Online Appendix

A Comment on "Asynchronous Choice in Repeated Coordination Games"

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December 12, 2025

Appendix B provides formal statements and proofs of the results mentioned in Remarks 2, 3 and 4. Appendix C contains calculations for the proof of Proposition 2. Appendix D provides detailed discussions on what we call the "period-1 onward" model.

B Formal Results for Remarks

B.1 Formal Result for Remark 2: Characterization of the Payoff Structure

As we mentioned in Remark 2 of the main text, the example used to prove Proposition 1 features an extreme payoff structure. Here, we provide a characterization of the payoff structure for which action profiles other than s^* are played for the first K periods.

Proposition 6. *Let $K \in \mathbb{N} \setminus \{1\}$ and $a \in [0, 1)$. Then, $\frac{K-1}{K} \leq a$ if and only if there are a pure coordination game G with $\alpha = a$ and $\overline{\delta} < 1$ such that, for any $\delta \in (\overline{\delta}, 1)$, the asynchronously repeated game $\mathrm{AR}(G, \delta)$ has a perfect equilibrium in which the players do not play s^* for the first K periods.*

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Note that, for there to be a perfect equilibrium in which s^* is not chosen for the first K periods, the value a needs to be large when K is large.[1] The intuition is that if K is large, then there are many periods in which the players do not receive the best feasible payoff. This means that each player i has a stronger incentive to deviate to s_i^*, which guarantees that in the next period she receives the best feasible payoff (by Theorem 1). To prevent this deviation, the payoff that she receives during the first K periods needs to be sufficiently high, which requires a to be high.

The condition in the case of $K = 2$, $\frac{1}{2} \leq a$, may not seem consistent with the condition $\delta \leq \frac{1+c}{2}$ that we stated in explaining the intuition for Proposition 1 in the main text.[2] This is because, for the latter condition to hold for a sufficiently large $\delta < 1$, we need $\frac{1+c}{2} \geq 1$, which is equivalent to $c \geq 1$, or $\alpha = \frac{1-(-c)}{2-(-c)} = \frac{1+c}{2+c} \geq \frac{2}{3}$. This discrepancy occurs because in proving Proposition 6, we have full freedom in choosing the payoff structure (among pure coordination games), while we only had one parameter c to vary the payoff structure when we explained the intuition for Proposition 1.

We also note that, if we replaced "pure coordination game" in the statement of Proposition 6 with "game of common interest," then the "only if" part would hold as a corollary of Proposition 6. However, the "if" part would fail because, even if $a < \frac{K-1}{K}$ holds, we can construct the following perfect equilibrium due to the folk theorem for asynchronously repeated games (Dutta, 1995; Yoon, 2001): Players play actions other than s^* for the first K periods and then play s^* thereafter. If some player deviates during those K periods, they play a perfect equilibrium in the continuation game that induces a payoff profile that is bounded away from u^* (such a payoff profile is Pareto dominated by u^*). We can find such a perfect equilibrium in the continuation game due to the folk theorem.

Proof of Proposition 6.

The "Only If" Part:

[1] Note that we exclude $K = 1$ in the statement of the result because there is no perfect equilibrium in which an action profile other than s^* is chosen exactly one period: If such an action profile $\hat{s}$ is chosen, then it has to be at period 0. Note that player 1's payoff in period must be at least as good as the payoff she receives in period 1, which is u^*. This implies that the payoff from action profile $\hat{s}$ is u^*, contradicting our assumption that s^* is the only action profile that induces a payoff of u^*. If we were to include $K = 1$ in our statement of the result, we would change the condition to $\frac{\max\{2,K\}-1}{\max\{2,K\}} \leq a$.

[2] Recall that $\delta > \frac{1+c}{2}$ was the condition for player 2 to have an incentive to deviate from a strategy profile in which s^* is not played for some periods.

	s^1	s^2	s^3
s^1	$1, 1$	a, a	$0, 0$
s^2	$0, 0$	a, a	a, a
s^3	$0, 0$	$0, 0$	a, a

	s^1	s^2	s^3
s^1	$1, 1$	$0, 0$	$0, 0$
s^2	a, a	a, a	$0, 0$
s^3	$0, 0$	a, a	a, a
s^4	$0, 0$	$0, 0$	a, a

Table 5: The examples for $K = 4$ (Left) and $K = 5$ (Right). Note that $(s_1^*, s_2^*) = (s^1, s^1)$.

<u>Case 1.</u> We start with the case in which K is even. Let player i's action set be $S_i = \{s^k\}_{k=1}^m$, where $m = \frac{K}{2} + 1$. Define player i's utility function as

$$u_i(s^k, s^\ell) \equiv \begin{cases} 1 & \text{if } k = \ell = 1 \\ a & \text{if } \ell - k \in \{0, 1\} \text{ and } \ell \neq 1 \\ 0 & \text{otherwise} \end{cases} .$$

Note that $(s_1^*, s_2^*) = (s^1, s^1)$. The left panel of Table 5 presents the case of $K = 4$.

Consider the following strategy profile f^*:

$$f_1^*(h) \equiv \begin{cases} s^m & \text{if } h = e \\ s^{k-1} & \text{if } h \neq e \text{ and } s_2(t-1) = s^k \text{ for some } k \in \{2, \ldots, m\} \\ s^1 & \text{if } h \neq e \text{ and } s_2(t-1) = s^1 \end{cases} ; \text{ and}$$

$$f_2^*(h) \equiv \begin{cases} s^m & \text{if } h = e \\ s^k & \text{if } h \neq e \text{ and } s_1(t-1) = s^k \text{ for some } k \in \{1, \ldots, m\} \end{cases} .$$

Note that, on the equilibrium path of play, (s^m, s^m) is chosen in period 0, and then the sequence of action profiles afterward is

$$(s^{m-1}, s^m), \quad (s^{m-1}, s^{m-1}), \quad (s^{m-2}, s^{m-1}), \quad \ldots, \quad (s^1, s^1).$$

The action profile stays at (s^1, s^1) thereafter. Hence, the action profile played is not $(s^1, s^1) = s^*$ for the first K periods.

We show that f^* is a perfect equilibrium when $\delta \in (0, 1)$ is sufficiently large. First, note that given any history and the opponent's strategy f_{-i}^*, player i does not strictly improve her payoff by playing an action that results in an instantaneous payoff of 0.

To see this, notice that i's payoff from such a deviation is at most $(1-\delta)\cdot 0+\delta\cdot 1$ while the payoff from following f_i^* is at least $(1-\delta^{2(m-1)})\cdot a+\delta^{2(m-1)}\cdot 1$. Hence, following f_i^* is weakly better if $(1-\delta^{2(m-1)})\cdot a+\delta^{2(m-1)}\cdot 1 \geq (1-\delta)\cdot 0+\delta\cdot 1$, or

$$a \geq \frac{\delta-\delta^{2(m-1)}}{1-\delta^{2(m-1)}} = \frac{\delta+\delta^2+\cdots+\delta^{2m-3}}{1+\delta+\cdots+\delta^{2m-3}} = 1-\frac{1}{1+\delta+\cdots+\delta^{2m-3}}.$$

Since the rightmost side is increasing in δ and tends to $\frac{2m-3}{2(m-1)}$ as $\delta \to 1$, it follows that no player strictly gains by deviating to an action that results in an instantaneous payoff of 0 because we assumed $a \geq \frac{K-1}{K}$ and $\delta < 1$, where $\frac{K-1}{K} = \frac{2m-3}{2(m-1)}$ holds by the definition of m.

Second, given any history at time $t \geq 1$, the opponent's strategy f_{-i}^* and $k \in \{1,\ldots,m-1\}$, if two actions s^k and s^{k+1} of the moving player i result in instantaneous payoffs that are not 0, then i would be better off choosing s^k, which is to follow f_i^*. To see this, note that the payoff from playing s^k is $(1-\delta^{\ell_k})\cdot a+\delta^{\ell_k}\cdot 1$ for some $\ell_k \in \mathbb{N}\cup\{0\}$, and given this, the payoff from playing s^{k+1} is at most $(1-\delta^{\ell_k+2})\cdot a+\delta^{\ell_k+2}\cdot 1$. The former is greater than the latter because $a<1$.

What remains is to show that, at time 0, player 1's payoff from playing s^m, which is to follow f_1^*, is no less than his payoff from playing s^{m-1}.[3] To see this, note that both payoffs are $(1-\delta^{2(m-1)})\cdot a+\delta^{2(m-1)}\cdot 1$. Hence, player 1 does not have a strict incentive to deviate.

Case 2. Next, we consider the case in which K is odd. Let the players' action sets be $S_1 = \{s^k\}_{k=1}^{m+1}$ and $S_2 = \{s^k\}_{k=1}^{m}$, where $m = \frac{K+1}{2}$. Note that $m \geq 2$ holds because $K \neq 1$. Define player i's utility function as

$$u_i(s^k, s^\ell) \equiv \begin{cases} 1 & \text{if } k=\ell=1 \\ a & \text{if } k-\ell \in \{0,1\} \text{ and } k \neq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Note that $(s_1^*, s_2^*) = (s^1, s^1)$. The right panel of Table 5 presents the case of $K=5$.

[3]Note that we have already checked player 2's incentive and player 1's other deviations at time 0.

Consider the following strategy profile f^*:

$$f_1^*(h) \equiv \begin{cases} s^{m+1} & \text{if } h = e \\ s^k & \text{if } h \neq e \text{ and } s_2(t-1) = s^k \text{ for some } k \in \{1, \dots, m\} \end{cases}; \text{ and}$$

$$f_2^*(h) \equiv \begin{cases} s^m & \text{if } h = e \\ s^{k-1} & \text{if } h \neq e \text{ and } s_1(t-1) = s^k \text{ for some } k \in \{2, \dots, m\} \\ s^1 & \text{if } h \neq e \text{ and } s_1(t-1) = s^1 \end{cases}.$$

Note that, on the equilibrium path of play, (s^{m+1}, s^m) is chosen in period 0, and then the sequence of action profiles afterward is

$$(s^m, s^m), \quad (s^m, s^{m-1}), \quad (s^{m-1}, s^{m-1}), \quad \dots, \quad (s^1, s^1).$$

The action profile stays at (s^1, s^1) thereafter. Hence, the action profile played is not $(s^1, s^1) = s^*$ for the first K periods.

We show that f^* is a perfect equilibrium when $\delta \in (0, 1)$ is sufficiently large. First, note that given any history and the opponent's strategy f_{-i}^*, player i does not strictly improve her payoff by playing an action that results in an instantaneous payoff of 0. To see this, notice that i's payoff from such a deviation is at most $(1-\delta)\cdot 0 + \delta \cdot 1$ while the payoff from following f_i^* is at least $(1-\delta^{2m-1}) \cdot a + \delta^{2m-1} \cdot 1$. Hence, following f_i^* is weakly better if $(1-\delta^{2m-1}) \cdot a + \delta^{2m-1} \cdot 1 \geq (1-\delta) \cdot 0 + \delta \cdot 1$, or

$$a \geq \frac{\delta - \delta^{2m-1}}{1 - \delta^{2m-1}} = \frac{\delta + \delta^2 + \cdots + \delta^{2m-2}}{1 + \delta + \cdots + \delta^{2m-2}} = 1 - \frac{1}{1 + \delta + \cdots + \delta^{2m-2}}.$$

Since the rightmost side is increasing in δ and tends to $\frac{2m-2}{2m-1}$ as $\delta \to 1$, it follows that no player strictly gains by deviating to an action that results in an instantaneous payoff of 0 because we assumed $a \geq \frac{K-1}{K}$ and $\delta < 1$, where $\frac{K-1}{K} = \frac{2m-2}{2m-1}$ holds by the definition of m.

Second, given any history at time $t \geq 1$, the opponent's strategy f_{-i}^* and $k \in \{1, \dots, m-1\}$, if two actions s^k and s^{k+1} of the moving player i result in instantaneous payoffs that are not 0, then i would be better off choosing s^k, which is to follow f_i^*. To see this, note that the payoff from playing s^k is $(1-\delta^{\ell_k}) \cdot a + \delta^{\ell_k} \cdot 1$ for some $\ell_k \in \mathbb{N} \cup \{0\}$, and given this, the payoff from playing s^{k+1} is at most $(1-\delta^{\ell_k+2}) \cdot a + \delta^{\ell_k+2} \cdot 1$. The former is greater than the latter because $a < 1$.

What remains is to show that, at time 0, player 1's payoff from playing s^{m+1}, which is to follow f_1^*, is no less than his payoff from playing s^m.[4] To see this, note that both payoffs are $(1-\delta^{2m-1}) \cdot a + \delta^{2m-1} \cdot 1$.

<u>The "If" Part:</u> Suppose that there are a pure coordination game G with $\alpha = a$ and $\bar{\delta} < 1$ such that, for any $\delta \in (\bar{\delta}, 1)$, the asynchronously repeated game $\mathrm{AR}(G, \delta)$ has a perfect equilibrium in which s^* is not played for the first K periods. Fix such G and a perfect equilibrium f^*.

By assumption, player 1's continuation payoff under f^* at the empty history is at most $(1-\delta^K) \cdot a + \delta^K \cdot 1$. In contrast, if she deviates, her continuation payoff is at least $(1-\delta) \cdot 0 + \delta \cdot 1$ by Theorem 1. Since f^* is a perfect equilibrium, the former is no less than the latter, that is,

$$(1-\delta^K) \cdot a + \delta^K \cdot 1 \geq (1-\delta) \cdot 0 + \delta \cdot 1.$$

Thus,

$$a \geq \frac{\delta - \delta^K}{1-\delta^K} = 1 - \frac{1}{1+\delta+\cdots+\delta^{K-1}}.$$

Since the rightmost side is increasing in δ and tends to $\frac{K-1}{K}$ as $\delta \to 1$, it follows that

$$a \geq \frac{K-1}{K}.$$

□

B.2 Formal Result for Remark 3: Eventual Convergence to the Best Action Profile

As we mentioned in Remark 3 of the main text, our results so far concern whether there is a period in which s^* is not chosen. In our example given by Table 2, under some discount factors, there exist perfect equilibria in which s^* is never chosen. The following result provides a necessary and sufficient condition on the discount factor under which s^* is never chosen.

Proposition 7. *Let $a \in [0, 1)$. Then, $\delta \in (a, 1)$ if and only if, for any pure coordination game G with $\alpha = a$ and any perfect equilibrium of the asynchronously repeated*

[4] Again, we have already checked player 2's incentive and player 1's other deviations at time 0.

game $\mathrm{AR}(G, \delta)$*, there is* $t' < \infty$ *such that the players always play* s^* *after period* t'*.*

The intuition for the result is as follows. If the discount factor is sufficiently high, then the players are willing to accept a short-run loss to gain a long-run benefit in the future. If, in contrast, the discount factor is low, then they are not willing to do so. The proposition shows that the parameter α of the payoff structure characterizes the cutoff value of the discount factor where the incentives change.

We note that, if we replaced "pure coordination game" in the statement of Proposition 7 with "game of common interest, then the "if" part would hold as a corollary of Proposition 7. However, the "only if" part fails: in the strategy profile f^{***} in the proof of Proposition 3, s^* is never taken, and it is a perfect equilibrium when the discount factor is sufficiently large.[5]

Proof of Proposition 7. For the "if" part, suppose to the contrary that $a \geq \delta$. We consider the stage game depicted in Table 6. Consider the following strategy profile:

$$f_i^*(h) \equiv \begin{cases} s^1 & \text{if } h \neq e \text{ and } s_{-i}(t-1) = s^1 \\ s^2 & \text{if } h = e \text{ or } s_{-i}(t-1) = s^2 \end{cases}.$$

Note that, on the equilibrium path of play, (s^2, s^2) is chosen in period 0, and the action profile stays at (s^2, s^2) thereafter. Hence, the action profile played is never $(s^1, s^1) = s^*$.

At the empty history e or at any time $t \geq 1$ at which $s_{-i}(t-1) = s^2$, player i's payoff from following f_i^* against f_{-i}^* is a. If she deviates, then her payoff is $(1-\delta) \cdot 0 + \delta \cdot 1 = \delta \leq a$. Thus, each player i does not have a strict incentive to deviate. At any time $t \geq 1$ at which $s_{-i}(t-1) = s^1$, it is player i's best response to follow f_i^* as it yields a continuation payoff of 1, the maximum possible payoff. In this perfect equilibrium, the players never take s^* on the equilibrium path.

For the "only if" part, let $\delta > a$, and suppose to the contradiction that there are a pure coordination game G with $\alpha = a$ and a perfect equilibrium in which the action profile s^* is never taken in the asynchronously repeated game $\mathrm{AR}(G, \delta)$. For

[5] Note that we cannot directly invoke the folk theorem for asynchronously repeated games (Dutta, 1995; Yoon, 2001) to show that the "only if" direction fails (as we did in the case of Proposition 6). The reason is that, even if an inefficient payoff profile can be the limit of perfect equilibrium payoff profiles as $\delta \to 1$, the sequence of perfect equilibria may involve increasingly many periods of initial inefficient action profiles, followed by a perpetual repetition of s^*.

	s^1	s^2
s^1	$1, 1$	$0, 0$
s^2	$0, 0$	a, a

Table 6: The stage game for the "if" part. Note that $(s_1^*, s_2^*) = (s^1, s^1)$.

each player and at any history, her equilibrium payoff is at most a. If she deviates by taking s^1, then she can guarantee a payoff of at least $(1-\delta)\cdot 0 + \delta \cdot 1 = \delta > a$ by Theorem 1, a contradiction. $\square$

B.3 Formal Result for Remark 4: n-Player Extension of Proposition 2

As discussed in Remark 4 of the main text, the condition on α in Proposition 2 can be extended to the case of n players.

Proposition 8. *Consider an n-player model as described in Remark 5. Fix $\delta \in (0,1)$. Then, $a < \frac{(1-\delta)\delta^{n-1}}{1-\delta^n}$ if and only if, for every pure coordination game with $\alpha = a$, the optimality result holds, i.e., players play s^* at every period in any perfect equilibrium of the asynchronously repeated game with discount factor δ.*

We omit the formal proof as it is a straightforward generalization of the proof of Proposition 2. For the "only if" direction, we construct a stage game in which each player has two actions s^1 and s^2, where

$$u_i(s) \equiv \begin{cases} 1 & \text{if } s = (s^1, \dots, s^1) \\ a & \text{if } s = (s^1, \dots, s^1, s^2, \dots, s^2) \text{ or } s = (s^2, \dots, s^2) \\ 0 & \text{otherwise} \end{cases},$$

and a perfect equilibrium in which player i choose s^1 if and only if all players $1, \dots, i-1$ are currently choosing s^1. The proof for the "if" direction relies on the n-player version of Theorem 1, which Lagunoff and Matsui (1995) provide.

Remark 4 also mentions an extension of Corollary 1, which can be obtained as a corollary of Proposition 8 above. This corollary is presented below.

Corollary 2. *Consider an n-player model as described in Remark 5. Fix a pure coordination game and suppose that $\alpha < \frac{1}{n}$. There exists $\overline{\delta} \in (0,1)$ such that for* *all $\delta \in (\overline{\delta}, 1)$, the optimality result holds for the asynchronously repeated game with discount factor δ.*

C Calculation for Proof of Proposition 2

We show that f^* is a perfect equilibrium when $a \geq \frac{\delta}{1+\delta}$. Consider player 1. At $t = 0$, if player 1 follows the strategy f_1^* to play s^2, then his payoff is $(1-\delta)(a+\delta a)+\delta^2$. If he deviates by taking s^1 in period 0, then his payoff is $(1-\delta)(1+\delta)a+\delta^2$. These two payoffs are equal with each other.

At $t \geq 1$, by Theorem 1, it suffices to consider a history $h \in H_1 \setminus \{e\}$ in which player 2 has taken s^2 in period $t-1$. If player 1 takes s^1, then his payoff is $(1-\delta)a+\delta$. If he deviates by taking s^2 at h, then his payoff is

$$(1-\delta)\left(a+\delta \cdot a+\delta^2 \cdot a+\frac{\delta^3}{1-\delta} \cdot 1\right).$$

The former payoff is no less than the latter when $a \geq \frac{\delta}{1+\delta}$.

Consider player 2. At period $t = 0$, if player 2 follows the strategy f_2^* to play s^2, then her payoff is $(1-\delta)(a+\delta a)+\delta^2$. If she deviates by taking s^1, then her payoff is $(1-\delta) \cdot 0+\delta$. Thus, player 2 does not have a strict incentive to deviate in period 0 if

$$(1-\delta)(a+\delta a)+\delta^2 \geq (1-\delta) \cdot 0+\delta.$$

This holds when $a \geq \frac{\delta}{1+\delta}$.

Suppose $t \geq 2$. By Theorem 1, it suffices to consider the case where player 1 has taken s^2 in period $t-1$. If player 2 follows f_2^* to play s^2, then her payoff is $(1-\delta)(a+\delta a)+\delta^2$. If she deviates by taking s^1, then her payoff is $(1-\delta) \cdot 0+\delta$. As in the preceding analysis, the former payoff is no less than the latter if $a \geq \frac{\delta}{1+\delta}$.

In sum, when $a \geq \frac{\delta}{1+\delta}$, the strategy profile f^* is a perfect equilibrium.

D The "Period-1 Onward" Model

As mentioned in footnote 1 of the main text, our formulation of payoffs is different from that of Lagunoff and Matsui (1997). We detail our interpretation of Lagunoff and Matsui (1997)'s formulation here.

In Lagunoff and Matsui (1997), given a strategy profile $f = (f_1, f_2)$, a history $h^t \in H$, and $\tau \in \mathbb{N} \cup \{0\}$, denote by $\tilde{s}^{t+\tau}(f \mid h^t)$ a (stochastic) action profile in the $(t+\tau)$-th period induced by f given history h^t (the only difference from our formulation is that $\tau = 0$ is allowed here).

Given $f = (f_1, f_2)$, player i's payoff at history h^t is:

$$V_i(f \mid h^t) \equiv (1-\delta) \sum_{\tau=0}^{\infty} \delta^{\tau} \mathbb{E}[u_i(\tilde{s}^{t+\tau}(f \mid h^t))], \quad \text{(Lagunoff and Matsui, 1997, p. 1469)} \tag{D.1}$$

where $\mathbb{E}$ is the expectation operator.

Lagunoff and Matsui (1997, p. 1469) state that they "let $u_i(\tilde{s}(e)) = 0$ to simplify notation," where they "write $\tilde{s}(h^t) = \tilde{s}^t(f \mid h^t)$." Since $\tilde{s}(\cdot)$ is defined only when its argument is of the form h^t, it is not defined when the argument is e (Lagunoff and Matsui (1997) do not let $e = h^{-1}$ as we do). We are then led to interpreting this specification as follows. Since equation (D.1) only defines the value conditional on h^t with $t \in \mathbb{N} \cup \{0\}$ (recall that h^0 contains the action profile taken at period 0), we define the value conditional on the empty history. To do this, we let $h^{-1} \equiv e$ in understanding Lagunoff and Matsui (1997) as well. With this notation, in equation (D.1), $V_i(f \mid e)$ is equal to $V_i(f \mid h^{-1})$, which can be written as the expectation of

$$(1-\delta)[u_i(\tilde{s}^{-1}(f \mid e)) + \delta u_i(\tilde{s}^0(f \mid e)) + \delta^2 u_i(\tilde{s}^1(f \mid e)) + \dots].$$

By definition, $\tilde{s}^0(f \mid e)$ is the action profile taken at period 0, $\tilde{s}^1(f \mid e)$ is the action profile taken at period 1, and so on. Our interpretation is that the "$u_i(\tilde{s}(e)) = 0$" in Lagunoff and Matsui (1997) is meant to let $u_i(\tilde{s}^{-1}(f \mid e))$ (the only undefined term) be equal to 0 (that is, given $e = h^{-1}$, we have $\tilde{s}(e) = \tilde{s}(h^{-1}) = \tilde{s}^{-1+0}(f \mid e)$). In summary, each player's value after the empty history is simply δ times the discounted sum of payoffs from period 0 onward, as written in (1) of the main text.[6] In this paper, we proceeded with the understanding that Lagunoff and Matsui (1997) consider this specification.

Summarizing our interpretation of Lagunoff and Matsui (1997)'s formulation, in this paper, whenever we mention the payoff at time $t+1$ after a history that has ended at time t, we mean the expectation of $(1-\delta)\sum_{\tau=1}^{\infty} \delta^{\tau-1} u_i(s(t+\tau))$ where $s(t')$

[6]We note that (1) appears in Lagunoff and Matsui (1997, p. 1468).

represents the action profile at time t'.[7] As discussed in footnote 1 of the main text, this is the expression of the payoff that the proofs in Lagunoff and Matsui (1997) use.[8]

An alternative interpretation would be that "$u_i(\tilde{s}(e)) = 0$" means that the players do not "count" the payoffs at period 0 and consider the payoffs from period 1 onward. Although such an interpretation makes unclear what the first term inside the expectation of (D.1) would be conditional on the empty history, we consider a model under such an interpretation (what we will term the "period-1 onward" model) in this section. We present such a model and show that the optimality result is restored in the two-player case. We then prove that the optimality result does not extend to the case with more than two players, despite Lagunoff and Matsui (1997)'s conjecture that it does. We note that, as we state in Remark 7, under the "period-1 onward" model, the conclusion of Theorem 4 of Lagunoff and Matsui (1997) would trivially hold without any sufficient condition.

D.1 The "Period-1 Onward" Model

Consider an alternative specification of payoffs where the value conditional on the empty history is represented as

$$V_i(f \mid e) \equiv (1-\delta)\mathbb{E}\left[\sum_{t=1}^{\infty} \delta^t u_i(\tilde{s}(t))\right],$$

where $\tilde{s}(t)$ is the (stochastic) action profile at time t that results from f conditional on e. That is, we do not count the payoffs that would realize given the action profile

[7] The value conditional on the history that has ended at time t defined in Lagunoff and Matsui (1997) is the expectation of $(1-\delta)\left[u_i(s(t)) + \delta \sum_{\tau=1}^{\infty} \delta^{\tau-1} u_i(s(t+\tau))\right]$ where $s(t)$ is a fixed action profile that is recorded in the history that has ended in time t.

[8] This specification also follows the one defined in the working paper version (Lagunoff and Matsui, 1995). In that paper, players receive flow payoffs in continuous time, and the payoff was defined to be

$$r \sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} e^{-r\tau} u_i(s^k) d\tau,$$

where r is the discount rate, $T_0 = 0$, T_k is the k-th revision opportunity after time 0, and s^k is the action profile between T_k and T_{k+1} (Lagunoff and Matsui, 1995, p. 5).

taken at time 0. We call this model the *"period-1 onward" model.*[9] In contrast, we call the model in the main text of the paper the *"period-0 onward" model.*

In this context, we formalize the optimality result as follows.

Proposition 9 (Optimality Result for the "Period-1 Onward" Model)**.** *Fix a two-player pure coordination game and consider the "period-1 onward" model. In any perfect equilibrium of the asynchronously repeated game with any discount factor, every player i plays s_i^* at every period except player 1's action at period 0.*

Note that player 1 is indifferent at period 0 and may take any action other than s_1^* because the period-0 payoffs are not counted.

Proof of Proposition 9. By Theorem 1, at any history h^0 at which player 2 has taken s_2^* in period 0, player 1 can guarantee himself the best payoff u^* in period 1. Thus, by choosing s_2^* in period 0, player 2 can guarantee herself a payoff of

$$(1-\delta)\cdot 0+\delta u^*,$$

which is the best possible payoff at the empty history. Since any other action would result in a strictly lower payoff, player 2 takes s_2^* at $t=0$ in any perfect equilibrium. Thus, each player i takes s_i^* from period 1 on in any perfect equilibrium. □

Remark 6. Proposition 9 implies that the strategy profile f^* constructed in Section 5 is not a perfect equilibrium in the "period-1 onward" model. To see when a player has an incentive to deviate, consider player 2. At period $t=0$, if player 2 follows the strategy f_2^* to play s^2 then her payoff is $0+2\delta^2$. If she deviates by taking s^1 in period 0, then her payoff is $0+2\delta$. Thus, player 2 has a strict incentive to deviate in period 0 in the "period-1 onward" model. Similarly, f^* constructed in the proof of Proposition 5 is not a perfect equilibrium in the "period-1 onward" model. To see this, consider player 2. At $t=0$, if player 2 follows the strategy f_2^* to play s^2, then her payoff is $0+2\delta^{2m-2}$. If she deviates by taking s^1 in period 0, then her payoff is $0+2\delta$. Thus, she has a strict incentive to deviate at the empty history in the "period-1 onward" model.

Remark 7. When a stage game is a game of common interest (defined in Section 6), the "period-1 onward" model has a unique perfect equilibrium (ignoring player

[9] It is not clear how to formulate such a model under the general moving structure in continuous time as in Section 3 of Lagunoff and Matsui (1997) (cf. footnote 8 of this Online Appendix).

1's period-0 action) if the discount factor is small enough. In the unique perfect equilibrium, every player i plays s_i^* at every period except player 1's action at period 0, irrespective of α. Thus, in the "period-1 onward" model, the sufficient condition in Theorem 4 of Lagunoff and Matsui (1997) would not play an important role in the sense that, for any $\alpha \in (0,1)$, there exists a range of discount factors in which there exists a unique perfect equilibrium (ignoring player 1's period-0 action) (where the parameter α is defined in Section 6). That is, the conclusion of Theorem 4 of Lagunoff and Matsui (1997) would trivially hold without any sufficient condition.

D.2 Optimality Result Does Not Hold for the n-Player Case

Consider the "period-1 onward" model. Although Lagunoff and Matsui (1997, p. 1473) conjecture that "[t]he same optimality result as before is obtained" for the case with more than two players, we show that the following claim holds. As in Remark 5, we suppose that players simultaneously choose an action at period 0, and player i has a chance to revise her action at period $kn + i$ for every $k \in \mathbb{N} \cup \{0\}$.

Proposition 10. *Fix any integers $n \geq 3$ and $K > 0$. There exist an n-player pure coordination game and $\overline{\delta} \in (0,1)$ such that the asynchronously repeated game under the "period-1 onward" model has a perfect equilibrium in which the players do not play s^* for at least the first K periods when the discount factor is $\delta \in (\overline{\delta}, 1)$.*

Appendix D.3 shows that the optimality result does not hold.

D.3 Two-Action n-Player Example

Appendix D.3.1 considers $K = 3$ with three players. Extending the argument to general K and a greater number of players is simple and is explained in Appendix D.3.2.

		s^1	
	s^1	s^2
s^1	$2, 2, 2$	$0, 0, 0$
s^2	$-10, -10, -10$	$1, 1, 1$

		s^2	
	s^1	s^2
s^1	$-10, -10, -10$	$-10, -10, -10$
s^2	$-10, -10, -10$	$-10, -10, -10$

Table 7: The three-player example. Player 1 chooses a table, player 2 chooses a row, and player 3 chooses a column. Note that $(s_1^*, s_2^*, s_3^*) = (s^1, s^1, s^1)$.

D.3.1 A Three-Player Example

Consider the three-player pure coordination game depicted in Table 7. Consider the following strategy profile f^* in the "period-1 onward" model:

$$f_1^*(h) \equiv s^1 \text{ for all } h, \qquad f_2^*(h) \equiv \begin{cases} s^2 & \text{if } h = e \\ s^1 & \text{if } h \neq e \end{cases}; \quad \text{and}$$

$$f_3^*(h) \equiv \begin{cases} s^2 & \text{if } h = e \text{ or if } h \neq e \text{ and } s_2(t-1) = s^2 \\ s^1 & \text{if } h \neq e \text{ and } s_2(t-1) = s^1 \end{cases}.$$

Note that, on the equilibrium path of play, (s^1, s^2, s^2) is chosen in period 0, and then the sequence of action profiles afterward is

$$(s^1, s^2, s^2), \quad (s^1, s^1, s^2), \quad \text{and} \quad (s^1, s^1, s^1).$$

The action profile stays at (s^1, s^1, s^1) thereafter. Hence, the action profile played is not $(s^1, s^1, s^1) = s^*$ for the first $K = 3$ periods.

We show that f^* is a perfect equilibrium when $\delta \in (0, 1)$ is sufficiently large. Consider player 1. Since s^1 weakly dominates s^2 for player 1 and the actions of players 2 and 3 do not depend on player 1's action, it is a best response for player 1 to follow f_1^* to play s^1 regardless of the history.

Consider player 2. Suppose that $t \geq 1$ and $(s_1(t-1), s_3(t-1)) \neq (s^1, s^2)$. Then, playing s^1 at t gives a weakly greater payoff than playing s^2 at every period starting with time t. Hence, it is a best response for player 2 to follow f_2^* to play s^1. Thus, it remains to consider the case with $t = 0$ and the case with $t \geq 1$ and $(s_1(t-1), s_3(t-1)) = (s^1, s^2)$.

At period 0, if player 2 follows f_2^* and takes s^2, then her payoff is $(1-\delta)\delta + 2\delta^3$. If she instead takes s^1, then her payoff is $2\delta^3$. Since the former payoff is greater than the latter payoff, player 2 does not have an incentive to deviate at period 0.

Suppose that $t \geq 1$ and $(s_1(t-1), s_3(t-1)) = (s^1, s^2)$. If player 2 follows f_2^* and takes s^1, then player 2 obtains a payoff of 2δ. If player 2 takes s^2 instead, then player 2 obtains a payoff of $(1-\delta)(1+\delta+\delta^2) + 2\delta^4$. Thus, player 2 has an incentive to follow f_2^* if

$$2\delta \geq (1-\delta)(1+\delta+\delta^2) + 2\delta^4, \text{ that is, } \delta \geq \frac{1}{2}.$$

Consider player 3. Suppose first that $t \geq 1$ and $s_2(t-1) \neq s^2$. Then, playing s^1 at t gives a weakly greater payoff than playing s^2 at every period starting with time t. Hence, it is a best response for player 3 to follow f_3^* to play s^1. Thus, it remains to consider the case with $t = 0$ and the case with $t \geq 1$ and $s_2(t-1) = s^2$.

At period 0, if player 3 follows f_3^* and takes s^2, then her payoff is $(1-\delta)\delta + 2\delta^3$. If player 3 takes s^1 then her payoff is $-10(1-\delta)\delta + 2\delta^2$. Thus, player 3 has an incentive to follow f_3^* if

$$(1-\delta)\delta + 2\delta^3 \geq -10(1-\delta)\delta + 2\delta^2,$$

which can be shown to hold for any $\delta \in (0,1)$.

Consider $t \geq 1$ and suppose $(s_1(t-1), s_2(t-1)) = (s^1, s^2)$. If player 3 follows f_3^* to play s^2, then player 3 gets a payoff of $(1-\delta)(1+\delta) + 2\delta^3$. If player 3 deviates by taking s^1, then player 3 gets a payoff of $-10(1-\delta)(1+\delta) + 2\delta^2$. Thus, player 3 has an incentive to follow f_3^* if

$$(1-\delta)(1+\delta) + 2\delta^3 \geq -10(1-\delta)(1+\delta) + 2\delta^2,$$

which can be shown to hold for any $\delta \in (0,1)$.

If $t \geq 1$ and $(s_1(t-1), s_2(t-1)) = (s^2, s^2)$, by following f_3^* to play s^2, player 3 gets a payoff of $(1-\delta)(-10+\delta) + 2\delta^3$. If player 3 takes s^1, then player 3 gets a payoff of $-10(1-\delta)(1+\delta) + 2\delta^2$. Thus, player 3 has an incentive to follow f_3^* if

$$(1-\delta)(-10+\delta) + 2\delta^3 \geq -10(1-\delta)(1+\delta) + 2\delta^2,$$

which can be shown to hold for any $\delta \in (0,1)$.

D.3.2 An n-Player Example

We show that we can extend the previous proof to general K and n players. To extend to general K, make the payoff function for players 2 and 3 when player 1 is choosing s^1 to be analogous to the one specified in Appendix A.1. To extend to n players, let the payoffs be such that players receive payoffs as specified in the previous sentence if all players in $\{4, \dots, n\}$ play s^1 and otherwise they receive a payoff of $-kL$ if k players in $\{4, \dots, n\}$ play s^2, where L is a large positive number.

References for the Online Appendix

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