule.

(1) Begin with an empty matching, that is, a matching μ such that $\mu_d = \emptyset$ for all $d \in D$.
(2) Choose a doctor d arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
(3) Let d apply to the most preferred hospital $\bar{h}$ at $\succ_d$ among the hospitals that have not rejected d so far. If d is unacceptable to $\bar{h}$, then reject this doctor and go back to Step 2. Otherwise, let r be the region such that $\bar{h} \in H_r$ and define vector $w = (w_h)_{h \in H_r}$ by
   (a) $w_{\bar{h}}$ is the number of doctors currently held at $\bar{h}$ plus one, and
   (b) w_h is the number of doctors currently held at h if $h \neq \bar{h}$.
(4) Each hospital $h \in H_r$ considers the new applicant d (if $h = \bar{h}$) *and* doctors who are temporarily held from the previous step together. It holds its $(\tilde{\text{Ch}}_r(w))_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to Step 2.

We define the **(generalized) flexible deferred acceptance mechanism** to be a mechanism that produces, for each input, the matching given at the termination of the above algorithm.

This algorithm is a generalization of the deferred acceptance algorithm of Gale and Shapley (1962) to the model with regional caps. The main differences are found in Steps

3 and 4. Unlike the deferred acceptance algorithm, this algorithm limits the number of doctors (tentatively) matched in each region r at q_r. This results in rationing of doctors across hospitals in the region, and the rationing rule is governed by regional preferences $\succeq_r$. Clearly, this mechanism coincides with the standard deferred acceptance algorithm if all the regional caps are large enough and hence non-binding.

We take the following result as the starting point of our analysis.

Proposition 0 (A Corollary of Theorem 1 of Kamada and Kojima (2018))**.** *Suppose that $\succeq_r$ is substitutable for every $r \in R$. Then the flexible deferred acceptance algorithm stops in finite steps. The mechanism produces a stable matching for any input and is group strategy-proof for doctors.*

This proposition offers a sense in which it is possible to design a desirable mechanism even under distributional constraints and various policy goals. As will be seen in subsequent sections, the class of substitutable regional preferences subsumes the "Rawlsian" regional preferences motivated by a residency matching application (Section 4.1) as well as others (Section 4.2). For each of these cases, the flexible deferred acceptance mechanism finds a stable matching that addresses a given policy goal, while inducing truthful reporting by doctors. Moreover, because stability implies efficiency (Kamada and Kojima, 2017), the algorithm produces an efficient matching.

A formal proof for a more general case can be found in Kamada and Kojima (2018), while a proof that is adapted to our present setting is in Appendix A.[17] We illustrate a sketch of the proof here.

Our proof strategy is to connect our matching model with constraints to the "matching with contracts" model (Hatfield and Milgrom, 2005).[18] More specifically, given the original matching model under constraints, we define an "associated model," a hypothetical matching model between doctors and *regions* instead of doctors and hospitals; In the associated model, we regard each region as a hypothetical consortium of hospitals that acts as one agent. By imagining that a region (hospital consortium) makes a coordinated employment decision, we can account for the fact that acceptance of a doctor by a hospital may depend on doctor applications to other hospitals in the same region, an inevitable

[17]The notation and concepts in that section are used for proofs for other results.

[18]Fleiner (2003) considers a framework that generalizes various mathematical results. A special case of his model corresponds to the model of Hatfield and Milgrom (2005), although not all results of the latter (e.g., those concerning incentives) are obtained in the former. See also Crawford and Knoer (1981) who observe that wages can represent general job descriptions in their model, given their assumption that firm preferences satisfy separability.

feature in markets under distributional constraints. This association necessitates, however, that we distinguish a doctor's placements in different hospitals in the given region. We account for this complication by defining a region's choice function over *contracts* rather than doctors, where a contract specifies a doctor-hospital pair to be matched. We construct such a choice function by using two pieces of information: the preferences of all the hospitals in the given region, and regional preferences. The idea is that each hospital's preferences are used for choosing doctors *given the number of allocated slots*, while regional preferences are used to regulate slots allocated to different hospitals in the region. In other words, regional preferences trade off multiple hospitals' desires to accept more doctors, when accepting more is in conflict with the regional cap. With the help of this association, we demonstrate that any stable allocation in the associate model with contracts induces a stable matching in the original model with distributional constraints (Proposition 9).

Once this association is established, with some work we show that the key conditions in the associated model—the substitutes condition and the law of aggregate demand—are satisfied (Proposition 8). This enables us to invoke existing results for matching with contracts, namely that an existing algorithm called the "cumulative offer process" finds a stable allocation, and it is (group) strategy-proof for doctors in the associated model (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2009; Hatfield and Kominers, 2012). Then, we observe that the outcome of the cumulative offer process corresponds to the matching produced by the flexible deferred acceptance algorithm in the original model with constraints (Remark 1). This correspondence establishes that the flexible deferred acceptance mechanism finds a stable matching in the original problem and this algorithm is group strategy-proof for doctors, proving Proposition 0.

The full proof, presented in Appendix A, formalizes this idea. The proof is somewhat involved because one needs to exercise some care when establishing correspondences between the two models and confirming that a property in one model induces the corresponding property in the other.

4. Results

4.1. **Stability in Kamada and Kojima (2015).** In this section we establish the main result of Kamada and Kojima (2015) by showing that their stability concept can be rewritten by using a substitutable regional preferences.

In Kamada and Kojima (2015), there is an exogenously given (government-imposed) nonnegative integer $\bar{q}_h \leq q_h$ called **target capacity**, for each hospital h such that

$\sum_{h\in H_r} \bar{q}_h \leq q_r$ for each region $r \in R$. Given a profile of target capacities, they define a stability concept. We refer to this concept as T-stability (where "T" signifies target capacities) to avoid confusion with stability defined earlier.

Definition 3. A matching μ is **T-stable** if it is feasible, individually rational, and if (d, h) is a blocking pair then (i) $|\mu_{r(h)}| = q_{r(h)}$, (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$, and

(iii) either $\mu_d \notin H_{r(h)}$ or $|\mu'_h| - \bar{q}_h > |\mu'_{\mu_d}| - \bar{q}_{\mu_d}$,

where μ' is the matching such that $\mu'_d = h$ and $\mu'_{d'} = \mu_{d'}$ for all $d' \neq d$.

Kamada and Kojima (2015) define the **flexible deferred acceptance algorithm** in their setting as follows. For each $r \in R$, specify an order of hospitals in region r: Denote $H_r = \{h_1, h_2, \ldots, h_{|H_r|}\}$ and order h_i earlier than h_j if $i < j$. Given this order, consider the following algorithm.

(1) Begin with an empty matching, that is, a matching μ such that $\mu_d = \emptyset$ for all $d \in D$.
(2) Choose a doctor d who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
(3) Let d apply to the most preferred hospital $\bar{h}$ at $\succ_d$ among the hospitals that have not rejected d so far. Let r be the region such that $\bar{h} \in H_r$.
(4) (a) For each $h \in H_r$, let D'_h be the entire set of doctors who have applied to but have not been rejected by h so far and are acceptable to h. For each hospital $h \in H_r$, choose the $\bar{q}_h$ best doctors according to $\succ_h$ from D'_h if they exist, and otherwise choose all doctors in D'_h. Formally, for each $h \in H_r$ choose D''_h such that $D''_h \subset D'_h$, $|D''_h| = \min\{\bar{q}_h, |D'_h|\}$, and $d \succ_h d'$ for any $d \in D''_h$ and $d' \in D'_h \setminus D''_h$.
    (b) Start with a tentative match D''_h for each hospital $h \in H_r$. Hospitals take turns to choose (one doctor at a time) the best remaining doctor in their current applicant pool. Repeat the procedure (starting with h_1, proceeding to $h_2, h_3, \ldots$ and going back to h_1 after the last hospital) until the regional quota q_r is filled or the capacity of the hospital is filled or no doctor remains to be matched. All other applicants are rejected.[19]

[19]Formally, let $\iota_i = 0$ for all $i \in \{1, 2, \ldots, |H_r|\}$. Let $i = 1$.

(i) If either the number of doctors already chosen by the region r as a whole equals q_r, or $\iota_i = 1$, then reject the doctors who were not chosen throughout this step and go back to Step 2.

Kamada and Kojima (2015) define the **flexible deferred acceptance mechanism** to be a mechanism that produces, for each input, the matching at the termination of the above algorithm.[20] The following proposition is stated as the main result of Kamada and Kojima (2015).

Proposition 1 (Theorem 2 of Kamada and Kojima (2015))**.** *In the setting of Kamada and Kojima (2015), the flexible deferred acceptance algorithm stops in finite steps. The mechanism produces a T-stable matching for any input and is group strategy-proof for doctors.*

In the remainder of this section, we establish this result as a corollary of the main result of the present paper, Proposition 0.

To start the analysis, fix a region r. Given the target capacity profile $(\bar{q}_h)_{h\in H_r}$ and the vector $w \in W_r$, define the **ordered excess weight vector** $\eta(w) = (\eta_1(w), ..., \eta_{|H_r|}(w))$ by setting $\eta_i(w)$ to be the i'th lowest value (allowing repetition) of $\{w_h - \bar{q}_h | h \in H_r\}$ (we suppress dependence of η on target capacities). For example, if $w = (w_{h_1}, w_{h_2}, w_{h_3}, w_{h_4}) = (2, 4, 7, 2)$ and $(\bar{q}_{h_1}, \bar{q}_{h_2}, \bar{q}_{h_3}, \bar{q}_{h_4}) = (3, 2, 3, 0)$, then $\eta_1(w) = -1, \eta_2(w) = \eta_3(w) = 2, \eta_4(w) = 4$.

Consider the regional preferences $\succeq_r$ that compare the excess weights lexicographically. More specifically, let $\succeq_r$ be such that $w \succ_r w'$ if and only if there exists an index $i \in \{1, 2, \ldots, |H_r|\}$ such that $\eta_j(w) = \eta_j(w')$ for all $j < i$ and $\eta_i(w) > \eta_i(w')$. The associated weak regional preferences $\succeq_r$ are defined by $w \succeq_r w'$ if and only if $w \succ_r w'$ or $\eta(w) = \eta(w')$. We call such regional preferences **Rawlsian**.

Proposition 2. *T-stability is a special case of stability such that the regional preferences of each region are Rawlsian.*

Proof. See Appendix B.1. □

Consider the (generalized) flexible deferred acceptance algorithm in a previous subsection. With the following quasi choice rule, this algorithm is equivalent to the flexible

(ii) Otherwise, let h_i choose the most preferred (acceptable) doctor in D'_{h_i} at $\succ_{h_i}$ among the doctors that have not been chosen by h_i so far, if such a doctor exists and the number of doctors chosen by h_i so far is strictly smaller than q_{h_i}.

(iii) If no new doctor was chosen at Step 4(b)ii, then set $\iota_i = 1$. If a new doctor was chosen at Step 4(b)ii, then set $\iota_j = 0$ for all $j \in \{1, 2, \ldots, |H_r|\}$. If $i < |H_r|$ then increment i by one and if $i = |H_r|$ then set i to be 1 and go back to Step 4(b)i.

[20]Propositions 0, 2, and 3 show that the algorithm stops in a finite number of steps.

deferred acceptance algorithm in Kamada and Kojima (2015): For each $w' \in W_r$,

$$\tilde{\mathrm{Ch}}_r(w') = \max_{\substack{w=w^k \text{ for some } k \\ \sum_{h\in H_r} w_h \le q_r}} w, \tag{4.1}$$

where $w^0 = (\min\{w'_h, \bar{q}_h\})_{h\in H_r}$ and $w^k \in W_r$ $(k = 1, 2, \dots)$ is defined by

$$w^k_{h_j} = \min\{w'_{h_j}, q_{h_j}, w^{k-1}_{h_j} + \mathbb{I}_{j\equiv k \pmod{|H_r|}}\} \quad \text{for each } j = 1, 2, \dots, |H_r|,$$

with "$\mathbb{I}$" being an indicator function. Note that there is a unique maximum in (4.1) because $w^{k'} \le w^{k'+1}$ for every $k' = 1, 2, \dots$ by the definition of w^k.[21]

Proposition 3. *Rawlsian preferences are substitutable with the associated quasi choice rule (4.1).*

Proof. See Appendix B.1. □

Propositions 0, 2, and 3 imply Theorem 2 of Kamada and Kojima (2015).

In Appendix D, we discuss how to allocate target capacities among hospitals in a region, within the Rawlsian framework. There we observe that the allocation problem is similar to the celebrated "bankruptcy problem," and consider several rules studied in that literature.

4.2. **Alternative Criteria.** Although Kamada and Kojima (2015) focus on a particular stability concept (T-stability) and corresponding regional preferences, called Rawlsian preferences, it is quite plausible that some societies may prefer to impose different criteria from the Rawlsian preferences. This section proposes other criteria that seem to be appealing. They are examples of regional preferences that satisfy substitutability defined in Definition 1. Therefore, the corresponding flexible deferred acceptance mechanisms find a stable matching with respect to those regional preferences.

In the following, we assume that $0 \succ_r w$ for any weight vector w such that $\sum_{h\in H_r} w_h > q_r$ or $w_h > q_h$ for some $h \in H_r$. Thus in (1) - (4) below, we assume that any weight vector w satisfies $\sum_{h\in H_r} w_h \le q_r$ and $w_h \le q_h$ for all $h \in H_r$.

(1) "Equal gains": Let the region prefer a distribution that equalizes the weights across hospitals in the region as much as possible. Formally, such a preference, which we call the **equal gains** preferences, can be expressed as the Rawlsian preferences for the special case in which the target capacity for every hospital is set at zero. Since Proposition 3 shows that the Rawlsian preferences are substitutable for any target capacity profile, the equal gains preferences satisfy substitutability.

[21]See footnote 10 for the definition of the order on W_r.

(2) "Equal Losses": Let the region prefer to equalize the "losses," that is, the differences between the (physical) capacities and the weights across hospitals in the region. More generally, one could consider the preferences for **equal losses above target capacities**, that is, the regional preferences first prefer to fill as many positions as possible to meet target capacities and then (lexicographically less importantly) prefer to equalize the losses. To formally define such preferences $\succ_r$, recall that $\eta(w)$ denotes the ordered excess weight vector as defined in Section 4.1, and define $\hat{\eta}(w)$ as a $|H_r|$-dimensional vector whose i'th component $\hat{\eta}_i(w)$ is the i'th highest value (allowing repetition) of $\{q_h - w_h | h \in H_r\}$. We let $w \succ_r w'$ if and only if
   (a) there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\min\{\eta_j(w), 0\} = \min\{\eta_j(w'), 0\}$ for all $j < i$ and $\min\{\eta_i(w), 0\} > \min\{\eta_i(w'), 0\}$, or
   (b) $\min\{\eta_i(w), 0\} = \min\{\eta_i(w'), 0\}$ for every index $i \in \{1, 2, \dots, |H_r|\}$, and there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\hat{\eta}_j(w), = \hat{\eta}_j(w')$ for all $j < i$ and $\hat{\eta}_i(w) < \hat{\eta}_i(w')$.

(3) "Proportional": The **proportional** regional preferences prefer to allocate positions to hospitals in a proportional manner subject to integer constraints. More precisely, define $\tilde{\eta}(w)$ as a $|H_r|$-dimensional vector whose i'th component $\tilde{\eta}_i(w)$ is the i'th lowest value (allowing repetition) of $\{w_h/q_h | h \in H_r\}$. We let $w \succ_r w'$ if there exists an index $i \in \{1, 2, \dots, |H_r|\}$ such that $\tilde{\eta}_j(w), = \tilde{\eta}_j(w')$ for all $j < i$ and $\tilde{\eta}_i(w) > \tilde{\eta}_i(w')$. As above, one could consider preferences for proportional losses as well. Also, these preferences can be generalized so that these concerns enter only above target capacities (this generalization is somewhat tedious but straightforward, and can be done as in Item 2). Finally, when constructing $\tilde{\eta}_i$, we can use a denominator different from q_h.[22]

(4) "Hospital-lexicographic": Let there be a pre-specified order over hospitals, and the region lexicographically prefers filling a slot in a higher-ranked hospital to filling that of a lower-ranked hospital. For instance, the region may desire to fill positions of hospitals that are underserved within the region (say, a prefecture may desire to fill positions of a hospital in a remote island within the prefecture before other hospitals). Formally, **hospital-lexicographic** regional preferences $\succ_r$ are defined as follows. Fix an order over hospitals in r, denoted by $h_1, h_2, \dots,$ and $h_{|H_r|}$. Let $w \succ_r w'$ if and only if there exists an index $i \in \{1, 2, \dots, |H_r|\}$

[22] Moreover, the generalizations mentioned above can be combined. For example, the region may desire to fill capacities above targets proportionally to $q_h - \bar{q}_h$.

such that $w_{h_j} = w'_{h_j}$ for all $j < i$ and $w_{h_i} > w'_{h_i}$. We note that one can also consider hospital-lexicographic preferences above targets by using the criterion for hospital-lexicographic preferences for weights above targets.

All the above regional preferences have associated quasi choice rules that satisfy the property that we call "order-respecting." To define this property, let there be a finite sequence of hospitals in region r such that each hospital h appears, potentially repeatedly, q_h times in the sequence, and the total size of the sequence is $\sum_{h \in H_r} q_h$. Consider a quasi choice rule that increases the weights of hospitals one by one following the specified order.[23] Formally, fix a vector $(h_1, h_2, \ldots, h_{\sum_{h\in H_r} q_h}) \in (H_r)^{\sum_{h\in H_r} q_h}$ such that $\#\{i \in \{1, 2, \ldots, \sum_{h\in H_r} q_h\} | h_i = h\} = q_h$ for each $h \in H_r$, and define $\tilde{\text{Ch}}_r(w)$ through the following algorithm:

(1) Let w^0 be the $|H_r|$-dimensional zero vector, indexed by hospitals in H_r.
(2) For any $t \geq 0$, if $\sum_{h\in H_r} w^t_h = q_r$ or $w^t_h = \min\{q_h, w_h\}$ for all $h \in H_r$, then stop the algorithm and define $\tilde{\text{Ch}}_r(w) = w^t$. If not, define w^{t+1} by:
   (a) If $w^t_{h_{t+1}} < \min\{q_{h_{t+1}}, w_{h_{t+1}}\}$, then let $w^{t+1}_{h_{t+1}} = w^t_{h_{t+1}} + 1$; otherwise, let $w^{t+1}_{h_{t+1}} = w^t_{h_{t+1}}$.
   (b) For every $h \neq h_{t+1}$, let $w^{t+1}_h = w^t_h$.

It is easy to see that any order-respecting quasi choice rule satisfies the condition in the definition of substitutability. Also it is easy to see that, for each of the above regional preferences (1) - (4), there exists an associated quasi choice rule that is order-respecting. By these observations, all of the above regional preferences are substitutable.

4.3. **Comparative Statics.** As illustrated in Section 3, our analytical approach is to construct an associated matching model with contracts and to utilize results from that model to obtain corresponding results in the original market. This connection enables us to exploit structural properties of stable allocations in the matching model with contracts. In particular, we obtain many comparative statics results as corollaries of a new general result in the matching with contract model (Lemma 1 in Appendix B.2).

We begin by stating various comparative statics results presented in Kamada and Kojima (2015). They formalize the current practice in Japan, the **Japan Residency Matching Program (JRMP) mechanism**. The JRMP mechanism is a rule that produces the matching resulting from the deferred acceptance algorithm except that, for each hospital h, it uses $\bar{q}_h \leq q_h$ instead of q_h as the hospital's capacity. In words, the JRMP mechanism pretends that the target capacities are actual physical capacities.

[23]Order-respecting quasi choice rules are similar to choice functions based on the precedence order of Kominers and Sönmez (2016), although we find no logical relationship between these two concepts.

The first result establishes comparisons across the flexible deferred acceptance, JRMP, and the (unconstrained) deferred acceptance algorithms:

Proposition 4 (Theorem 3 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015). For any preference profile,*

(1) *Each doctor $d \in D$ weakly prefers a matching produced by the deferred acceptance mechanism to the one produced by the flexible deferred acceptance mechanism to the one produced by the JRMP mechanism.*
(2) *If a doctor is unmatched in the deferred acceptance mechanism, she is unmatched in the flexible deferred acceptance mechanism. If a doctor is unmatched in the flexible deferred acceptance mechanism, she is unmatched in the JRMP mechanism.*

The next result pertains to the effect of changes in regional caps.

Proposition 5 (Proposition 3 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015). Fix a picking order in the flexible deferred acceptance mechanism. Let $(q_r)_{r\in R}$ and $(q'_r)_{r\in R}$ be regional caps such that $q'_r \leq q_r$ for each $r \in R$. Then the following statements hold.*

(1) *Each doctor $d \in D$ weakly prefers a matching produced by the flexible deferred acceptance mechanism under regional caps $(q_r)_{r\in R}$ to the one under $(q'_r)_{r\in R}$.*
(2) *For each region r such that $q_r = q'_r$, the number of doctors matched in r at a matching produced by the flexible deferred acceptance mechanism under regional caps $(q'_r)_{r\in R}$ is weakly larger than at the matching under $(q_r)_{r\in R}$.*

Another comparative statics result is about the changes in the imposed constraints under the JRMP mechanism.

Proposition 6 (Proposition 4 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015). Let $(\bar{q}_h)_{h\in H}$ and $(\bar{q}'_h)_{h\in H}$ be target capacities such that $\bar{q}'_h \leq \bar{q}_h$ for each $h \in H$. Then the following statements hold.*[24]

(1) *Each doctor $d \in D$ weakly prefers a matching produced by the JRMP mechanism under target capacities $(\bar{q}_h)_{h\in H}$ to the one under $(\bar{q}'_h)_{h\in H}$.*
(2) *Each hospital $h \in H$ such that $\bar{q}_h = \bar{q}'_h$ weakly prefers a matching produced by the JRMP mechanism under target capacities $(\bar{q}'_h)_{h\in H}$ to the one under $(\bar{q}_h)_{h\in H}$.*

[24] Since the JRMP mechanism is equivalent to the deferred acceptance mechanism with respect to the target capacities, this result can also be obtained by appealing to the "Capacity Lemma" by Konishi and Ünver (2006), although we obtain these results as corollaries of a more general result, Lemma 1.

Moreover, the number of doctors matched to any such h in the former matching is weakly larger than that in the latter.

The following result, also from Kamada and Kojima (2015), shows that, whenever a hospital or a region is underserved under the flexible deferred acceptance mechanism, the (unconstrained) deferred acceptance mechanism cannot improve the match at such a hospital or a region.

Proposition 7 (Proposition 2 of Kamada and Kojima (2015))**.** *Consider the model of Kamada and Kojima (2015).*

(1) *If the number of doctors matched with $h \in H$ in the flexible deferred acceptance mechanism is strictly less than its target capacity, then the set of doctors matched with h under the (unconstrained) deferred acceptance mechanism is a subset of the one under the flexible deferred acceptance mechanism.*
(2) *If the number of doctors matched in $r \in R$ in the flexible deferred acceptance mechanism is strictly less than its regional cap, then each hospital h in r weakly prefers a matching produced by the flexible deferred acceptance mechanism to the one under the (unconstrained) deferred acceptance mechanism. Moreover, the number of doctors matched to any such h in the former matching is weakly larger than that in the latter.*

We obtain all these results as corollaries of a single general comparative statics result in the matching with contracts model. More specifically, we establish that if the choice function of a region becomes larger in the set inclusion sense, then all doctors are made weakly better off and all other regions are made weakly worse off in the doctor-optimal stable allocation (Lemma 1 in Appendix B.2). Also, Hatfield and Milgrom (2005) show that the outcome of a cumulative offer process is a doctor-optimal allocation. Given these results, we can prove all the above results by demonstrating that all the comparisons above can be interpreted as comparisons of outcomes of cumulative offer processes under different choice functions of regions. The formal statement of the Lemma and proofs of all the results in this section can be found in Appendix B.2.

5. Conclusion

This paper presented a model of matching under distributional constraints. Building upon an approach of Kamada and Kojima (2015), we defined a stability concept that takes distributional constraints into account. We presented a general model to allow for a variety of policymaker preferences over doctor distributions. We showed the generalization subsumes the model of Kamada and Kojima (2015) as a special case, while admitting a number of other practically relevant cases. Using our general model, we proved policy-relevant comparative statics results.

We note that our analysis builds upon a new connection between matching with constraints and matching with contracts. After the first draft of this paper was written, a similar approach was adopted by other studies such as Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014), Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016), and Kojima, Tamura, and Yokoo (2018).[25]

In addition to its intrinsic theoretical interest, our major motivation for a general theory was the desire to accommodate various constraints and policy preferences in practice, thus enabling applications to diverse types of real problems. As already mentioned, geographic and other distributional constraints are prevalent in practice; Concrete examples include British and Japanese medical matches, Chinese graduate admission, European college admissions, and Scottish teacher allocation, just to name a few. Although all these markets are subject to distributional constraints, because of differences in details, the same mechanism may be suitable in one market while unfit in another. This is a major reason that a general theory is needed. Moreover, we are quite confident that there are many other markets with specific constraints which have yet to be recognized or addressed in the literature. We hope that this paper provides a useful building block for market design in those undiscovered markets, and stimulates further research in matching under constraints and, more generally, practical market design.

References

Abdulkadiroğlu, A. (2005): "College admissions with affirmative action," *International Journal of Game Theory*, 33(4), 535–549.

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Abraham, D. J., R. Irving, and D. Manlove (2007): "Two algorithms for the student-project allocation problem," *Journal of Discrete Algorithms*, 5, 73–90.

Alkan, A. (2001): "On preferences over subsets and the lattice structure of stable matchings," *Review of Economic Design*, 6(1), 99–111.

——— (2002): "A class of multipartner matching markets with a strong lattice structure," *Economic Theory*, 19(4), 737–746.

[25]See Sönmez and Switzer (2013) for a more direct application of matching with contracts model, where a cadet can be matched with a branch under one of two possible contracts. See also Sönmez (2013) and Kominers and Sönmez (2016).

Alkan, A., and D. Gale (2003): "Stable Schedule Matching Under Revealed Preferences," *Journal of Economic Theory*, 84, 73–94.

Aygün, O., and T. Sönmez (2012): "Matching with Contracts: The Critical Role of Irrelevance of Rejected Contracts," Discussion paper, Boston College Department of Economics.

Biró, P., T. Fleiner, R. W. Irving, and D. F. Manlove (2010): "The College Admissions problem with lower and common quotas," *Theoretical Computer Science*, 411, 3136–3153.

Blair, C. (1988): "The lattice structure of the set of stable matchings with multiple partners," *Mathematics of operations research*, 13(4), 619–628.

Boston Public Schools (2013): "BPS 5 Year Capital Facilities Master Plan, Phase I Fiscal Years 2014 - 2018 BPS 5 Year Capital Facilities Master Plan, Phase I Fiscal Years 2014-2018," http://bostonpublicschools.org/cms/lib07/MA01906464/Centricity/Domain/111/capital_facility_master_plan_fy2014-2018_volume_1_final.pdf.

Chambers, C., and B. Yenmez (2017): "Choice and Matching," *American Economic Journal: Microeconomics*, 9, 126–147.

Crawford, V. (1991): "Comparative statics in matching markets," *Journal of Economic Theory*, 54(2), 389–400.

Crawford, V., and E. M. Knoer (1981): "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49, 437–450.

Dubins, L. E., and D. A. Freedman (1981): "Machiavelli and the Gale-Shapley algorithm," *American Mathematical Monthly*, 88, 485–494.

Echenique, F., and B. Yenmez (2015): "How to control controlled school choice," *American Economic Review*, 105(8), 2679–2694.

Ehlers, L., I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim (2014): "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*.

Ergin, H., and T. Sönmez (2006): "Games of School Choice under the Boston Mechanism," *Journal of Public Economics*, 90, 215–237.

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28, 103–126.

Gale, D., and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

Gale, D., and M. A. O. Sotomayor (1985a): "Ms Machiavelli and the Stable matching problem," *American Mathematical Monthly*, 92, 261–268.

——— (1985b): "Some Remarks on the Stable Matching Problem," *Discrete Applied Mathematics*, 11, 223–232.

Goto, M., A. Iwasaki, Y. Kawasaki, R. Kurata, Y. Yasuda, and M. Yokoo (2016): "Strategyproof matching with regional minimum and maximum quotas," *Artificial Intelligence*, 235, 40–57.

Goto, M., A. Iwasaki, Y. Kawasaki, Y. Yasuda, and M. Yokoo (2014): "Improving Fairness and Efficiency in Matching Markets with Regional Caps: Priority-list based Deferred Acceptance Mechanism," mimeo (the latest version is available at http://mpra.ub.uni-muenchen.de/53409/).

Hafalir, I. E., M. B. Yenmez, and M. A. Yildirim (2013): "Effective affirmative action in school choice," *Theoretical Economics*, 8(2), 325–363.

Hatfield, J. W., and F. Kojima (2009): "Group Incentive Compatibility for Matching with Contracts," *Games and Economic Behavior*, pp. 745–749.

——— (2010): "Substitutes and Stability for Matching with Contracts," *Journal of Economic Theory*, pp. 1704–1723.

Hatfield, J. W., and S. D. Kominers (2012): "Matching in Networks with Bilateral Contracts," *American Economic Journal: Microeconomics*, 4(1), 176–208.

——— (2013): "Hidden Substitutes," Discussion paper, Working paper, Harvard Univ.

Hatfield, J. W., and P. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

Japan Residency Matching Program (2013): "Match Results by Residency Programs," http://www.jrmp.jp/oubosya/2013kekka.pdf.

Kamada, Y., and F. Kojima (2015): "Efficient Matching under Distributional Constraints: Theory and Applications," *American Economic Review*, 105(1), 67–99.

——— (2017): "Stability concepts in matching with distributional constraints," *Journal of Economic Theory*, 168, 107–142.

——— (2018): "Stability and Strategy-Proofness for Matching with Constraints: A Necessary and Sufficient Condition," *Theoretical Economics*, 13, 761–794.

Kelso, A., and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.

Kojima, F., and M. Manea (2010): "Axioms for Deferred Acceptance," *Econometrica*, 78(2), 633–653.

Kojima, F., A. Tamura, and M. Yokoo (2018): "Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis," *Journal of Economic Theory*, 176, 803–833.

Kominers, S. D., and T. Sönmez (2016): "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 11(2), 683–710.

Konishi, H., and M. U. Ünver (2006): "Games of Capacity Manipulation in the Hospital-Intern Market," *Social Choice and Welfare*, 27, 3–24.

Martinez, R., J. Masso, A. Neme, and J. Oviedo (2004): "On group strategy-proof mechanisms for a many-to-one matching model," *International Journal of Game Theory*, 33, 115–128.

Ministry of Health, Labour and Welfare (2014): "Frequently Asked Questions about the Medical Residency Program (For Hospitals)," http://www.mhlw.go.jp/stf/seisakunitsuite/bunya/kenkou_iryou/iryou/rinsyo/qa/byouin.html.

Roth, A. E. (1982): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7, 617–628.

——— (1985): "The College Admission Problem is not Equivalent to the Marriage Problem," *Journal of Economic Theory*, 36, 277–288.

——— (1991): "A Natural Experiment in the Organization of Entry Level Labor Markets: Regional Markets for New Physicians and Surgeons in the U.K.," *American Economic Review*, 81, 415–440.

Sönmez, T. (2013): "Bidding for army career specialties: Improving the ROTC branching mechanism," *Journal of Political Economy*, 121(1), 186–219.

Sönmez, T., and T. B. Switzer (2013): "Matching With (Branch-of-Choice) Contracts at the United States Military Academy," *Econometrica*, 81(2), 451–488.

Thomson, W. (2003): "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey," *Mathematical Social Sciences*, 45(3), 249–297.

Westkamp, A. (2013): "An Analysis of the German University Admissions System," *Economic Theory*, 53(3), 561–589.

Appendix A. Proof of Proposition 0

As stated earlier, Proposition 0 is a special case of Theorem 1 of Kamada and Kojima (2018). We provide the proof of Proposition 0 for the reader's convenience, while the language in the proof closely follows that of Kamada and Kojima (2018).

Let there be two types of agents, doctors in D and regions in R. Note that we regard a region, instead of a hospital, as an agent in this model. There is a set of contracts $X = D \times H$.

We assume that, for each doctor d, any set of contracts with cardinality two or more is unacceptable, that is, a doctor wants to sign at most one contract. For each doctor d, her preferences $\succ_d$ over $(\{d\} \times H) \cup \{\emptyset\}$ are given as follows.[26] We assume $(d,h) \succ_d (d,h')$ in this model if and only if $h \succ_d h'$ in the original model, and $(d,h) \succ_d \emptyset$ in this model if and only if $h \succ_d \emptyset$ in the original model.

For each region $r \in R$, we assume that the region has preferences $\succeq_r$ and its associated choice rule $\text{Ch}_r(\cdot)$ over all subsets of $D \times H_r$. For any $X' \subset D \times H_r$, let $w(X') := (w_h(X'))_{h \in H_r}$ be the vector such that $w_h(X') = |\{(d,h) \in X' | d \succ_h \emptyset\}|$. For each X', the chosen set of contracts $\text{Ch}_r(X')$ is defined by

(A.1)
$$\text{Ch}_r(X') = \bigcup_{h \in H_r} \left\{ (d,h) \in X' \;\middle|\; |\{d' \in D | (d',h) \in X', d' \succeq_h d\}| \le (\tilde{\text{Ch}}_r(w(X')))_h \right\}.$$

That is, each hospital $h \in H_r$ chooses its $(\tilde{\text{Ch}}_r(w(X')))_h$ most preferred contracts available in X'.

We extend the domain of the choice rule to the collection of all subsets of X by setting $\text{Ch}_r(X') = \text{Ch}_r(\{(d,h) \in X' | h \in H_r\})$ for any $X' \subseteq X$.

Definition 4 (Hatfield and Milgrom (2005))**.** Choice rule $\text{Ch}_r(\cdot)$ satisfies the **substitutes condition** if there does not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \text{Ch}_r(X' \cup \{x'\})$ and $x' \in \text{Ch}_r(X' \cup \{x, x'\})$.

In other words, contracts are substitutes if adding a contract to the choice set never induces a region to choose a contract it previously rejected. Hatfield and Milgrom (2005) show that there exists a stable allocation (defined in Definition 6) when contracts are substitutes for every region.

---

[26] We abuse notation and use the same notation $\succ_d$ for preferences of doctor d both in the original model and in the associated model with contracts.

Definition 5 (Hatfield and Milgrom (2005))**.** Choice rule $\mathrm{Ch}_r(\cdot)$ satisfies the **law of aggregate demand** if for all $X' \subseteq X'' \subseteq X$, $|\mathrm{Ch}_r(X')| \leq |\mathrm{Ch}_r(X'')|$.[27]

Proposition 8. *Suppose that $\succeq_r$ is substitutable. Then choice rule $\mathrm{Ch}_r(\cdot)$ defined above satisfies the substitutes condition and the law of aggregate demand.*[28]

Proof. Fix a region $r \in R$. Let $X' \subseteq X$ be a subset of contracts and $x = (d, h) \in X \setminus X'$ where $h \in H_r$. Let $w = w(X')$ and $w' = w(X' \cup x)$. To show that Ch_r satisfies the substitutes condition, we consider a number of cases as follows.

(1) Suppose that $\emptyset \succ_h d$. Then $w' = w$ and, for each $h' \in H_r$, the set of acceptable doctors available at $X' \cup x$ is identical to the one at X'. Therefore, by inspection of the definition of Ch_r, we have $\mathrm{Ch}_r(X' \cup x) = \mathrm{Ch}_r(X')$, satisfying the conclusion of the substitutes condition in this case.

(2) Suppose that $d \succ_h \emptyset$.

   (a) Consider a hospital $h' \in H_r \setminus h$. Note that we have $w'_{h'} = w_{h'}$. This and the inequality $[\tilde{\mathrm{Ch}}_r(w')]_{h'} \leq w'_{h'}$ (which always holds by the definition of $\tilde{\mathrm{Ch}}_r$) imply that $[\tilde{\mathrm{Ch}}_r(w')]_{h'} \leq w_{h'}$. Thus we obtain $\min\{[\tilde{\mathrm{Ch}}_r(w')]_{h'}, w_{h'}\} = [\tilde{\mathrm{Ch}}_r(w')]_{h'}$. Since $w' \geq w$ and condition (2.1) holds, this implies that

$$[\tilde{\mathrm{Ch}}_r(w)]_{h'} \geq [\tilde{\mathrm{Ch}}_r(w')]_{h'}. \tag{A.2}$$

   Also observe that the set $\{d' \in D | (d', h') \in X'\}$ is identical to $\{d' \in D | (d', h') \in X' \cup x\}$, that is, the sets of doctors that are available to hospital h' are identical under X' and $X' \cup x$. This fact, inequality (A.2), and the definition of Ch_r imply that if $x' = (d', h') \notin \mathrm{Ch}_r(X')$, then $x' \notin \mathrm{Ch}_r(X' \cup x)$, obtaining the conclusion for the substitute condition in this case.

   (b) Consider hospital h.

      (i) Suppose that $[\tilde{\mathrm{Ch}}_r(w)]_h \geq [\tilde{\mathrm{Ch}}_r(w')]_h$. In this case we follow an argument similar to (but slightly different from) Case (2a): Note that the set $\{d' \in D | (d', h) \in X'\}$ is a subset of $\{d' \in D | (d', h) \in X' \cup x\}$, that is, the set of doctors that are available to hospital h under X' is smaller than under $X' \cup x$. These properties and the definition of Ch_r imply

[27] Analogous conditions called cardinal monotonicity and size monotonicity are introduced by Alkan (2002) and Alkan and Gale (2003) for matching models without contracts.

[28] Note that choice rule $\mathrm{Ch}_r(\cdot)$ allows for the possibility that multiple contracts are signed between the same pair of a region and a doctor. Without this possibility, the choice rule may violate the substitutes condition (Sönmez and Switzer, 2013; Sönmez, 2013). Hatfield and Kominers (2013) explore this issue further.

that if $x' = (d', h) \in X' \setminus \text{Ch}_r(X')$, then $x' \in X' \setminus \text{Ch}_r(X' \cup x)$, obtaining the conclusion for the substitute condition in this case.

(ii) Suppose that $[\tilde{\text{Ch}}_r(w)]_h < [\tilde{\text{Ch}}_r(w')]_h$. This assumption and (2.2) imply $[\tilde{\text{Ch}}_r(w)]_h = w_h$. Thus, by the definition of Ch_r, any contract $(d', h) \in X'$ such that $d' \succ_h \emptyset$ is in $\text{Ch}_r(X')$. Equivalently, if $x' = (d', h) \in X' \setminus \text{Ch}_r(X')$, then $\emptyset \succ_h d'$. Then, again by the definition of Ch_r, it follows that $x' \notin \text{Ch}_r(X' \cup x)$ for any contract $x' = (d', h) \in X' \setminus \text{Ch}_r(X')$. Thus we obtain the conclusion of the substitute condition in this case.

To show that Ch_r satisfies the law of aggregate demand, simply note that $\tilde{\text{Ch}}_r$ is acceptant by assumption. This leads to the desired conclusion. □

A subset X' of $X = D \times H$ is said to be **individually rational** if (1) for any $d \in D$, $|\{(d, h) \in X' | h \in H\}| \leq 1$, and if $(d, h) \in X'$ then $h \succ_d \emptyset$, and (2) for any $r \in R$, $\text{Ch}_r(X') = X' \cap (D \times H_r)$.

Definition 6. A set of contracts $X' \subseteq X$ is a **stable allocation** if

(1) it is individually rational, and
(2) there exists no region $r \in R$, hospital $h \in H_r$, and a doctor $d \in D$ such that $(d, h) \succ_d x$ and $(d, h) \in \text{Ch}_r(X' \cup \{(d, h)\})$, where x is the contract that d receives at X' if any and $\emptyset$ otherwise.

When condition (2) is violated by some (d, h), we say that (d, h) is a **block** of X'. A **doctor-optimal stable allocation** in the matching model with contracts is a stable allocation that every doctor weakly prefers to every other stable allocation (Hatfield and Milgrom, 2005).

Given any individually rational set of contracts X', define a **corresponding matching** $\mu(X')$ in the original model by setting $\mu_d(X') = h$ if and only if $(d, h) \in X'$ and $\mu_d(X') = \emptyset$ if and only if no contract associated with d is in X'. Since each doctor regards any set of contracts with cardinality of at least two as unacceptable, each doctor receives at most one contract at X' and hence $\mu(X')$ is well defined for any individually rational X'.

Proposition 9. *If X' is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

Proof. Suppose that X' is a stable allocation in the associated model with contracts and denote $\mu := \mu(X')$. Individual rationality of μ is obvious from the construction of μ. Suppose that (d, h) is a blocking pair of μ. Denoting $r := r(h)$, by the definition of stability, it suffices to show that the following conditions (A.3) and (A.4) hold if $\mu_d \notin H_r$, and (A.3), (A.4) and (A.5) hold if $\mu_d \in H_r$:

$$|\mu_{H_r}| = q_r, \tag{A.3}$$

$$d' \succ_h d \text{ for all } d' \in \mu_h, \tag{A.4}$$

$$w \succeq_r w', \tag{A.5}$$

where $w = (w_h)_{h \in H_r}$ is defined by $w_{h'} = |\mu_{h'}|$ for all $h' \in H_r$ while $w' = (w'_h)_{h \in H_r}$ is defined by $w'_h = w_h + 1$, $w'_{\mu_d} = w_{\mu_d} - 1$ (if $\mu_d \in H_r$) and $w'_{h'} = w_{h'}$ for all other $h' \in H_r$.

Claim 1. *Conditions (A.3) and (A.4) hold (irrespectively of whether $\mu_d \in H_r$ or not).*

Proof. First note that the assumption that $h \succ_d \mu_d$ implies that $(d, h) \succ_d x$ where x denotes the (possibly empty) contract that d signs under X'. Let $w'' = (w''_h)_{h \in H_r}$ be defined by $w''_h = w_h + 1$ and $w''_{h'} = w_{h'}$ for all other $h' \in H_r$.

(1) Assume by contradiction that condition (A.4) is violated, that is, $d \succ_h d'$ for some $d' \in \mu_h$. First, by consistency of $\tilde{\text{Ch}}_r$, we have $[\tilde{\text{Ch}}_r(w'')]_h \geq [\tilde{\text{Ch}}_r(w)]_h$.[29] That is, weakly more contracts involving h are signed at $X' \cup (d, h)$ than at X'. This property, together with the assumptions that $d \succ_h d'$ and that $(d', h) \in X'$ imply that $(d, h) \in \text{Ch}_r(X' \cup (d, h))$.[30] Thus, together with the above-mentioned property that $(d, h) \succ_d x$, (d, h) is a block of X' in the associated model of matching with contracts, contradicting the assumption that X' is a stable allocation.

(2) Assume by contradiction that condition (A.3) is violated, so that $|\mu_{H_r}| \neq q_r$. Then, since $|\mu_{H_r}| \leq q_r$ by the construction of μ and the assumption that X' is individually rational, it follows that $|\mu_{H_r}| < q_r$. Then $(d, h) \in \text{Ch}_r(X' \cup (d, h))$ because,
   (a) $d \succ_h \emptyset$ by assumption,
   (b) since $\sum_{h \in H_r} w_h = \sum_{h \in H_r} |\mu_h| = |\mu_{H_r}| < q_r$, it follows that $\sum_{h \in H_r} w''_h = \sum_{h \in H_r} w_h + 1 \leq q_r$. Moreover, $|\mu_h| < q_h$ because (d, h) is a blocking pair by

[29]To show this claim, assume for contradiction that $[\tilde{\text{Ch}}_r(w'')]_h < [\tilde{\text{Ch}}_r(w)]_h$. Then, $[\tilde{\text{Ch}}_r(w'')]_h < [\tilde{\text{Ch}}_r(w)]_h \leq w_h$. Moreover, since $w''_{h'} = w_{h'}$ for every $h' \neq h$ by construction of w'', it follows that $[\tilde{\text{Ch}}_r(w'')]_{h'} \leq w''_{h'} = w_{h'}$. Combining these inequalities, we have that $\tilde{\text{Ch}}_r(w'') \leq w$. Also we have $w \leq w''$ by the definition of w'', so it follows that $\tilde{\text{Ch}}_r(w'') \leq w \leq w''$. Thus, by consistency of $\tilde{\text{Ch}}_r$, we obtain $\tilde{\text{Ch}}_r(w'') = \tilde{\text{Ch}}_r(w)$, a contradiction to the assumption $[\tilde{\text{Ch}}_r(w'')]_h < [\tilde{\text{Ch}}_r(w)]_h$.

[30]The proof of this claim is as follows. $\text{Ch}_r(X')$ induces hospital h to select its $[\tilde{\text{Ch}}_r(w)]_h$ most preferred contracts while $\text{Ch}_r(X' \cup (d, h))$ induces h to select a weakly larger number $[\text{Ch}_r(w'')]_h$ of its most preferred contracts. Since (d', h) is selected as one of the $[\tilde{\text{Ch}}_r(w)]_h$ most preferred contracts for h at X' and $d \succ_h d'$, we conclude that (d, h) should be one of the $[\text{Ch}_r(w'')]_h \geq [\tilde{\text{Ch}}_r(w)]_h$ most preferred contracts at $X' \cup (d, h)$, thus selected at $X' \cup (d, h)$.

assumption and (A.4) holds, so $w''_h = |\mu_h| + 1 \leq q_h$. These properties and the assumption that $\tilde{\text{Ch}}_r$ is acceptant imply that $\tilde{\text{Ch}}_r(w'') = w''$. In particular, this implies that all contracts $(d', h) \in X' \cup (d, h)$ such that $d' \succ_h \emptyset$ is chosen at $\text{Ch}_r(X' \cup (d, h))$.

Thus, together with the above-mentioned property that $(d, h) \succ_d x$, (d, h) is a block of X' in the associated model of matching with contract, contradicting the assumption that X' is a stable allocation.

□

To finish the proof of the proposition suppose that $\mu_d \in H_r$ and by contradiction that (A.5) fails, that is, $w' \succ_r w$. Then it should be the case that $[\tilde{\text{Ch}}_r(w'')]_h = w''_h = w_h + 1 = |\mu_h| + 1$.[31] Also we have $|\mu_h| < q_h$ and hence $|\mu_h| + 1 \leq q_h$ and $d \succ_h \emptyset$, so

$$(d, h) \in \text{Ch}_r(X' \cup (d, h)).$$

This relationship, together with the assumption that $h \succ_d \mu_d$, and hence $(d, h) \succ_d x$, is a contradiction to the assumption that X' is stable in the associated model with contracts.

□

Remark 1. Each step of the flexible deferred acceptance algorithm corresponds to a step of the cumulative offer process (Hatfield and Milgrom, 2005), that is, at each step, if doctor d proposes to hospital h in the flexible deferred acceptance algorithm, then at the same step of the cumulative offer process, contract (d, h) is proposed. Moreover, the set of doctors accepted for hospitals at a step of the flexible deferred acceptance algorithm corresponds to the set of contracts held at the corresponding step of the cumulative offer process. Therefore, if X' is the allocation that is produced by the cumulative offer process, then $\mu(X')$ is the matching produced by the flexible deferred acceptance algorithm.

Proof of Proposition 0. By Proposition 8, the choice function of each region satisfies the substitutes condition and the law of aggregate demand in the associate model of matching with contracts. By Hatfield and Milgrom (2005), Hatfield and Kojima (2009), and

[31]To show this claim, assume by contradiction that $[\tilde{\text{Ch}}_r(w'')]_h \leq w_h$. Then, since $w''_{h'} = w_{h'}$ for any $h' \neq h$ by the definition of w'', it follows that $\tilde{\text{Ch}}_r(w'') \leq w \leq w''$. Thus by consistency of $\tilde{\text{Ch}}_r$, we obtain $\tilde{\text{Ch}}_r(w'') = \tilde{\text{Ch}}_r(w)$. But $\tilde{\text{Ch}}_r(w) = w$ because X' is a stable allocation in the associated model of matching with contracts, so $\tilde{\text{Ch}}_r(w'') = w$. This is a contradiction because $w' \leq w''$ and $w' \succ_r w$ while $\tilde{\text{Ch}}_r(w'') \in \arg\max_{\succeq_r} \{w''' | w''' \leq w''\}$.

Hatfield and Kominers (2012), the cumulative offer process with choice functions satisfying these conditions produces a stable allocation and is (group) strategy-proof.[32] The former fact, together with Remark 1 and Proposition 9, implies that the outcome of the flexible deferred acceptance algorithm is a stable matching in the original model. The latter fact and Remark 1 imply that the flexible deferred acceptance mechanism is (group) strategy-proof for doctors. □

Appendix B. Proofs for Section 4

B.1. **Proofs for Section 4.1.**

Proof of Proposition 2. Let μ be a matching and w be defined by $w_{h'} = |\mu_{h'}|$ for each $h' \in H_r$ and w' by $w'_h = w_h + 1$, $w'_{\mu_d} = w_{\mu_d} - 1$, and $w'_{h'} = w_{h'}$ for all $h' \in H_r \setminus \{h, \mu_d\}$. It suffices to show that $w \succeq_r w'$ if and only if $|\mu_h| + 1 - \bar{q}_h > |\mu_{\mu_d}| - 1 - \bar{q}_{\mu_d}$.

Suppose that $|\mu_h|+1-\bar{q}_h > |\mu_{\mu_d}|-1-\bar{q}_{\mu_d}$. This means that $w_h+1-\bar{q}_h > w_{\mu_d}-1-\bar{q}_{\mu_d}$, which is equivalent to either $w_h - \bar{q}_h = w_{\mu_d} - 1 - \bar{q}_{\mu_d}$ or $w_h - \bar{q}_h \geq w_{\mu_d} - \bar{q}_{\mu_d}$. In the former case, obviously $\eta(w) = \eta(w')$, so $w \succeq_r w'$. In the latter case, $\{h'|w'_{h'} - \bar{q}_{h'} < |\mu_{\mu_d}| - \bar{q}_{\mu_d}\} = \{h'|w_{h'} - \bar{q}_{h'} < |\mu_{\mu_d}| - \bar{q}_{\mu_d}\} \cup \{\mu_d\}$, and $w_{h'} = w'_{h'}$ for all $h' \in \{h'|w_{h'} - \bar{q}_{h'} < |\mu_{\mu_d}| - \bar{q}_{\mu_d}\}$. Thus we obtain $w \succ_r w'$.

If $|\mu_h|+1-\bar{q}_h \leq |\mu_{\mu_d}|-1-\bar{q}_{\mu_d}$, then obviously $w' \succ_r w$. This completes the proof. □

Proof of Proposition 3. It is clear that the quasi choice rule $\tilde{\text{Ch}}_r$ defined in (4.1) satisfies the condition (2.1) for substitutability (as well as consistency and acceptance). Thus in the following, we will show that $\tilde{\text{Ch}}_r$ indeed satisfies $\tilde{\text{Ch}}_r(w) \in \arg\max_{\succeq_r}\{x|x \leq w\}$ for each w. Let $w' = \tilde{\text{Ch}}_r(w)$. Assume by contradiction that $w' \notin \arg\max_{\succeq_r}\{x|x \leq w\}$ and consider an arbitrary $w'' \in \arg\max_{\succeq_r}\{x|x \leq w\}$. Then we have $w'' \succ_r w'$, so there exists i such that $\eta_j(w'') = \eta_j(w')$ for every $j < i$ and $\eta_i(w'') > \eta_i(w')$. Consider the following cases.

(1) Suppose $\sum_j \eta_j(w'') > \sum_j \eta_j(w')$. First note that $\sum_j \eta_j(w'') + \sum_h \bar{q}_h = \sum_h w''_h \leq q_r$ because $w'' \in \arg\max_{\succeq_r}\{x|x \leq w\}$. Thus $\sum_h w'_h = \sum_j \eta_j(w') + \sum_h \bar{q}_h < \sum_j \eta_j(w'') + \sum_h \bar{q}_h \leq q_r$. Moreover, the assumption implies that there exists a hospital h such that $w'_h < w''_h \leq \min\{q_h, w_h\}$. These properties contradict the construction of $\tilde{\text{Ch}}_r$.

[32] Aygün and Sönmez (2012) point out that a condition called path-independence (Fleiner, 2003) or irrelevance of rejected contracts (Aygün and Sönmez, 2012) is needed for these conclusions. Aygün and Sönmez (2012) show that the substitutes condition and the law of aggregate demand imply this condition. Since the choice rules in our context satisfy the substitutes condition and the law of aggregate demand, the conclusions go through.

(2) Suppose $\sum_j \eta_j(w'') < \sum_j \eta_j(w')$. First note that $\sum_j \eta_j(w') + \sum_h \bar{q}_h = \sum_h w'_h \leq q_r$ by construction of $\tilde{\mathrm{Ch}}_r$. Thus $\sum_h w''_h = \sum_j \eta_j(w'') + \sum_h \bar{q}_h < \sum_j \eta_j(w') + \sum_h \bar{q}_h \leq q_r$. Moreover, the assumption implies that there exists a hospital h such that $w''_h < w'_h \leq \min\{q_h, w_h\}$. Then, w''' defined by $w'''_h = w''_h + 1$ and $w'''_{h'} = w''_{h'}$ for all $h' \neq h$ satisfies $w''' \leq w$ and $w''' \succ_r w''$, contradicting the assumption that $w'' \in \arg\max_{\succeq_r} \{x | x \leq w\}$.

(3) Suppose that $\sum_j \eta_j(w'') = \sum_j \eta_j(w')$. Then there exists some k such that $\eta_k(w'') < \eta_k(w')$. Let $l = \min\{k | \eta_k(w'') < \eta_k(w')\}$ be the smallest of such indices. Then since $l > i$, we have $\eta_i(w') < \eta_i(w'') \leq \eta_l(w'') < \eta_l(w')$. Thus it should be the case that $\eta_i(w') + 2 \leq \eta_l(w')$. By the construction of $\tilde{\mathrm{Ch}}_r$, this inequality holds only if $w'_h = \min\{q_h, w_h\}$, where h is an arbitrarily chosen hospital such that $w'_h - \bar{q}_h = \eta_i(w')$. Now it should be the case that $w''_h = \min\{q_h, w_h\}$ as well, because otherwise $w'' \notin \arg\max_{\succeq_r} \{x | x \leq w\}$.[33] Thus $w'_h = w''_h$. Now consider the modified vectors of both w' and w'' that delete the entries corresponding to h. All the properties described above hold for these new vectors. Proceeding inductively, we obtain $w'_h = w''_h$ for all h, that is, $w' = w''$. This is a contradiction to the assumption that $w' \notin \arg\max_{\succeq_r} \{x | x \leq w\}$ and $w'' \in \arg\max_{\succeq_r} \{x | x \leq w\}$.

The above cases complete the proof. □

B.2. **Proofs for Section 4.3.** The following result, which applies not only to matching with contract models defined over the set of contracts $D \times H$ but also to those defined over general environments, proves useful.

Lemma 1. *Consider a model of matching with contracts. Fix the set of doctors and regions as well as doctor preferences. Assume that choice rules* $Ch := (Ch_r)_{r \in R}$ *and* $Ch' := (Ch'_r)_{r \in R}$ *satisfy* $Ch'_r(X') \subseteq Ch_r(X')$ *for every subset of contracts* X' *and region* r*. Then the following two statements hold:*

(1) *Each doctor weakly prefers the outcome of the cumulative offer process with respect to* Ch *to the result with respect to* Ch'*. Hence each doctor weakly prefers the doctor-optimal stable allocation under* Ch *to the doctor-optimal stable allocation under* Ch'*.*

[33] The proof that $w'' \notin \arg\max_{\succeq_r} \{x | x \leq w\}$ if $w''_h < \min\{q_h, w_h\}$ is as follows. Suppose that $w''_h < \min\{q_h, w_h\}$. Consider w''' defined by $w'''_h = w''_h + 1$, $w'''_{h'} = w''_{h'} - 1$ for some h' such that $w''_{h'} - \bar{q}_{h'} = \eta_i(w'')$, and $w'''_{h''} = w''_{h''}$ for all $h'' \in H_r \setminus \{h, h'\}$. Then we have $w'''_h - \bar{q}_h = w''_h - \bar{q}_h + 1 \leq w'_h - \bar{q}_h < w''_{h'} - \bar{q}_{h'}$, where the weak inequality follows because $w''_h < \min\{q_h, w_h\} = w'_h$. The strict inequality implies that $w'_h - \bar{q}_h \leq w''_{h'} - 1 - \bar{q}_{h'} = w'''_{h'} - \bar{q}_{h'}$. Hence $w'''_h - \bar{q}_h \leq w'''_{h'} - \bar{q}_{h'}$, which implies $w''' \succ_r w''$.

(2) *The set of contracts that have been offered up to and including the terminal step of the cumulative offer process under Ch is a subset of the corresponding set under Ch'.*

Proof. Let Y_d and Y'_d be the contracts allocated to d by the cumulative offer processes under Ch and Ch', respectively. Also, let $C(t)$ be the set of contracts that have been offered up to and including step t of the cumulative offer process under Ch, and $C'(t)$ be the corresponding set for the cumulative offer process under Ch'. Let T and T' be the terminal steps for the cumulative offer processes under Ch and Ch', respectively. We first prove Part 2 of the lemma, and then show Part 1.

Part 2: Suppose the contrary, i.e., that $C(T) \not\subseteq C'(T')$. Then there exists a step t' such that $C(t) \subseteq C'(T')$ for all $t < t'$ and $C(t') \not\subseteq C'(T')$ holds. That is, t' is the first step such that an application not made in the cumulative offer process under Ch' is made in the cumulative offer process under Ch. Let x be the contract that d offers in this step under Ch. Notice that $Y'_d \succ_d x$. This implies that $Y'_d \neq \emptyset$ and that Y'_d is rejected by r' in some steps of the cumulative offer process under Ch, where r' is the region associated with Y'_d. Let the first of such steps be t''. Since in the cumulative offer process doctors make offers in order of their preferences, $Y'_d \succ_d x$ implies that $t'' < t'$, which in turn implies $C(t'') \subseteq C'(T')$ by the definition of t'.

Now, we show that the set of contracts accepted by r' at step t'' of the cumulative offer process under Ch is a superset of the set of contracts accepted by r' from the application pool $C(t'')$ (which is a subset of $C'(T')$) at step T' of the cumulative offer process under Ch'. To see this, note that if the same application pool $C'(T')$ is given, the set of contracts accepted by r' in the cumulative offer process under Ch is weakly larger than that under Ch' by the assumption that $Ch'_r(X') \subseteq Ch_r(X')$ for all $X' \subseteq X$ and $r \in R$. Since Ch is substitutable, subtracting applications in $C'(T') \setminus C(t'')$ does not shrink the set of contracts accepted by r' within $C(t'')$ at step t'' of the cumulative offer process under Ch, which establishes our claim.

However, this contradicts our earlier conclusion that Y'_d is rejected by r' at step t'' of the cumulative offer process under Ch while she is allocated Y'_d in the cumulative offer process under Ch'. Hence we conclude that $C(T) \subseteq C'(T')$.

Part 1: Now, since in the cumulative offer process each doctor d make offers of contracts in order of her preferences, Y_d is $\emptyset$ or the worst contract for d in the set of contracts associated with d in $C(T)$. Similarly, for each doctor d, Y'_d is $\emptyset$ or the worst contract for d in the set of contracts associated with d in $C'(T')$. If $Y_d \neq \emptyset$, this and $C(T) \subseteq C'(T')$ imply that $Y_d \succeq_d Y'_d$. If $Y_d = \emptyset$, d has applied to all acceptable contracts in the cumulative offer process under Ch. Thus $C(T) \subseteq C'(T')$ implies that she has applied to all acceptable contracts in the algorithm under Ch', too. Let x' be the worst acceptable contract in X for d, and r be a region associated with x'. At this point we already know that Y'_d is either x' or $\emptyset$, and we will show that $Y'_d = \emptyset$ in what follows. Again, $C(T) \subseteq C'(T')$ implies that all applications associated with r in $C(T)$ is in $C'(T')$. In particular, d's application to x' is in $C'(T')$. Since Ch is substitutable, subtracting applications in $C'(T') \setminus C(T)$ does not shrink the set of doctors accepted by r within $C(T)$ at step T of the deferred acceptance, so d not being accepted by r from $C(T)$ at step T of the cumulative offer process under Ch implies that she is not accepted by r from $C'(T')$ in step T' of the process under Ch' either. But since we have shown that d's offer of contract x' to r is in $C'(T')$, this implies that in the cumulative offer process under Ch', x' is rejected by r. Because x' is the worst acceptable contract for d and d's applications are made in order of her preferences, we conclude that $Y'_d = \emptyset$, thus in particular $Y_d \succeq_d Y'_d$.

This shows that each doctor $d \in D$ weakly prefers a contract allocated by the cumulative offer process under Ch to the one under Ch'.

Since the outcome of the cumulative offer process is the doctor-optimal stable allocation, the preceding proof has also shown that the doctor-optimal stable allocation under Ch is weakly more preferred to the doctor-optimal stable allocation under Ch'. □

Lemma 1 is a generalization of a number of existing results. Gale and Sotomayor (1985a,b) establish comparative statics results in one-to-one and many-to-one matching with respect to the extension of an agent's list of her acceptable partners or an addition of an agent to the market, and Crawford (1991) generalizes the results to many-to-many matching. Konishi and Ünver (2006) consider many-to-one matching and obtain a comparative statics result with respect to the changes of hospital capacities.[34] All these changes are special cases of changes in the choice rules, so these results are corollaries of Lemma 1.

Lemma 1 may be of independent interest as the most general comparative statics result known to date. In addition, the lemma implies various results that are directly relevant to the current study of regional caps, such as Propositions 4, 5, 6, and 7 in the main text.

[34]See also Kelso and Crawford (1982) who derive comparative statics results in a matching model with wages, and Hafalir, Yenmez, and Yildirim (2013) and Ehlers, Hafalir, Yenmez, and Yildirim (2014) who study comparative statics in the context of diversity in school choice. Echenique and Yenmez (2015) and Chambers and Yenmez (2017) independently obtain similar results to ours in a framework based on choice functions as primitives.

Proof of Proposition 4. **Part 1:** Let $\mathrm{Ch}^F = (\mathrm{Ch}^F_r)_{r\in R}$ be the choice rule associated with the flexible deferred acceptance as defined earlier, that is, for each region $r \in R$ and subset of contracts $X' \subseteq X = D \times H$, the chosen set of contracts $\mathrm{Ch}^F_r(X')$ is defined by

$$\mathrm{Ch}^F_r(X') = \bigcup_{h\in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D|(d',h) \in X', d' \succeq_h d\}| \le (\tilde{\mathrm{Ch}}_r(w(X')))_h\right\},$$

where $\tilde{\mathrm{Ch}}_r$ corresponds to a Rawlsian regional preference of region r and $w(X') = (w_h(X'))_{h\in H_r}$ is the vector such that $w_h(X') = |\{(d,h) \in X'|d \succ_h \emptyset\}|$ (this is a special case of the choice rule (A.1)).

Moreover, consider choice rules $Ch^D = (Ch^D_r)_{r\in R}$ and $Ch^J = (Ch^J_r)_{r\in R}$ such that, for each X' and r,

$$\mathrm{Ch}^D_r(X') = \bigcup_{h\in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D|(d',h) \in X', d' \succeq_h d\}| \le q_h\right\},$$
$$\mathrm{Ch}^J_r(X') = \bigcup_{h\in H_r} \left\{(d,h) \in X' \;\middle|\; |\{d' \in D|(d',h) \in X', d' \succeq_h d\}| \le \bar{q}_h\right\}.$$

Clearly, both Ch^D and Ch^J satisfy the substitute condition and the law of aggregate demand. Moreover, the matchings corresponding to the results of the cumulative offer processes under Ch^D and Ch^J are identical to the results of the deferred acceptance algorithm and the JRMP mechanism, respectively. Because $\min\{\bar{q}_h, w_h\} \le (\tilde{\mathrm{Ch}}_r(w(X')))_h \le q_h$ for all $h \in H_r$ and X', by inspection of the above definitions of the choice rules we obtain $\mathrm{Ch}^J_r(X') \subseteq \mathrm{Ch}^F_r(X') \subseteq \mathrm{Ch}^D_r(X')$ for all X' and r. Thus the desired conclusion follows by Part 1 of Lemma 1.

Part 2: This is a direct corollary of Part 1 and the fact that none of the algorithms considered here matches a doctor to an unacceptable hospital. □

Proof of Proposition 5. Let $\mathrm{Ch} = (\mathrm{Ch}_r)_{r\in R}$ and $\mathrm{Ch}' = (\mathrm{Ch}'_r)_{r\in R}$ be the choice rules associated with the flexible deferred acceptance mechanisms (as defined in the proof of Proposition 4) with respect to $(q_r)_{r\in R}$ and $(q'_r)_{r\in R}$, respectively.

Part 1: Because $q'_r \le q_r$ for each $r \in R$, the definition of these choice rules implies $\mathrm{Ch}'_r(X') \subseteq \mathrm{Ch}_r(X')$ for all X' and r. Hence the desired conclusion follows by Part 1 of Lemma 1.

Part 2: Since $\mathrm{Ch}'_r(X') \subseteq \mathrm{Ch}_r(X')$ for all X' and r as mentioned in the proof of Part 1, Part 2 of Lemma 1 implies that $C(T) \subseteq C'(T')$, where C, T, C', and T' are as defined in Part 2 of the lemma. Note that the sets of contracts allocated to hospitals in r at the conclusions of the cumulative offer processes under Ch and Ch' are given as r's choice from contracts associated with r in $C(T)$ and $C'(T')$, respectively. Because the choice rules satisfy the law of aggregate demand and the set-inclusion relationship $C(T) \subseteq C'(T')$ holds, for any r such that $q_r = q'_r$, the number of doctors matched in r under a matching produced by the flexible deferred acceptance mechanism under regional caps $(q'_r)_{r\in R}$ is weakly larger than in the matching under $(q_r)_{r\in R}$, completing the proof. □

Proof of Proposition 6. Let $\text{Ch} = (\text{Ch}_r)_{r\in R}$ and $\text{Ch}^{'} = (\text{Ch}^{'}_r)_{r\in R}$ be the choice rules associated with the JRMP mechanisms (as defined in the proof of Proposition 4) with respect to $(\bar{q}_h)_{h\in H}$ and $(\bar{q}'_h)_{h\in H}$, respectively.

Part 1: Because $\bar{q}'_h \leq \bar{q}_h$ for each $h \in H$, the definition of these choice rules implies $\text{Ch}'_r(X') \subseteq \text{Ch}_r(X')$ for all X' and r. Hence the desired conclusion follows by Part 1 of Lemma 1.

Part 2: Since $\text{Ch}'_r(X') \subseteq \text{Ch}_r(X')$ for all X' and r as mentioned in the proof of Part 1, Part 2 of Lemma 1 implies that $C(T) \subseteq C'(T')$, where C, T, C', and T' are as defined in Part 2 of Lemma 1. Note that the matchings for h at the conclusions of the cumulative offer processes under Ch and Ch' are given as h's most preferred acceptable doctors up to $\bar{q}_h = \bar{q}'_h$ from contracts associated with h in $C(T)$ and $C'(T')$, respectively. Thus the set-inclusion relationship $C(T) \subseteq C'(T')$ implies both of the statements of Part 2. □

Proof of Proposition 7. **Part 1:** First, by Part 2 of Lemma 1 and the proof of Proposition 4, the set of contracts that have been offered up to and including the terminal step under the deferred acceptance mechanism is a subset of the one under the flexible deferred acceptance mechanism. Second, by the construction of the flexible deferred acceptance algorithm, and the assumption that hospital h's target capacity is not filled, under the flexible deferred acceptance mechanism h is matched to every doctor who is acceptable to h and who applied to h in some step of the algorithm. These two facts imply the conclusion.

Part 2: First, by Part 2 of Lemma 1 and the proof of Proposition 4, the set of contracts that have been offered up to and including the terminal step under the deferred acceptance mechanism is a subset of the one under the flexible deferred acceptance mechanism. Second, by the construction of the flexible deferred acceptance algorithm, and the assumption that region r's regional cap is not filled, under the flexible deferred acceptance mechanism any hospital h in region r is matched to every doctor who is acceptable and who is among the most preferred q_h doctors who applied to h in some step of the algorithm. These two facts imply the conclusion. □

Appendix C. Further Statistics on Heterogeneity of Capacities

Across prefectures in Japan, the mean and the median of the ratios of the maximum and the minimum hospital capacities are 20.98 and 19, respectively (see Figure 3). The mean and the median of the Gini coefficients across prefectures are both 0.48, showing that the heterogeneity of hospital capacities is quite significant.[35] Capacities differ substantially in the school choice context as well; see Table 1 that reports data from Boston Public Schools (2013). The ratios of the maximum and the minimum of school capacities range from 1.80 to 16.19 with the median of 5.28, and all the Gini coefficients are no less than 0.10.

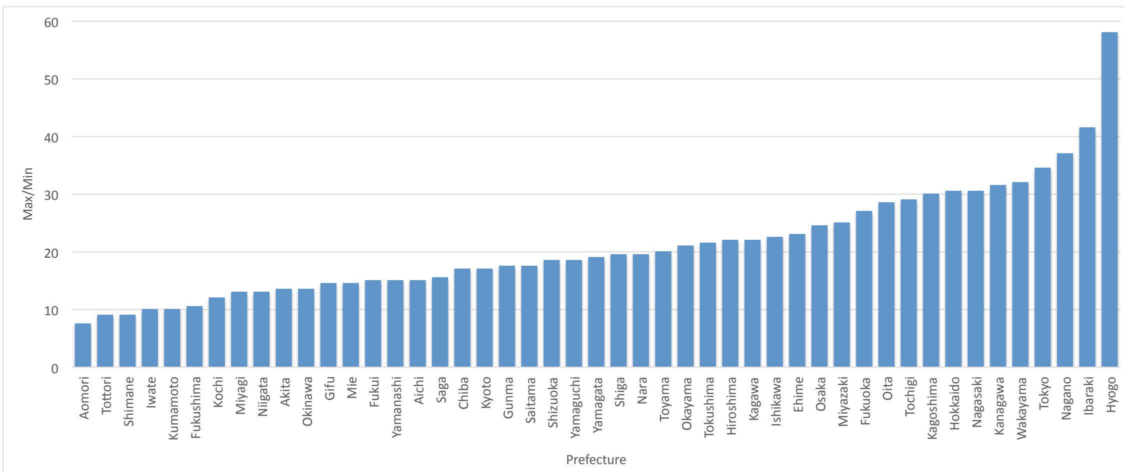

Figure 3. The ratios of the maximum and the minimum hospital capacities across prefectures.

Category	Maximum	Minimum	Max/Min	Gini Coefficient
Early Learning Center	234	109	2.15	0.10
Elementary School	871	165	5.28	0.25
Exam School	2323	1291	1.80	0.13
High School	1457	90	16.19	0.32
Kindergarten–Eight	960	132	7.27	0.27
Middle School	760	288	2.64	0.15
Special/Alternative Education	297	25	11.88	0.31

Table 1. Heterogeneity in size by school category.

[35]The data are taken from Japan Residency Matching Program (2013)

Appendix D. Allocating Target Capacities

A problem related to, but distinct from, our discussion in Section 4.2 is how to allocate target capacities among hospitals in a region, within the simple, Rawlsian framework of Kamada and Kojima (2015). We will not try to provide a final answer to the normative question of how to do so for several reasons. First, there may be different ways to specify the quasi choice rule even given the same target capacity profile, as we have seen in this section, and in fact there may be reasonable quasi choice rules that do not even presuppose the existence of target capacities. Second, even if we fix a quasi choice rule, the relation between target capacities and the desirability of the resulting outcome is ambiguous. An example in Kamada and Kojima (2015) shows that the effect on hospital welfare is ambiguous.[36]

Despite these reservations, hospitals may still find having higher targets intuitively appealing in practice, so the problem seems to be practically important. Motivated by this observation, we present several methods to allocate target capacities that seem to be reasonable.

To do so, our starting point is to point out that the problem of allocating target capacities is similar to the celebrated "bankruptcy problem" (see Thomson (2003)). This is a useful association in the sense that, in the bankruptcy problem, there are known analyses (e.g., axiomatic characterizations) of various allocation rules, which can be utilized to judge which rule is appropriate for a given application.

To make this association, recall that in the standard bankruptcy problem, there is a divisible asset and agents whose claims sum up to (weakly) more than the amount of the available asset. Our problem is a discrete analogue of the bankruptcy problem. The regional cap q_r is an asset, hospitals in region r are agents, and physical capacity q_h is the claim of hospital h. Just as in the bankruptcy problem, the sum of the physical capacities may exceed the available regional cap, so the target capacity profile $(\bar{q}_h)_{h\in H_r}$ needs to be decided subject to the constraint $\sum_{h\in H_r} \bar{q}_h \leq q_r$.

This association suggests adaptations of well-known solutions in the bankruptcy problem to our problem, with the modification due to the fact that both the asset and the claims are discrete in our problem. The following are a few examples (in the following, we assume $\sum_{h\in H_r} q_h \geq q_r$; otherwise set $\bar{q}_h = q_h$ for all h).

[36]Example 9 in Kamada and Kojima (2015) shows that the effect on hospital welfare is ambiguous. In fact, Example 15 in Kamada and Kojima (2015) shows that the same conclusion holds even if hospitals or doctors have homogeneous preferences, which are strong restrictions that often lead to strong conclusions in matching.

(1) "Constrained Equal Awards Rule": This solution allocates the targets as equally as possible except that, for any hospital, it does not allocate a target larger than the capacity. This rule is called the **constrained equal awards rule** in the literature on the bankruptcy problem. In our context, this solution should be modified because all the targets need to be integers. Formally, a constrained equal awards rule in our context can be defined as follows:
   (a) Find λ that satisfies $\sum_{h \in H_r} \min\{\lambda, q_h\} = q_r$.
   (b) For each $h \in H_r$, if $\lambda > q_h$, then set $\bar{q}_h = q_h$. Otherwise, set $\bar{q}_h$ to be either $\lfloor \lambda \rfloor$ (the largest integer no larger than λ) or $\lfloor \lambda \rfloor + 1$, subject to the constraint that $\sum_{h \in H_r} \bar{q}_h = q_r$.

   The rule to decide which hospital receives $\lfloor \lambda \rfloor$ or $\lfloor \lambda \rfloor + 1$ is arbitrary: For any decision rule, the resulting target profiles satisfy conditions assumed in Kamada and Kojima (2015). The decision can also use randomization, which may help achieve ex ante fairness.

(2) "Constrained Equal Losses Rule": This solution allocates the targets in such a way that it equates losses (that is, differences between the capacities and the targets) as much as possible, except that none of the allocated targets can be strictly smaller than zero. This rule is called the **constrained equal losses rule** in the literature on the bankruptcy problem. As in the case of the constrained equal awards rule, this solution should be modified because all the targets need to be integers. Formally, a constrained equal losses rule in our context can be defined as follows:
   (a) Find λ that satisfies $\sum_{h \in H_r} \max\{q_h - \lambda, 0\} = q_r$.
   (b) For each $h \in H_r$, if $q_h - \lambda < 0$, then set $\bar{q}_h = 0$. Otherwise, set $\bar{q}_h$ to be either $q_h - \lfloor \lambda \rfloor$ or $q_h - \lfloor \lambda \rfloor - 1$, subject to the constraint that $\sum_{h \in H_r} \bar{q}_h = q_r$.

   As in the constrained equal awards rule, the rule to decide which hospital receives $q_h - \lfloor \lambda \rfloor$ or $q_h - \lfloor \lambda \rfloor - 1$ is arbitrary: For any decision rule, the resulting target profiles satisfy conditions assumed in Kamada and Kojima (2015). The decision can also use randomization, which may help achieve ex ante fairness.

(3) "Proportional Rule": This solution allocates the targets in a manner that is as proportional as possible to the hospitals' capacities. This rule is called the **proportional rule** in the literature on the bankruptcy problem. As in the case of the previous rules, this solution should be modified because all the targets need to be integers. Formally, a proportional rule in our context can be defined as follows:
   (a) Find λ that satisfies $\sum_{h \in H_r} \lambda q_h = q_r$.

(b) For each $h \in H_r$, set $\bar{q}_h$ be either $\lfloor \lambda q_h \rfloor$ or $\lfloor \lambda q_h \rfloor + 1$, subject to the constraint that $\sum_{h \in H_r} \bar{q}_h = q_r$.

As in the previous cases, the rule to decide which a hospital receives $\lfloor \lambda q_h \rfloor$ or $\lfloor \lambda q_h \rfloor + 1$ is arbitrary: For any decision rule, the resulting target profiles satisfy conditions assumed in Kamada and Kojima (2015). The decision can also use randomization, which may help achieve ex ante fairness.

The proportional rule seems to be fairly appealing in practice. This rule is used in Japanese residency match and Chinese graduate school admission, for example.