# A Comment on "Asynchronous Choice in Repeated Coordination Games"*

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**Abstract**

Lagunoff and Matsui (1997), "Asynchronous Choice in Repeated Coordination Games," *Econometrica* 65 (6): 1467–1477, assert the following optimality result: asynchronously repeated pure coordination game has a unique perfect equilibrium when players are patient, in which every player always takes the best action on the equilibrium path. We provide a counterexample to this optimality result and a sufficient condition under which the result is restored. We also provide a counterexample to Theorem 4 of Lagunoff and Matsui (1997) that states a sufficient condition under which the optimality result holds for games of common interest.

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# 1 Introduction

Lagunoff and Matsui (1997) consider asynchronous repeated games where the stage game is a two-player pure coordination game, i.e., a game in which the players have the same payoff function. Lagunoff and Matsui (1997) establish that players will keep playing the best (i.e., Pareto-dominating) action profile $s^*$ once some player $i$ chooses $s_i^*$ (Theorem 1), and the equilibrium payoff approaches the payoff from $s^*$ as the discount factor approaches 1 (Theorem 2). From these two results, they also state the following optimality result: when players are patient, there is a unique perfect equilibrium, where players always play $s^*$ from period 0 on.

While their Theorems 1 and 2 are correct, we show by means of examples that the optimality result does not hold. Specifically, we show that for any $K > 0$, there exists a pure coordination game such that the asynchronous repetition of that game has an equilibrium in which patient players do not take the best action for at least the first $K$ periods. We also provide a sufficient condition on the stage-game payoffs under which the optimality result is restored. This sufficient condition uses a condition more stringent than what is used in the proof of Theorem 4 in Lagunoff and Matsui (1997), which provides a sufficient condition for equilibrium uniqueness in games of common interest. We then provide a counterexample to Theorem 4 in Lagunoff and Matsui (1997) which relies on this discrepancy, and prove a weaker version of the statement of the theorem.

The rest of this paper is structured as follows. Section 2 sets up the model of Lagunoff and Matsui (1997). Section 3 revisits their Theorems 1 and 2. Section 4 revisits their optimality result. Section 5 provides the counterexamples. Section 6 provides sufficient conditions on the stage-game payoffs under which the optimality result is restored and provides a counterexample to Theorem 4 in Lagunoff and Matsui (1997).

# 2 The Lagunoff and Matsui (1997) Model

This section revisits the baseline two-player model of Lagunoff and Matsui (1997). A stage game is given by $G = \langle S_1, S_2, u_1, u_2 \rangle$ where $S_i$ is player $i$'s finite action set with $|S_i| \geq 2$ and $u_i : S_1 \times S_2 \to \mathbb{R}$ is her utility function.

Time is discrete, starting from 0. At time 0, both players simultaneously choose their action. Subsequently, player 1 has a chance to revise his action at every odd-numbed period $t \in \{1, 3, 5, \dots\}$, while player 2 has a chance to revise her action at every even-numbered period $t \in \{2, 4, 6, \dots\}$. Letting $s(t)$ be the action profile at time $t$, that is, the action profile in which each player's action is the most recent action choice (including at time $t$), player $i$'s payoff at time $t$ is defined as $u_i(s(t))$.

Given a sequence of action profiles $(s(t))_{t=0}^{\infty}$ and letting $\delta \in (0, 1)$ be a common discount factor, each player $i$ seeks to maximize the normalized discounted sum

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(s(t)). \qquad \text{(Lagunoff and Matsui, 1997, p. 1468)} \tag{1}$$

Denote by $e$ an empty history. Denote by $H_i$ the set of all histories after which player $i \in \{1, 2\}$ moves. Let $H = H_1 \cup H_2$. Note that $H_1 \cap H_2 = \{e\}$ because the two players simultaneously move only after the empty history. Denote by $h^t$ a history ending in period $t$ (including the action profile in $t$). Note that $h^0 \neq e$, and we let $h^{-1} \equiv e$ to simplify the exposition.

A *strategy* of player $i$ is a function $f_i : H_i \to \Delta(S_i)$. Given a strategy profile $f = (f_1, f_2)$, a history $h^t \in H$, and $\tau \in \mathbb{N}$, denote by $\tilde{s}^{t+\tau}(f \mid h^t)$ a (stochastic) action profile in the $(t + \tau)$-th period induced by $f$ after history $h^t$.

Given $f = (f_1, f_2)$, player $i$'s payoff at history $h^t$ is:

$$V_i(f \mid h^t) \equiv (1-\delta)\sum_{\tau=1}^{\infty}\delta^{\tau-1}\mathbb{E}[u_i(\tilde{s}^{t+\tau}(f \mid h^t))],$$

where $\mathbb{E}$ is the expectation operator.[1]

A strategy profile $f^* = (f_1^*, f_2^*)$ is a *perfect equilibrium* (PE) if, for each $i \in \{1, 2\}$, $f_i^*$ is a best response to $f_j^*$ (with $j \neq i$) after every history $h \in H$:

$$V_i(f^* \mid h) \geq V_i(f_i, f_j^* \mid h) \text{ for any of player } i\text{'s strategies } f_i.$$

Lagunoff and Matsui (1997, Theorem 0) show that, for any stage game $G$, the

[1]This formulation is different from that of Lagunoff and Matsui (1997), who defined $V_i(f \mid h^t)$ to be $(1-\delta)\times$ [the payoff from the fixed action profile at time $t$] $+\ \delta\times$ [our $V_i(f \mid h^t)$]. Since maximizing our $V_i(f \mid h^t)$ is equivalent to maximizing Lagunoff and Matsui (1997)'s $V_i(f \mid h^t)$, and the proofs in Lagunoff and Matsui (1997) write payoffs conditional on histories using our formulation, we use our version of $V_i(f \mid h^t)$ here. We detail our interpretation of Lagunoff and Matsui (1997)'s formulation in the Appendix.

asynchronous-move repetition of $G$ has a perfect equilibrium.

Finally, we introduce pure coordination games. A stage game $G$ is a *pure coordination game* if $u_1 = u = u_2$ for some $u$. Denote by $s^*$ the unique action profile that gives each player her highest payoff $u^* := \max_{s \in S_1 \times S_2} u(s)$. The uniqueness of such an action profile is not assumed in Lagunoff and Matsui (1997), but we make that assumption to simplify our exposition.[2]

# 3 Revisiting Lagunoff and Matsui (1997, Theorems 1 and 2)

This section revisits Theorems 1 and 2 of Lagunoff and Matsui (1997), which are correct. Theorem 1 states that, in any perfect equilibrium, conditional on any history that ends with $s^*$, both players' payoffs are $u^*$. Let $\tilde{s}(h)$ be the action profile at the end of history $h \in H \setminus \{e\}$.

**Theorem 1** (Theorem 1 of Lagunoff and Matsui (1997))**.** *If $f$ is a perfect equilibrium of an asynchronous-move pure-coordination game, then, for any history $h \in H \setminus \{e\}$ with $\tilde{s}(h) = s^*$,*

$$V_i(f \mid h) = u^* \text{ for each } i \in \{1, 2\}.$$

Note that Theorem 1 holds for any $\delta \in (0, 1)$.

Theorem 2 states that for any fixed asynchronous-move pure-coordination game, if the players are patient enough then their equilibrium payoffs can be made arbitrary close to $u^*$.

**Theorem 2** (Theorem 2 of Lagunoff and Matsui (1997))**.** *Fix any asynchronous-move pure-coordination game. Fix any $\varepsilon > 0$. Then, there exists $\overline{\delta} \in (0, 1)$ such that, if $\delta \in (\overline{\delta}, 1)$ then for any perfect equilibrium $f$ of the asynchronous-move game and for any history $h \in H$,*

$$V_i(f \mid h) > u^* - \varepsilon \text{ for each } i \in \{1, 2\}$$

*and the action profile $s^*$ reaches in a finite number of periods.*

[2] If we allow for multiple action profiles that result in a payoff of $u^*$, then one can construct an example of a pure coordination game and a perfect equilibrium in its asynchronously repeated game where the players mix in period 0 and hence there is a positive probability that they do not receive $u^*$ in period 0.

| | $s^1$ | $s^2$ |
|---|---|---|
| $s^1$ | $2, 2$ | $0, 0$ |
| $s^2$ | $-10, -10$ | $1, 1$ |

Table 1: The example for $K = 2$. Note that $(s_1^*, s_2^*) = (s^1, s^1)$.

# 4 Optimality Result and Our Main Point

Lagunoff and Matsui (1997, p. 1471) state the following:

> "Theorems 1 and 2 together with the existence result jointly establish an optimality result. In every equilibrium, players choose $s^*$ at the beginning of the game and never depart. Note that Theorem 2 seems only to suggest an approximation to $s^*$. However, this is because we start the process from an arbitrary state. In equilibrium, the "initial" state $h^0$ or the state after $e$ is determined by players' simultaneous choice. In determining $h^0$, they follow a reasoning process similar to the one in the proof of Theorem 2. As a result, they take $s^*$ from the beginning."

The main point of this paper is that this optimality result does not hold.

**Proposition 1.** *For any positive integer $K > 0$, there exist a pure-coordination game $G$ and $\overline{\delta} \in (0, 1)$ such that, if $\delta \in (\overline{\delta}, 1)$, the asynchronous-move game with its stage game being $G$ has a perfect equilibrium in which the players do not play $s^*$ for at least the first $K$ periods.*

# 5 Counterexamples

For any positive integer $K$, we construct a two-player pure-coordination game such that the players do not take $s^*$ for at least the first $K$ periods. Section 5.1 presents a construction for $K = 2$. Section 5.2 extends the argument to general $K$.

## 5.1 The Argument for $K = 2$

Consider the $2 \times 2$ pure-coordination game depicted in Table 1.

We consider the following strategy profile $f^*$. Player 1 takes $s^2$ at the empty history; and in period $t \geq 1$, player 1 takes $s^1$ at any $h \in H_1$. Player 2 takes $s^2$ at the empty history; and in period $t \geq 2$, player 2 takes $s^1$ at $h \in H_2$ if and only if $s_1(t-1) = s^1$ (we denote by $s_i(\tau)$ the action taken by player $i$ in the action profile $s(\tau)$).

Note that, on the equilibrium path of play, $(s^2, s^2)$ is chosen in period 0, player 1 chooses $s^1$ in period 1 so the action profile becomes $(s^1, s^2)$, and then player 2 chooses $s^1$ in period 2 so the action profile becomes $(s^1, s^1)$. The action profile stays at $(s^1, s^1)$ thereafter. Hence, the action profile played is not $(s^1, s^1) = s^*$ for the first two periods.

We show that $f^*$ is a perfect equilibrium when $\delta \in (0,1)$ is sufficiently large. Consider player 1. At $t = 0$, if player 1 follows the strategy $f_1^*$ to play $s^2$, then his payoff is $(1-\delta) + 2\delta^2$. If he deviates by taking $s^1$ in period 0, then his payoff is $2\delta^2$, which is lower than $(1-\delta) + 2\delta^2$.

At $t \geq 1$, by Theorem 1, it suffices to consider a history $h \in H_1$ in which player 2 has taken $s_2(t-1) = s^2$. If player 1 takes $s^1$, then his payoff is $2\delta$. If he deviates by taking $s^2$ at $h$, then his payoff is

$$(1-\delta)\left(1 + \delta \cdot 1 + \delta^2 \cdot 0 + \frac{\delta^3}{1-\delta} \cdot 2\right).$$

Thus, player 1 does not have an incentive to deviate if

$$2\delta \geq (1-\delta)(1+\delta) + 2\delta^3, \text{ that is, } \delta \geq \frac{1}{2}.$$

Consider player 2. At period $t = 0$, if player 2 follows the strategy $f_2^*$ to play $s^2$, then her payoff is $(1-\delta) + 2\delta^2$. If she deviates by taking $s^1$, then her payoff is $-10(1-\delta) + 2\delta$. Thus, player 2 does not have an incentive to deviate in period 0 if

$$(1-\delta) + 2\delta^2 \geq -10(1-\delta) + 2\delta,$$

which holds for any $\delta \in (0,1)$.

Suppose $t \geq 2$. By Theorem 1, it suffices to consider the case where player 1 has taken $s_1(t-1) = s^2$. If player 2 follows $f_2^*$ to play $s^2$, then her payoff is $(1-\delta) + 2\delta^2$. If she deviates by taking $s^1$, then her payoff is $-10(1-\delta) + 2\delta$. As in the preceding analysis, we have $(1-\delta) + 2\delta^2 \geq -10(1-\delta) + 2\delta$ for all $\delta \in (0,1)$.

In sum, when $\delta \geq \frac{1}{2}$, the strategy profile $f^*$ is a perfect equilibrium.

| | $s^1$ | $s^2$ | $s^3$ |
|---|---|---|---|
| $s^1$ | $2, 2$ | $0, 0$ | $-M, -M$ |
| $s^2$ | $-M, -M$ | $0, 0$ | $0, 0$ |
| $s^3$ | $-M, -M$ | $-M, -M$ | $1, 1$ |

Table 2: The example for $K = 4$. Note that $(s_1^*, s_2^*) = (s^1, s^1)$.

## 5.2 The Argument for General $K$

Let $m$ be the minimum integer with $2(m-1) \geq K$ and player $i$'s action set be $S_i = \{s^k\}_{k=1}^m$. Define player $i$'s utility function as

$$u_i(s^k, s^\ell) \equiv \begin{cases} 2 & \text{if } k = \ell = 1 \\ 1 & \text{if } k = \ell = m \\ 0 & \text{if } k = \ell \in \{2, \dots, m-1\} \text{ or } k+1 = \ell \in \{2, \dots, m\} \\ -M & \text{otherwise} \end{cases},$$

where $M \geq 4m - 6$. Note that $(s_1^*, s_2^*) = (s^1, s^1)$. Table 2 presents the case of $K = 4$, and thus $m = 3$.

Consider the following strategy profile $f^*$:

$$f_1^*(h) \equiv \begin{cases} s^m & \text{if } h = e \\ s^{k-1} & \text{if } h \neq e \text{ and } s_2(t-1) = s^k \text{ for some } k \in \{2, \dots, m\} \\ s^1 & \text{if } h \neq e \text{ and } s_2(t-1) = s^1 \end{cases}; \text{ and}$$

$$f_2^*(h) \equiv \begin{cases} s^m & \text{if } h = e \\ s^k & \text{if } h \neq e \text{ and } s_1(t-1) = s^k \text{ for some } k \in \{1, \dots, m\} \end{cases}.$$

Note that, on the equilibrium path of play, $(s^m, s^m)$ is chosen in period 0, and then the sequence of action profiles afterward is

$$(s^{m-1}, s^m), \quad (s^{m-1}, s^{m-1}), \quad (s^{m-2}, s^{m-1}), \quad \dots, \quad (s^1, s^1).$$

The action profile stays at $(s^1, s^1)$ thereafter. Hence, the action profile played is not $(s^1, s^1) = s^*$ for the first $2m - 1 (\geq K)$ periods. Note also that $(u_1, u_2)$ and $f^*$ defined above boil down to $(u_1, u_2)$ and $f^*$ defined in Section 5.1, respectively, when $K = 2$.

We show that $f^*$ is a perfect equilibrium when $\delta \in (0,1)$ is sufficiently large. First, note that given any history and the opponent's strategy $f^*_{-i}$, player $i$ has no incentive to play an action that results in the instantaneous payoff of $-M$. To see this, notice that $i$'s payoff from such a deviation is at most $(1-\delta)(-M) + \delta \cdot 2$ while the payoff from following $f^*_i$ is at least $\delta^{2(m-1)} \cdot 2$. Hence, following $f^*_i$ is better if $\delta^{2(m-1)} \cdot 2 \geq (1-\delta)(-M) + \delta \cdot 2$, or

$$M \geq 2\frac{\delta(1-\delta^{2m-3})}{1-\delta} = 2\delta(1+\delta+\delta^2+\cdots+\delta^{2m-4}).$$

Since the right-hand side is less than $2(2m-3)$, which is equal to $4m-6$, it follows that no player has an incentive to play an action that results in the instantaneous payoff of $-M$ because we assumed $M \geq 4m-6$.

Second, given any history at time $t \geq 1$, the opponent's strategy $f^*_{-i}$ and $k \in \{1,\ldots,m-1\}$, if two actions $s^k$ and $s^{k+1}$ of the moving player $i$ result in the instantaneous payoff that is not $-M$, then $i$ would be better off choosing $s^k$, which is to follow $f^*_i$. To see this, note that the payoff from playing $s^k$ is $\delta^{\ell_k} \cdot 2$ for some $\ell_k \in \mathbb{N} \cup \{0\}$, and given this, the payoff from playing $s^{k+1}$ is at most $(1-\delta)(1+\delta) + \delta^{\ell_k+2} \cdot 2$. The former is greater than the latter if

$$2\delta^{\ell_k} \geq (1-\delta)(1+\delta) + 2\delta^{\ell_k+2},$$

which is equivalent to $2\delta^{\ell_k}(1-\delta^2) \geq (1-\delta)(1+\delta)$, or $\delta \geq \left(\frac{1}{2}\right)^{\frac{1}{\ell_k}}$. Letting $\bar{\ell} \equiv \max\limits_{1 \leq k \leq m-1} \ell_k$, this inequality holds for all $k \in \{1,\ldots,m-1\}$ if $\delta \geq \left(\frac{1}{2}\right)^{\frac{1}{\bar{\ell}}}$.

What remains is to show that, at time 0, player 1's payoff from playing $s^m$, which is to follow $f^*_1$, is greater than his payoff from playing $s^{m-1}$.[3] To see this, note that the former payoff is $(1-\delta) + 2\delta^{2m-2}$ while the latter payoff is $2\delta^{2m-2}$. The former is greater than the latter. This completes the proof.

## 5.3 An $n$-Player Example

After stating their optimality result for the two-player case, Lagunoff and Matsui (1997, p. 1473) state that "[t]he same optimality result as before is obtained" for $n$ players.

[3] Note that we have already checked player 2's incentive and player 1's other deviations at time 0 in the first paragraph of this proof.

We argue that we can extend the argument in Section 5.2 to the $n$-player case, in which players simultaneously choose an action at period 0, and player $i$ has a chance to revise her action at period $kn + i$ for every $k \in \mathbb{N} \cup \{0\}$. For each player $i \in \{1, \ldots, n\}$, suppose that her payoff $u_i$ is given as in Section 5.2 when all players other than players 1 and 2 choose $s^1$, while the payoffs are $-M$ otherwise. We define $(f_i^*)_{i=1}^2$ so that it does not depend on the actions $(s_i)_{i=3}^n$ while it depends on $(s_i)_{i=1}^2$ in the same way as in Section 5.2. The strategies $(f_i^*)_{i=3}^n$ choose $s^1$ under any history. For any positive integer $K$, this specification allows us to have an outcome induced by $f^*$ such that $s^*$ is not played for at least the first $K$ periods. An analogous argument can be made for more general timing structures (such as Poisson processes) described in Lagunoff and Matsui (1997).

# 6 Sufficient Condition

We derive a sufficient condition under which the optimality result holds. For this purpose, we introduce the notation defined in Lagunoff and Matsui (1997). Let $u_* \equiv \min_s u(s)$ and $\overline{u} \equiv \max_{s \neq s^*} u(s)$. Define:

$$\alpha \equiv \frac{\overline{u} - u_*}{u^* - u_*}.$$

**Proposition 2.** *Fix a pure coordination game and suppose that $\alpha < \frac{1}{2}$. There exists $\overline{\delta} \in (0, 1)$ such that for all $\delta \in (\overline{\delta}, 1)$, the optimality result holds, i.e., players play $s^*$ at every period in any perfect equilibrium of the asynchronously repeated game.*

*Proof of Proposition 2.* Consider any history at which player $i$ has a move. If $i$ plays $s_i^*$, then her payoff is at least $(1-\delta)u_* + \delta u^*$. If she instead chooses a different action, her payoff is at most $(1 - \delta)(\overline{u} + \delta\overline{u}) + \delta^2 u^*$. The former is greater than the latter if

$$(1 - \delta)u_* + \delta u^* > (1 - \delta)(\overline{u} + \delta\overline{u}) + \delta^2 u^*, \tag{2}$$

which is equivalent to $\delta > \frac{\overline{u}-u_*}{u^*-\overline{u}}$. Thus, there exists $\overline{\delta} \in (0, 1)$ such that this inequality holds for all $\delta \in (\overline{\delta}, 1)$ if $\frac{\overline{u}-u_*}{u^*-\overline{u}} < 1$, which is equivalent to $\alpha < \frac{1}{2}$. □

**Remark 1.** The condition on $\alpha$ in Proposition 2 can be naturally extended to the case of $n$ players. The incentive constraint (2) becomes

$$(1-\delta)(1+\delta+\cdots+\delta^{n-2})u_* + \delta^{n-1}u^* > (1-\delta)(1+\delta+\cdots+\delta^{n-1})\overline{u} + \delta^n u^*,$$

which can be shown to hold for a sufficiently high $\delta \in (0,1)$ if $\alpha < \frac{1}{n}$.

Lagunoff and Matsui (1997, the proof of Theorem 4) present a different condition for the same purpose as condition (2), which is "$(1-\delta)u_* + \delta u^* > \overline{u}$." This condition is stated as one that guarantees that the moving player $i$ wants to switch to $s_i^*$ under the assumption that $s^*$ is kept being played once some player $j$ chooses $s_j^*$. In fact, however, any stage game satisfies this condition for a sufficiently high $\delta \in (0,1)$ because the condition is equivalent to $\frac{\delta}{1-\delta} > \frac{\overline{u}-u_*}{u^*-\overline{u}}$,[4] but the game in Table 1 has a perfect equilibrium in which $s^*$ is not chosen for some periods on the path of play when $\delta$ is high as argued in Section 5.1. Moreover, there exists a counterexample to Theorem 4 of Lagunoff and Matsui (1997) because of this discrepancy. To explain this point, say that $G$ is a *game of common interest* if there exists an action profile $s^*$ such that $u_i(s^*) > u_i(s)$ for all $i \in \{1,2\}$ and $s \neq s^*$. Note that two players' payoff functions can be different from each other. Normalize the payoffs so that the two players' best payoffs are both $u^*$ and the worst payoffs are both $u_*$. We generalize the definitions of $\overline{u}$ and $\alpha$ as follows:

$$\overline{u} \equiv \max_{s \neq s^*}\{u_1(s), u_2(s)\} \quad \text{and} \quad \alpha \equiv \frac{\overline{u}-u_*}{u^*-u_*}.$$

Theorem 4 of Lagunoff and Matsui (1997) states the following: In an asynchronous-move game of common interest, there exists a neighborhood of discount factors $\delta \in (0,1)$ under which the optimal outcome $s^*$ is the unique outcome if $2\alpha + \alpha^2 - \alpha^3 < 1$.

We, however, show that the following claim holds true.

**Proposition 3.** *For any $a \in (0,1)$ that satisfies $a^2 - 3a + 1 \leq 0$, there exists a game of common interest $G$ with $\alpha = a$ such that, for any $\delta \in (0,1)$, asynchronous-move game with its stage game being $G$ has a perfect equilibrium in which $s^*$ is not played for multiple periods.*

Note that the condition on $a$ reduces to $a \geq \frac{3-\sqrt{5}}{2} \simeq 0.382$. Thus, this proposition

[4] Recall that the condition we derived in the proof of Proposition 2 is $\delta > \frac{\overline{u}-u_*}{u^*-\overline{u}}$ and it is possible that this does not hold for any $\delta \in (0,1)$ as the right-hand side can be no less than 1.

| | $s^1$ | $s^2$ | $s^3$ |
|---|---|---|---|
| $s^1$ | $1, 1$ | $0, a$ | $0, a$ |
| $s^2$ | $a, 0$ | $a, a$ | $0, 0$ |
| $s^3$ | $a, 0$ | $0, 0$ | $0, 0$ |

Table 3: An Example of a Game of Common Interest, where $a \in (0, 1)$. Note that $(s_1^*, s_2^*) = (s^1, s^1)$.

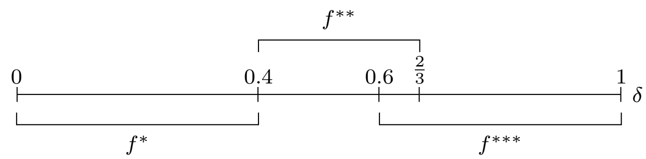


Figure 1: The Space of $\delta \in (0, 1)$. For each strategy profile $f^*$, $f^{**}$, or $f^{***}$, the figure depicts the range of $\delta$ under which it is a perfect equilibrium when $a = 0.4$.

constitutes a counterexample to Theorem 4 of Lagunoff and Matsui (1997), as $2\alpha + \alpha^2 - \alpha^3 < 1$ holds when $\alpha < 0.445$.

To prove Proposition 3, we consider the game depicted in Table 3. Notice that this is a game of common interest and $\alpha = a$.

In what follows, we define three strategies, $f^*$, $f^{**}$, and $f^{***}$ and show that they are a perfect equilibrium if $\delta \leq a$, $a \leq \delta \leq \frac{a}{1-a}$, and $\max\{1 - a, \frac{1}{2}\} \leq \delta$, respectively. The case for $a = 0.4$ is depicted in Figure 1.

We first show that the following strategy profile $f^*$ is a perfect equilibrium if $\delta \leq a$. For each $i$, $f_i^*(h) \equiv s^2$ if $h = e$ or $s_{-i}(t-1) = s^2$. Otherwise, $f_i^*(h) \equiv s^1$.

Note that, on the path of play of $f^*$, $(s^2, s^2)$ is chosen in period 0, and the action profile stays at $(s^2, s^2)$ thereafter.

To see that $f^*$ is a perfect equilibrium if $\delta \leq a$, notice that each $i$'s strategy gives rise to, given $f_{-i}^*$, a weakly better payoff than any deviation at every period except at the history $h$ such that $h = e$ or $s_{-i}(t-1) = s^2$. At such a history, if $i$ follows $f_i^*$ to choose $s^2$, her payoff is $a$. If she instead deviates to $s^1$ and $s^3$, respectively, her payoffs are $(1-\delta) \cdot 0 + \delta \cdot 1$ and $(1-\delta) \cdot 0 + \delta((1-\delta)a + \delta \cdot 1)$, where the former is greater. Thus, player $i$ has an incentive to follow $f_i^*$ if

$$a \geq (1-\delta) \cdot 0 + \delta \cdot 1,$$

which is equivalent to $\delta \leq a$.

We next show that the following strategy profile $f^{**}$ is a perfect equilibrium if $a \leq \delta \leq \frac{a}{1-a}$.

$$f_1^{**}(h) \equiv s^2 \text{ if } h = e, \text{ and } f_1^{**}(h) \equiv s^1 \text{ otherwise.}$$
$$f_2^{**}(h) \equiv s^2 \text{ if } h = e \text{ or } s_1(t-1) = s^2, \text{ and } f_2^{**}(h) \equiv s^1 \text{ otherwise.}$$

Note that, on the path of play of $f^{**}$, the players choose $(s^2, s^2)$ in period 0, player 1 chooses $s^1$ in period 1 so the action profile becomes $(s^1, s^2)$, and then player 2 chooses $s^1$ in period 2 so the action profile becomes $(s^1, s^1)$. The action profile stays at $(s^1, s^1)$ thereafter.

To see that $f^{**}$ is a perfect equilibrium if $a \leq \delta \leq \frac{a}{1-a}$, notice that each $i$'s strategy gives rise to, given $f_{-i}^{**}$, a weakly better payoff than any deviation at every period except at the history $h$ such that $h = e$ for player 2 or $s_{-i}(t-1) = s^2$ for players 1 and 2. Thus, we consider such histories.

Consider player 1 and suppose $s_2(t-1) = s^2$. If he follows $f_1^{**}$ to play $s^1$, his payoff is $(1-\delta) \cdot 0 + \delta \cdot 1$. If he deviates to play $s^2$ and $s^3$, respectively, his payoffs are $(1-\delta)(a + \delta a) + \delta^3 \cdot 1$ and $(1-\delta) \cdot 0 + \delta((1-\delta)a + \delta \cdot 1)$. The latter is obviously less than the payoff from $s^1$. Thus, player 1 has an incentive to follow $f_1^{**}$ if

$$(1-\delta) \cdot 0 + \delta \cdot 1 \geq (1-\delta)(a + \delta a) + \delta^3 \cdot 1,$$

which is equivalent to $\delta \geq a$.

Next, consider player 2. If $h = e$ or $s_1(t-1) = s^2$, if she follows $f_2^{**}$ to play $s^2$, her payoff is $(1-\delta)(a + \delta a) + \delta^2 \cdot 1$. If she instead plays $s^1$ and $s^3$, respectively, her payoffs are $(1-\delta) \cdot 0 + \delta \cdot 1$ and $(1-\delta) \cdot 0 + \delta((1-\delta)a + \delta \cdot 1)$. The latter is obviously less than the payoff from the former. Thus, player 2 has an incentive to follow $f_2^{**}$ if

$$(1-\delta)(a + \delta a) + \delta^2 \cdot 1 \geq (1-\delta) \cdot 0 + \delta \cdot 1,$$

which is equivalent to $\delta \leq \frac{a}{1-a}$.

Finally, we show that the following strategy profile $f^{***}$ is a perfect equilibrium if $\max\{1-a, \frac{1}{2}\} \leq \delta$.

For each $i$, let $f_i^{***}(e) \equiv s^1$. For any history $h$ under which the action profiles that have been played are $((s^1, s^1))$ or $((s^1, s^1), (s^2, s^1), L)$ where $L$ is the repetition of $\ell$ times of $(s^2, s^2)$ where $\ell \in \mathbb{N} \cup \{0\}$, we let $f_i^{***}(h) \equiv s^2$ for each $i$. Let $\tilde{H}^{1,j}$ be the set of histories such that there has been exactly one period in which some player has deviated and the deviating player was $j$, where if both players have deviated at period 0, then we set $j = 2$. For any $h \in \tilde{H}^{1,j}$, we let $f_{-j}^{***}(h) = s^3$ and $f_j^{***}(h) = s^1$.[5] Now we recursively define $\tilde{H}^{k,j}$, which is the set of histories such that there have been exactly $k$ periods in which some player has deviated and the most recent player who has deviated was $j$. For any $h \in \tilde{H}^{k,j}$, we let $f_{-j}^{***}(h) = s^3$ and $f_j^{***}(h) = s^1$.

Note that, on the path of play of $f^{***}$, $(s^1, s^1)$ is chosen in period 0, player 1 chooses $s^2$ in period 1 so the action profile becomes $(s^2, s^1)$, and then player 2 chooses $s^2$ in period 2 so the action profile becomes $(s^2, s^2)$. The action profile stays at $(s^2, s^2)$ thereafter.

To see that $f^{***}$ is a perfect equilibrium if $\max\{1 - a, \frac{1}{2}\} \leq \delta$, consider first the incentive at period 0. For each player, following $f_i^{***}$ to play $s^1$ results in the payoff of $(1 - \delta) \cdot 1 + \delta a$ while deviating leads to the payoff no greater than $a$. Thus, player $i$ has an incentive to follow $f_i^{***}$.

For any other periods, following $f_i^{***}$ induces a payoff greater than the payoff from deviating at every history except when $h \in \tilde{H}^{k,-i}$ with $s_{-i}(t-1) \in \{s^1, s^2\}$.

Suppose $s_{-i}(t-1) = s^1$. If player $i$ follows $f_i^{***}$ to play $s^3$, her payoff is $a$. If she instead deviates to play $s^1$ and $s^2$, respectively, her payoffs are $(1 - \delta) \cdot 1 + \delta \cdot 0$ and $(1 - \delta) \cdot a + \delta \cdot 0$, where the latter is obviously smaller than the former. Hence, player $i$ has an incentive to follow $f_i^{***}$ if

$$a \geq (1 - \delta) \cdot 1 + \delta \cdot 0,$$

which is equivalent to $\delta \geq 1 - a$.

Suppose $s_{-i}(t-1) = s^2$. If player $i$ follows $f_i^{***}$ to play $s^3$, her payoff is $(1 - \delta) \cdot 0 + \delta a$. If she instead deviates to play $s^1$ and $s^2$, respectively, her payoffs are 0 and $(1 - \delta)a + \delta \cdot 0$, where the former is obviously smaller than the latter. Hence, player $i$ has an incentive to follow $f_i^{***}$ if

$$(1 - \delta) \cdot 0 + \delta a \geq (1 - \delta)a + \delta \cdot 0,$$

which is equivalent to $\delta \geq \frac{1}{2}$ because $a > 0$.

[5] Note that if it is $j$'s turn to move at a history $h \in \tilde{H}^{1,j}$ in period $t \geq 2$, then the opponent $-j$ has been playing $s^3$ after $j$'s deviation and thus $s_{-j}(t-1) = s^3$ must hold.

Overall, there is a perfect equilibrium in which $s^*$ is not played for multiple periods if

$$\max\left\{1-a, \frac{1}{2}\right\} \leq \frac{a}{1-a}.$$

First, note that, since $a \in (0,1)$, $1-a \leq \frac{a}{1-a}$ is equivalent to $a^2 - 3a + 1 \leq 0$, and $\frac{1}{2} \leq \frac{a}{1-a}$ is equivalent to $a \geq \frac{1}{3}$. Since $a^2-3a+1 \leq 0$ reduces to $a \geq \frac{3-\sqrt{5}}{2} \simeq 0.382 > \frac{1}{3}$, we conclude that, for any $\delta \in (0,1)$, there is a perfect equilibrium in which $s^*$ is not played for multiple periods if $a^2 - 3a + 1 \leq 0$.

We remark that Proposition 3 is not inconsistent with Proposition 2, which establishes the optimality result for pure coordination games. This is because a game of common interest depicted in Table 3 becomes a pure coordination game only when $a = 0$, in which case the condition $a^2 - 3a + 1 \leq 0$ is violated.[6]

The following result imposes a stronger condition on $\alpha$ than in Theorem 5, and shows that there exists a neighborhood of discount factors under which the optimality result holds.

**Proposition 4.** *Fix a game of common interest and suppose that $\alpha < \frac{1}{3}$. There exists a neighborhood of discount factors $\delta \in (0,1)$ under which the optimality result holds, i.e., players play $s^*$ at every period in any perfect equilibrium of the asynchronously repeated game.*

*Proof of Proposition 4.* As in the proof of Proposition 2, starting from any action profile that is not equal to $s^*$, it is sufficient for condition (2) to hold for the optimality result. Starting from $s^*$, it is sufficient for

$$(1-\delta)u^* + \delta u_* > (1-\delta^2)\overline{u} + \delta^2 u^*$$

to hold for the optimality result (note that this inequality is equivalent to condition (10) of Lagunoff and Matsui (1997)).

Without loss, let $u^* = 1$ and $u_* = 0$. Then, $\overline{u} = \alpha$. The two conditions can be rewritten as

$$\delta > \frac{\alpha}{1-\alpha} =: f(\alpha)$$

[6]When $a = 0$, the ranges of discount factors for which $f^*$, $f^{**}$, and $f^{***}$ are shown to be a perfect equilibrium reduce to, respectively, $\delta \leq 0$, $0 \leq \delta \leq 0$, and $1 \leq \delta$.

and

$$\delta < \frac{1}{2} \cdot \frac{(-1) + \sqrt{4(1-\alpha)^2 + 1}}{1-\alpha} =: g(\alpha),$$

respectively. Note that $f$ is increasing in $\alpha \in [0, 1)$ and $g$ is decreasing in $\alpha \in [0, 1)$ because

$$g'(\alpha) = \frac{-1 + \frac{1}{\sqrt{4(\alpha-1)^2+1}}}{2(1-\alpha)^2} < 0.$$

Since $g(0) = \frac{-1+\sqrt{5}}{2} (\approx 0.618) > 0 = f(0)$ and $f(\frac{1}{3}) = \frac{1}{2} = g(\frac{1}{3})$, it follows that $g(\alpha) > f(\alpha)$ for all $\alpha < \frac{1}{3}$. The proof is complete. □

# References


Lagunoff, R. and A. Matsui (1995): "An "Anti-Folk Theorem" for a Class of Asynchronously Repeated Games," Working paper.

——— (1997): "Asynchronous Choice in Repeated Coordination Games," *Econometrica*, 65, 1467–1477.


# A Appendix

As mentioned in footnote 1, our formulation of payoffs is different from that of Lagunoff and Matsui (1997). We detail our interpretation of Lagunoff and Matsui (1997)'s formulation here.

In Lagunoff and Matsui (1997), given a strategy profile $f = (f_1, f_2)$, a history $h^t \in H$, and $\tau \in \mathbb{N} \cup \{0\}$, denote by $\tilde{s}^{t+\tau}(f \mid h^t)$ a (stochastic) action profile in the $(t+\tau)$-th period induced by $f$ given history $h^t$ (the only difference from our formulation is that $\tau = 0$ is allowed here).

Given $f = (f_1, f_2)$, player $i$'s payoff at history $h^t$ is:

$$V_i(f \mid h^t) \equiv (1-\delta) \sum_{\tau=0}^{\infty} \delta^\tau \mathbb{E}[u_i(\tilde{s}^{t+\tau}(f \mid h^t))], \quad \text{(Lagunoff and Matsui, 1997, p. 1469)} \tag{3}$$

where $\mathbb{E}$ is the expectation operator.

Lagunoff and Matsui (1997, p. 1469) states that they "let $u_i(\tilde{s}(e)) = 0$ to simplify notation," where they "write $\tilde{s}(h^t) = \tilde{s}^t(f \mid h^t)$." Since $\tilde{s}(\cdot)$ is defined only when its argument is of the form $h^t$, it is not defined when the argument is $e$ (Lagunoff and Matsui (1997) do not let $e = h^{-1}$ as we do). We are then led to interpreting this specification as follows. Since equation (3) only defines the value conditional on $h^t$ with $t \in \mathbb{N} \cup \{0\}$ (recall that $h^0$ contains the action profile taken at period 0), we define the value conditional on the empty history. To do this, we let $h^{-1} \equiv e$ in understanding Lagunoff and Matsui (1997) as well. With this notation, in equation (3), $V_i(f \mid e)$ is equal to $V_i(f \mid h^{-1})$, which can be written as the expectation of

$$(1-\delta)[u_i(\tilde{s}^{-1}(f \mid e)) + \delta u_i(\tilde{s}^0(f \mid e)) + \delta^2 u_i(\tilde{s}^1(f \mid e)) + \dots].$$

By definition, $\tilde{s}^0(f \mid e)$ is the action profile taken at period 0, $\tilde{s}^1(f \mid e)$ is the action profile taken at period 1, and so on. Our interpretation is that the "$u_i(\tilde{s}(e)) = 0$" in Lagunoff and Matsui (1997) is meant to let $u_i(\tilde{s}^{-1}(f \mid e))$ (the only undefined term) be equal to 0 (that is, given $e = h^{-1}$, we have $\tilde{s}(e) = \tilde{s}(h^{-1}) = \tilde{s}^{-1+0}(f \mid e)$). In summary, each player's value after the empty history is simply $\delta$ times the discounted sum of payoffs from period 0 onward, as written in equation (1). In this paper, we proceeded with the understanding that Lagunoff and Matsui (1997) consider this specification.

Summarizing our interpretation of Lagunoff and Matsui (1997)'s formulation, in this paper, whenever we mention the payoff at time $t+1$ after a history that has ended at time $t$, we mean the expectation of $(1-\delta)\sum_{\tau=1}^{\infty} \delta^{\tau-1} u_i(s(t+\tau))$ where $s(t')$ represents the action profile at time $t'$.[7] As discussed in footnote 1, this is the expression of the payoff that the proofs in Lagunoff and Matsui (1997) use.[8]

An alternative interpretation would be that "$u_i(\tilde{s}(e)) = 0$" means that the players

[7]The value conditional on the history that has ended at time $t$ defined in Lagunoff and Matsui (1997) is the expectation of $(1-\delta)\left[u_i(s(t)) + \delta \sum_{\tau=1}^{\infty} \delta^{\tau-1} u_i(s(t+\tau))\right]$ where $s(t)$ is a fixed action profile that is recorded in the history that has ended in time $t$.

[8]This specification also follows the one defined in the working paper version (Lagunoff and Matsui, 1995). In that paper, players receive flow payoffs in continuous time, and the payoff was defined to be

$$r \sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} e^{-r\tau} u_i(s^k) d\tau,$$

where $r$ is the discount rate, $T_0 = 0$, $T_k$ is the $k$-th revision opportunity after time 0, and $s^k$ is the action profile between $T_k$ and $T_{k+1}$ (Lagunoff and Matsui, 1995, p. 5).

do not "count" the payoffs at period 0 and consider the payoffs from period 1 onward. Although such an interpretation makes unclear what the first term inside the expectation of (3) would be conditional on the empty history, we consider a model under such an interpretation (what we will term the "period-1 onward" model) in this Appendix. We present such a model and show that the optimality result is restored in the two-player case. We then prove that the optimality result does not extend to the case with more than two players, despite Lagunoff and Matsui (1997)'s claim that it does. We note that, as we state in Remark 3, under the "period-1 onward" model, the conclusion of Theorem 4 of Lagunoff and Matsui (1997) would trivially hold without any sufficient condition.

## A.1 The "Period-1 Onward" Model

Consider an alternative specification of payoffs where the value conditional on the empty history is represented as

$$V_i(f \mid e) \equiv (1-\delta)\mathbb{E}\left[\sum_{t=1}^{\infty} \delta^t u_i(\tilde{s}(t))\right],$$

where $\tilde{s}(t)$ is the (stochastic) action profile at time $t$ that results from $f$ conditional on $e$. That is, we do not count the payoffs that would realize given the action profile taken at time 0. We call this model the *"period-1 onward" model.*[9] In contrast, we call the model in the main section the *"period-0 onward" model.*

In this context, we formalize the optimality result as follows.

**Proposition 5** (Optimality Result)**.** *In the "period-1 onward" model, in any perfect equilibrium of a two-player asynchronous-move pure-coordination game, every player $i$ plays $s_i^*$ at every period except player 1's action at period 0.*

Note that player 1 is indifferent at period 0 and may take any action other than $s_1^*$ because the period-0 payoffs are not counted.

*Proof of Proposition 5.* By Theorem 1, at any history $h^0$ at which player 2 has taken $s_2^*$ in period 0, player 1 can guarantee himself the best payoff $u^*$ in period 1. Thus,

[9] It is not clear how to formulate such a model under the general moving structure in continuous time as in Section 3 of Lagunoff and Matsui (1997) (cf. Footnote 8).

by choosing $s_2^*$ in period 0, player 2 can guarantee herself a payoff of

$$(1-\delta)\cdot 0 + \delta u^*,$$

which is the best possible payoff at the empty history. Since any other action would result in a strictly lower payoff, player 2 takes $s_2^*$ at $t=0$ in any perfect equilibrium. Thus, each player $i$ takes $s_i^*$ from period 1 on in any perfect equilibrium. $\square$

**Remark 2.** Proposition 5 implies that the strategy profile $f^*$ constructed in Section 5.1 is not a perfect equilibrium in the "period-1 onward" model. To see when a player has an incentive to deviate, consider player 2. At period $t=0$, if player 2 follows the strategy $f_2^*$ to play $s^2$ then her payoff is $0+2\delta^2$. If she deviates by taking $s^1$ in period 0, then her payoff is $0+2\delta$. Thus, player 2 has an incentive to deviate in period 0 in the "period-1 onward" model. Similarly, $f^*$ constructed in Section 5.2 is not a perfect equilibrium in the "period-1 onward" model. To see this, consider player 2. At $t=0$, if player 2 follows the strategy $f_2^*$ to play $s^2$, then her payoff is $0+2\delta^{2m-2}$. If she deviates by taking $s^1$ in period 0, then her payoff is $0+2\delta$. Thus, she has an incentive to deviate at the empty history in the "period-1 onward" model.

**Remark 3.** When a stage game is a game of common interest (defined in Section 6), the "period-1 onward" model has a unique perfect equilibrium (ignoring player 1's period-0 action) if the discount factor is small enough. In the unique perfect equilibrium, every player $i$ plays $s_i^*$ at every period except player 1's action at period 0, irrespective of $\alpha$. Thus, in the "period-1 onward" model, the sufficient condition in Theorem 4 of Lagunoff and Matsui (1997) would not play an important role in the sense that there exists a range of discount factors in which there exists a unique perfect equilibrium (ignoring player 1's period-0 action) for any $\alpha \in (0,1)$ (defined in Section 6). That is, the conclusion of Theorem 4 of Lagunoff and Matsui (1997) would trivially hold without any sufficient condition.

## A.2 A Counterexample

Consider the "period-1 onward" model. Although Lagunoff and Matsui (1997, p. 1473) claim that "[t]he same optimality result as before is obtained" for the case with more than two players, we show that the following claim holds. As in Section 5.3, we

| | | $s^1$ | |
|---|---|---|---|
| | | $s^1$ | $s^2$ |
| | $s^1$ | $2, 2, 2$ | $0, 0, 0$ |
| | $s^2$ | $-10, -10, -10$ | $1, 1, 1$ |

| | | $s^2$ | |
|---|---|---|---|
| | | $s^1$ | $s^2$ |
| | $s^1$ | $-10, -10, -10$ | $-10, -10, -10$ |
| | $s^2$ | $-10, -10, -10$ | $-10, -10, -10$ |

Table 4: A Three-Player Example. Player 1 chooses a table, player 2 chooses a row, and player 3 chooses a column. Note that $(s_1^*, s_2^*, s_3^*) = (s^1, s^1, s^1)$.

suppose that players simultaneously choose an action at period 0, and player $i$ has a chance to revise her action at period $kn + i$ for every $k \in \mathbb{N} \cup \{0\}$.

**Proposition 6.** *In the "period-1 onward" model with more than two players, for any positive integer $K > 0$, there exist a pure-coordination game $G$ and $\overline{\delta} \in (0, 1)$ such that, if $\delta \in (\overline{\delta}, 1)$, the asynchronous-move game with its stage game being $G$ has a perfect equilibrium in which the players do not play $s^*$ for at least the first $K$ periods.*

To show this result, in Appendix A.3 we provide a counterexample for the optimality result.

## A.3 Two-Action $n$-Player Example

Appendix A.3.1 considers $K = 3$ with three players. Extending the argument to general $K$ and a greater number of players is simple and is explained in Appendix A.3.2.

### A.3.1 A Three-Player Example

Consider the three-player pure-coordination game depicted in Table 4. Consider the following strategy profile $f^*$.

$$f_1^*(h) \equiv s^1 \text{ for all } h, \qquad f_2^*(h) \equiv \begin{cases} s^2 & \text{if } h = e \\ s^1 & \text{if } h \neq e \end{cases}; \qquad \text{and}$$

$$f_3^*(h) \equiv \begin{cases} s^2 & \text{if } h = e \text{ or if } h \neq e \text{ and } s_2(t-1) = s^2 \\ s^1 & \text{if } h \neq e \text{ and } s_2(t-1) = s^1 \end{cases}.$$

We show that $f^*$ is a perfect equilibrium when $\delta \in (0, 1)$ is sufficiently large. Consider player 1. Since $s^1$ weakly dominates $s^2$ for player 1 and the actions of players 2 and 3 do not depend on player 1's action, it is a best response for player 1 to follow $f_1^*$ to play $s^1$ regardless of the history.

Consider player 2. Suppose that $t \geq 1$ and $(s_1(t-1), s_3(t-1)) \neq (s^1, s^2)$. Then, playing $s^1$ at $t$ gives a weakly greater payoff than playing $s^2$ at every period starting with time $t$. Hence, it is a best response for player 2 to follow $f_2^*$ to play $s^1$. Thus, it remains to consider the case with $t = 0$ and the case with $t \geq 1$ and $(s_1(t-1), s_3(t-1)) = (s^1, s^2)$.

At period 0, if player 2 follows $f_2^*$ and takes $s^2$, then her payoff is $(1-\delta)\delta + 2\delta^3$. If she instead takes $s^1$, then her payoff is $2\delta^3$. Since the former payoff is greater than the latter payoff, player 2 does not have an incentive to deviate at period 0.

Suppose that $t \geq 1$ and $(s_1(t-1), s_3(t-1)) = (s^1, s^2)$. If player 2 follows $f_2^*$ and takes $s^1$, then player 2 obtains a payoff of $2\delta$. If player 2 takes $s^2$ instead, then player 2 obtains a payoff of $(1-\delta)(1+\delta+\delta^2) + 2\delta^4$. Thus, player 2 has an incentive to follow $f_2^*$ if

$$2\delta \geq (1-\delta)(1+\delta+\delta^2) + 2\delta^4, \text{ that is, } \delta \geq \frac{1}{2}.$$

Consider player 3. Suppose first that $t \geq 1$ and $s_2(t-1) \neq s^2$. Then, playing $s^1$ at $t$ gives a weakly greater payoff than playing $s^2$ at every period starting with time $t$. Hence, it is a best response for player 3 to follow $f_3^*$ to play $s^1$. Thus, it remains to consider the case with $t = 0$ and the case with $t \geq 1$ and $s_2(t-1) = s^2$.

At period 0, if player 3 follows $f_3^*$ and takes $s^2$, then her payoff is $(1-\delta)\delta + 2\delta^3$. If player 3 takes $s^1$ then her payoff is $-10(1-\delta)\delta + 2\delta^2$. Thus, player 3 has an incentive to follow $f_3^*$ if

$$(1-\delta)\delta + 2\delta^3 \geq -10(1-\delta)\delta + 2\delta^2,$$

which can be shown to hold for any $\delta \in (0, 1)$.

Consider $t \geq 1$ and suppose $(s_1(t-1), s_2(t-1)) = (s^1, s^2)$. If player 3 follows $f_3^*$ to play $s^2$, then player 3 gets a payoff of $(1-\delta)(1+\delta) + 2\delta^3$. If player 3 deviates by taking $s^1$, then player 3 gets a payoff of $-10(1-\delta)(1+\delta) + 2\delta^2$. Thus, player 3 has an incentive to follow $f_3^*$ if

$$(1-\delta)(1+\delta) + 2\delta^3 \geq -10(1-\delta)(1+\delta) + 2\delta^2,$$

which can be shown to hold for any $\delta \in (0, 1)$.

If $t \geq 1$ and $(s_1(t-1), s_2(t-1)) = (s^2, s^2)$, by following $f_3^*$ to play $s^2$, player 3 gets a payoff of $(1-\delta)(-10+\delta)+2\delta^3$. If player 3 takes $s^1$, then player 3 gets a payoff of $-10(1-\delta)(1+\delta)+2\delta^2$. Thus, player 3 has an incentive to follow $f_3^*$ if

$$(1-\delta)(-10+\delta)+2\delta^3 \geq -10(1-\delta)(1+\delta)+2\delta^2,$$

which can be shown to hold for any $\delta \in (0,1)$.

### A.3.2 An $n$-Player Example

We show that we can extend the previous proof to general $K$ and $n$ players. To extend to general $K$, make the payoff function for players 2 and 3 when player 1 is choosing $s^1$ to be analogous to the one specified in Section 5.2. To extend to $n$ players, let the payoffs be such that players receive payoffs as specified in the previous sentence if all players in $\{4, \ldots, n\}$ play $s^1$ and they receive $-M$ otherwise, where $M$ is a large positive number.